

Common fixed point theorems of integral type in \mathcal{G} -metric space via control function

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Abstract

In this paper, we establish fixed point results for two pairs of functions with the assistance of CLR property in the context of \mathcal{G} -metric space. Our sequel generalizes various existing fixed-point results that are given in the literature. An illustrative example is likewise given to demonstrate that our speculation from metric space to \mathcal{G} -metric spaces is genuine.

Keywords: CLR property, G-metric space, common fixed point, weakly compatible map
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1 Introduction

In 2002, Branciari [4] gave the perception of new kind of contraction known as integral type contraction in the framework of complete metric space. Afterwards many authors established common fixed point theorems for integral type contraction in metric space, fuzzy metric space and cone metric spaces (see [6]-[7]). In 2006, Mustafa and Sims [14] proposed the perception of G-metric spaces as a generalization of metric space. Afterwards Mustafa et al. [15] proved fixed point results for one map satisfying condition in the frame of complete G-metric space. In 2015, Sarwar et al. [19] established fixed point results with the aid of CLR property in the frame of metric spaces. In 2016, Rahman et al. [17] gave common fixed point results of Altman integral type for four self-maps in the setting of S-metric space. In 2018, Panda et al. [16] generalized results of [4] with the assistance of a continuous cyclic map and established fixed point results for weaker integral contraction. In 2019, Kumar et al. [8] established common fixed point theorems in symmetrical G-metric space. In 2020, Arora et al. conferred common fixed point results for modified β -admissible contraction and almost Z-contraction in the edge of metric space and G-metric space (see [1, 9]). Afterwards, Debnath et al. [5] studied the existence and uniqueness of common fixed point theorems for Kannan, Reich and Chatterjea type pairs of self-maps in the context of complete metric space. In 2021, Arora [2] established some common fixed point results for four self maps in the context of Gs-metric space via CLR property. Recently, Murthy et al. [13] investigated common fixed point results for two covariant functions with the assistance of upper semi-continuous function in the frame of bipolar metric space.

Now, we present the significant definitions and theorems which are favourable in the proof of our sequel.

Theorem 1.1. [4] Let (X, σ) be a metric space, $m < 1$ and S_1 be self-mapping such that for every $x, y \in X$,

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$$\int_0^{\sigma(S_1x, S_1y)} \mu(s) ds \leq m \int_0^{\sigma(x, y)} \mu(s) ds$$

where $\mu : R^+ \rightarrow R$ is Lebesgue integrable mapping which is summable and non-negative, satisfying $\int_0^\ell \mu(s) ds > 0$, for each $\ell > 0$, then S_1 has a unique fixed point.

Definition 1.2. [14] Let X be a non empty set and $\mathcal{G} : X^3 \rightarrow [0, \infty)$ be a map which fulfils the following conditions:

- (i) $\mathcal{G}(x_1, y_1, z_1) = 0$ if $x_1 = y_1 = z_1$;
- (ii) $0 < \mathcal{G}(x_1, x_1, y_1)$ whenever $x_1 \neq y_1$, for all $x_1, y_1 \in X$;
- (iii) $\mathcal{G}(x_1, x_1, y_1) \leq \mathcal{G}(x_1, y_1, z_1)$, $y_1 \neq z_1$;
- (iv) $\mathcal{G}(x_1, y_1, z_1) = \mathcal{G}(x_1, z_1, y_1) = \mathcal{G}(y_1, x_1, z_1) = \mathcal{G}(z_1, x_1, y_1) = \mathcal{G}(y_1, z_1, x_1) = \mathcal{G}(z_1, y_1, x_1)$;
- (v) $\mathcal{G}(x_1, y_1, z_1) \leq \mathcal{G}(x_1, a_1, a_1) + \mathcal{G}(a_1, y_1, z_1)$;

for every $x_1, y_1, z_1, a_1 \in X$, then the function \mathcal{G} is said to be \mathcal{G} -metric on X and (X, \mathcal{G}) is known as \mathcal{G} -metric space.

Definition 1.3. [18] Let Ψ be a family of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following properties:

- (i) ψ is upper semi-continuous, strictly increasing;
- (ii) $\{\psi^n(t)\}_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$, for all $t > 0$;
- (iii) $\psi(t) < t$, for every $t > 0$.

These functions are known as comparison functions.

Definition 1.4. [3] Two self mappings A and B on (X, \mathcal{G}) are said to be weakly compatible if they commute at coincident points.

Definition 1.5. Let (X, \mathcal{G}) be a \mathcal{G} metric space and f, g, P, Q be self maps on X . The pairs (P, Q) and (R, S) are satisfy the joint common limit in the range of mappings (JCLR) property if there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = \lim_{n \rightarrow \infty} Rx_n = \lim_{n \rightarrow \infty} Sx_n = Qv = Rv$ for some $v \in X$.

Definition 1.6. [11] Let (X, \mathcal{G}) be a \mathcal{G} metric space and f, g, h and j be four self maps. The pairs (f, j) and (g, h) satisfy common limit range property with respect to j and h , denoted by CLR_{jh} if there exists two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} jx_n = \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} hy_n = \ell \in j(X) \cap h(X).$$

The main aim of our paper is to establish fixed point theorems with the help of control function and CLR property in the context of generalized metric space. Our results enhance and unify the results established by [16] and various other results in the literature.

2 Main Results

Theorem 2.1. Let (X, \mathcal{G}) be \mathcal{G} -metric space and S_1, S_2, S_3, S_4 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x, S_2y, S_2y)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right)$$

where $\mu : R^+ \rightarrow R$ is Lebesgue integrable mapping which is summable and non-negative,

$$\sigma = \{\sigma/\sigma : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi continuous, } \sigma(0) = 0 \text{ and } \sigma(s) < s \text{ for each } s > 0\}$$

and

$$\nu(x, y, z) = \max\{\mathcal{G}(S_3x, S_4y, S_4y), \mathcal{G}(S_3x, S_2y, S_2y), \mathcal{G}(S_4y, S_2y, S_2y), \mathcal{G}(S_1x, S_3x, S_3x)\}.$$

Also, satisfying the following condition that the pairs (S_1, S_3) and (S_2, S_4) satisfy CLR property with respect to maps S_3 and S_4 . Further if both the pairs (S_1, S_3) and (S_2, S_4) are weakly compatible, then S_1, S_2, S_3, S_4 have a unique common fixed point in X .

Proof . Let (S_1, S_3) and (S_2, S_4) satisfy $CLR_{S_3S_4}$ property, therefore there exist two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n \rightarrow \infty} S_1 u_n = \lim_{n \rightarrow \infty} S_3 u_n = \lim_{n \rightarrow \infty} S_2 v_n = \lim_{n \rightarrow \infty} S_4 v_n = w. \quad (2.1)$$

for some $w \in S_4(X) \cap S_3(X)$. Since $w \in S_3(X)$, so there exists a point $z \in X$ such that $S_3 z = w$. Taking (2.1) into account, we obtain

$$\lim_{n \rightarrow \infty} S_1 u_n = \lim_{n \rightarrow \infty} S_3 u_n = \lim_{n \rightarrow \infty} S_2 v_n = \lim_{n \rightarrow \infty} S_4 v_n = w = S_3 z. \quad (2.2)$$

Next, we claim that $S_1 z = S_3 z$. Suppose on the contrary that $S_1 z \neq S_3 z$. Substituting $x = z$ and $y = v_n$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1 z, S_2 v_n, S_2 v_n)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(z, v_n, v_n)} \mu(s) ds \right),$$

where

$$\nu(z, v_n, v_n) = \max\{\mathcal{G}(S_3 z, S_4 v_n, S_4 v_n), \mathcal{G}(S_3 z, S_2 v_n, S_2 v_n), \mathcal{G}(S_4 v_n, S_2 v_n, S_2 v_n), \mathcal{G}(S_1 z, S_3 z, S_3 z)\}.$$

Letting $n \rightarrow \infty$ in the above inequalities and using (2.2), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu(z, v_n, v_n) &= \max\{\mathcal{G}(w, w, w), \mathcal{G}(w, w, w), \mathcal{G}(w, w, w), \mathcal{G}(S_1 z, w, w)\} \\ &= \mathcal{G}(S_1 z, w, w) \end{aligned}$$

and

$$\begin{aligned} \int_0^{\mathcal{G}(S_1 z, w, w)} \mu(s) ds &= \lim_{n \rightarrow \infty} \sup \int_0^{\mathcal{G}(S_1 z, S_2 v_n, S_2 v_n)} \mu(s) ds \\ &\leq \lim_{n \rightarrow \infty} \sup \sigma \left(\int_0^{\lambda(z, v_n, v_n)} \mu(s) ds \right) \\ &\leq \sigma \left(\lim_{n \rightarrow \infty} \sup \int_0^{\lambda(z, v_n, v_n)} \mu(s) ds \right) \\ &= \sigma \left(\int_0^{\mathcal{G}(S_1 z, w, w)} \mu(s) ds \right) \\ &< \int_0^{\mathcal{G}(S_1 z, w, w)} \mu(s) ds, \end{aligned}$$

which is a contradiction. Therefore, $S_1 z = S_3 z$. Hence,

$$S_1 z = S_3 z = w. \quad (2.3)$$

Since $w \in j(X)$, there exists $\rho \in X$ such that $j\rho = w$. Substituting, $j\rho = w$ in (2.1), we obtain

$$\lim_{n \rightarrow \infty} S_1 u_n = \lim_{n \rightarrow \infty} S_3 u_n = \lim_{n \rightarrow \infty} S_2 v_n = \lim_{n \rightarrow \infty} S_4 v_n = w = j\rho. \quad (2.4)$$

Now, we want to prove that $S_2 \rho = S_4 \rho$. Suppose that $S_2 \rho \neq S_4 \rho$. Substituting $x = u_n$ and $y = \rho$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1 u_n, S_2 \rho, S_2 \rho)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(u_n, \rho, \rho)} \mu(s) ds \right),$$

where

$$\nu(u_n, \rho, \rho) = \max\{\mathcal{G}(S_3 u_n, S_4 \rho, S_4 \rho), \mathcal{G}(S_3 u_n, S_2 \rho, S_2 \rho), \mathcal{G}(S_4 \rho, S_2 \rho, S_2 \rho), \mathcal{G}(S_1 u_n, S_3 u_n, S_3 u_n)\}.$$

Letting $n \rightarrow \infty$ and using (2.4), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu(u_n, \rho, \rho) &= \max\{\mathcal{G}(w, w, w), \mathcal{G}(w, w, w), \mathcal{G}(w, w, w), \mathcal{G}(S_1 u_n, w, w)\} \\ &= \mathcal{G}(S_1 u_n, w, w) \end{aligned}$$

and

$$\begin{aligned} \int_0^{\mathcal{G}(S_1 u_n, w, w)} \mu(s) ds &= \limsup_{n \rightarrow \infty} \int_0^{\mathcal{G}(S_1 u_n, S_2 \rho, S_2 \rho)} \mu(s) ds \\ &\leq \limsup_{n \rightarrow \infty} \sigma \left(\int_0^{\lambda(u_n, \rho, \rho)} \mu(s) ds \right) \\ &\leq \sigma \left(\limsup_{n \rightarrow \infty} \int_0^{\lambda(u_n, \rho, \rho)} \mu(s) ds \right) \\ &= \sigma \left(\int_0^{\mathcal{G}(S_1 u_n, w, w)} \mu(s) ds \right) \\ &< \int_0^{\mathcal{G}(S_1 u_n, w, w)} \mu(s) ds, \end{aligned}$$

which is a contradiction. Therefore, $S_2 \rho = S_4 \rho$. Hence,

$$S_2 \rho = S_4 \rho = w. \quad (2.5)$$

From (2.3) and (2.5), we obtain

$$S_1 z = S_3 z = S_2 \rho = S_4 \rho = w. \quad (2.6)$$

Next, we show that w is common fixed point of S_1, S_2, S_3, S_4 .

Since (S_1, S_3) and (S_2, S_4) are weakly compatible, $S_1 z = S_3 z$ implies that $S_3 S_1 z = S_1 S_3 z$. Now, using (2.6), we obtain

$$S_1 w = S_3 w. \quad (2.7)$$

Also, $S_2 \rho = S_4 \rho$ implies $S_4 S_2 \rho = S_2 S_4 \rho$. Now, using (2.6), we obtain

$$S_2 w = S_4 w. \quad (2.8)$$

Next, we claim that $S_1 w = w$. Let us suppose contrary that $S_1 w \neq w$. Substituting $x = w$ and $y = \rho$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1 w, S_2 \rho, S_2 \rho)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(w, \rho, \rho)} \mu(s) ds \right)$$

where

$$\nu(w, \rho, \rho) = \max\{\mathcal{G}(S_3 w, S_4 \rho, S_4 \rho), \mathcal{G}(S_3 w, S_2 \rho, S_2 \rho), \mathcal{G}(S_4 \rho, S_2 \rho, S_2 \rho), \mathcal{G}(S_1 w, S_3 w, S_3 w)\}.$$

Letting $n \rightarrow \infty$ and using (2.6), (2.7), (2.8), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu(w, \rho, \rho) &= \max\{\mathcal{G}(S_1 w, w, w), \mathcal{G}(S_1 w, w, w), \mathcal{G}(w, w, w), \mathcal{G}(S_1 w, S_1 w, S_1 w)\} \\ &= \mathcal{G}(S_1 w, w, w) \end{aligned}$$

and

$$\begin{aligned}
\int_0^{\mathcal{G}(S_1 w, w, w)} \mu(s) ds &= \limsup_{n \rightarrow \infty} \int_0^{\mathcal{G}(S_1 w, S_2 \rho, S_2 \rho)} \mu(s) ds \\
&\leq \limsup_{n \rightarrow \infty} \sigma \left(\int_0^{\lambda(w, \rho, \rho)} \mu(s) ds \right) \\
&\leq \sigma \left(\limsup_{n \rightarrow \infty} \int_0^{\lambda(w, \rho, \rho)} \mu(s) ds \right) \\
&= \sigma \left(\int_0^{\mathcal{G}(S_1 w, w, w)} \mu(s) ds \right) \\
&< \int_0^{\mathcal{G}(S_1 w, w, w)} \mu(s) ds,
\end{aligned}$$

which is a contradiction. Therefore, $S_1 w = w$. From (2.7), we obtain

$$S_1 w = S_3 w = w. \quad (2.9)$$

In the similar way by substituting $x = z, y = w$ in assumption of Theorem 2.1 and using (2.6), (2.8), we obtain

$$S_2 w = S_4 w = w. \quad (2.10)$$

Combining (2.9) and (2.10), we obtain

$$S_1 w = S_3 w = w = S_2 w = S_4 w.$$

Consequently, w is a common fixed point of S_1, S_2, S_3 and S_4 . Lastly, we shall examine the uniqueness of common fixed point of S_1, S_2, S_3 and S_4 . Let us assume that ρ_1 and ρ_2 are two common fixed points of S_1, S_2, S_3 and S_4 . Substituting $x = \rho_1$ and $y = \rho_2$ in the presumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1 \rho_1, S_2 \rho_2, S_2 \rho_2)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(\rho_1, \rho_2, \rho_2)} \mu(s) ds \right),$$

where

$$\begin{aligned}
\nu(\rho_1, \rho_2, \rho_2) &= \max\{\mathcal{G}(S_3 \rho_1, S_4 \rho_2, S_4 \rho_2), \mathcal{G}(S_3 \rho_1, S_2 \rho_2, S_2 \rho_2), \mathcal{G}(S_4 \rho_2, S_2 \rho_2, S_2 \rho_2), \mathcal{G}(S_1 \rho_1, S_3 \rho_1, S_3 \rho_1)\} \\
&= \max\{\mathcal{G}(\rho_1, \rho_2, \rho_2), \mathcal{G}(\rho_1, \rho_2, \rho_2), \mathcal{G}(\rho_2, \rho_2, \rho_2), \mathcal{G}(\rho_1, \rho_1, \rho_1)\} \\
&= \mathcal{G}(\rho_1, \rho_2, \rho_2).
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\mathcal{G}(\rho_1, \rho_2, \rho_2)} \mu(s) ds &\leq \sigma \left(\int_0^{\mathcal{G}(\rho_1, \rho_2, \rho_2)} \mu(s) ds \right) \\
&< \int_0^{\mathcal{G}(\rho_1, \rho_2, \rho_2)} \mu(s) ds,
\end{aligned}$$

which is a logical inconsistency. Therefore, $\rho_1 = \rho_2$. Consequently, w is a unique common fixed point of S_1, S_2, S_3 and S_4 . \square

Theorem 2.2. Let (X, \mathcal{G}) be \mathcal{G} -metric space and S_1, S_2, S_3, S_4 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1 x, S_2 y, S_2 y)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right)$$

where $\mu : R^+ \rightarrow R$ is Lebesgue integrable mapping which is summable and non-negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi continuous, } \sigma(0) = 0 \text{ and } \sigma(s) < s \text{ for each } s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}(S_3x, S_4y, S_4y), \mathcal{G}(S_3x, S_2y, S_2y), \mathcal{G}(S_4y, S_2y, S_2y), \mathcal{G}(S_1x, S_3x, S_3x)\}.$$

Also satisfying the following condition that the pairs (S_1, S_3) and (S_2, S_4) satisfy E.A property such that S_3 or S_4 is closed subspace of X . Further if both the pairs (S_1, S_3) and (S_2, S_4) are weakly compatible, then S_1, S_2, S_3, S_4 have a unique common fixed point in X .

Corollary 2.3. Let (X, \mathcal{G}) be \mathcal{G} -metric space and S_1, S_2 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x, S_1y, S_1z)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right)$$

where $\mu : R^+ \rightarrow R$ is Lebesgue integrable mapping which is summable and non-negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi continuous, } \sigma(0) = 0 \text{ and } \sigma(s) < s \text{ for each } s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}((S_2x, S_2y, S_2z), \mathcal{G}((S_2x, S_1x, S_1x), \mathcal{G}(S_2y, S_1y, S_1y), \mathcal{G}(S_2z, S_1z, S_1z)\}.$$

Also satisfying the following condition that the pair (S_1, S_2) satisfy CLR property with respect to map S_2 , then S_1 and S_2 have a coincidence point in X . Further, if the pair (S_1, S_2) is weakly compatible, then S_1, S_2 have a unique common fixed point in X .

Corollary 2.4. Let (X, \mathcal{G}) be \mathcal{G} -metric space and S_1, S_2, S_3 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x, S_2y, S_2y)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right)$$

where $\mu : R^+ \rightarrow R$ is Lebesgue integrable mapping which is summable and non-negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \rightarrow \mathbb{R} \text{ is upper semi continuous, } \sigma(0) = 0 \text{ and } \sigma(s) < s \text{ for each } s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}((S_3x, S_2y, S_2y), \mathcal{G}((S_3y, S_1x, S_1x), \mathcal{G}(S_3x, S_1y, S_1y), \mathcal{G}(S_3y, S_2x, S_2x)\}.$$

Also satisfying the following condition that the pairs (S_1, S_2) and (S_1, S_3) satisfy CLR property with respect to maps S_2 and S_3 . Further, if the pair (S_1, S_2) and (S_1, S_3) are weakly compatible, then S_1, S_2 and S_3 have a unique common fixed point in X .

Example 2.5. Let (X, \mathcal{G}) be \mathcal{G} -metric space with the metric $\mathcal{G}(x, y, z) = |x-y| + |y-z| + |z-x|$, for all x, y, z in $X = (0, 5)$. Let S_1, S_2, S_3 and S_4 be four self-mappings such that for every $x, y \in X, t > 0$,

$$S_1x = \begin{cases} 1 & \text{if } x \in (0, 4] \\ \frac{1}{8} & \text{otherwise} \end{cases}; \quad S_2x = \begin{cases} 1 & \text{if } x \in (0, 4] \\ \frac{1}{9} & \text{otherwise} \end{cases}$$

$$S_3x = \begin{cases} 1 & \text{if } x \in (0, 4] \\ \frac{1}{5} & \text{otherwise} \end{cases} \quad \text{and} \quad S_4x = \begin{cases} 1 & \text{if } x \in (0, 4] \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Let $\{u_n\} = \frac{1}{n}$ and $\{v_n\} = \frac{1}{n+1}$ be two sequences in X . Then,

$$\lim_{n \rightarrow \infty} S_1u_n = \lim_{n \rightarrow \infty} S_1\left(\frac{1}{n}\right) = 1;$$

$$\lim_{n \rightarrow \infty} S_2v_n = \lim_{n \rightarrow \infty} S_2\left(\frac{1}{n+1}\right) = 1;$$

$$\lim_{n \rightarrow \infty} S_3v_n = \lim_{n \rightarrow \infty} S_3\left(\frac{1}{n+1}\right) = 1;$$

$$\lim_{n \rightarrow \infty} S_4u_n = \lim_{n \rightarrow \infty} S_4\left(\frac{1}{n}\right) = 1.$$

Therefore,

$$\lim_{n \rightarrow \infty} S_1 u_n = \lim_{n \rightarrow \infty} S_2 v_n = \lim_{n \rightarrow \infty} S_3 v_n = \lim_{n \rightarrow \infty} S_4 u_n = 1.$$

So, (S_1, S_3) and (S_2, S_4) enjoys the $CLR_{S_3 S_4}$ property. Let us define $\mu(s) = 4s$, and $\sigma(s) = \frac{s}{4}$. Whenever, $x, y \in [0, 4]$, then, $S_1 x = S_2 y = S_3 y = S_4 x = 1$, which implies that

$$\nu(x, y, z) = 0 = \max\{\mathcal{G}(S_3 x, S_4 y, S_4 y), \mathcal{G}(S_3 x, S_2 y, S_2 y), \mathcal{G}(S_4 y, S_2 y, S_2 y), \mathcal{G}(S_1 x, S_3 x, S_3 x)\}.$$

Also, $\mathcal{G}(S_1 x, S_2 y, S_2 y) = 0$. Therefore,

$$\int_0^{\mathcal{G}(S_1 x, S_2 y, S_2 y)} \mu(s) ds = 0 = \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right)$$

Whenever, $x, y \in [4, 5]$, then $S_1 x = \frac{1}{8}$, $S_2 y = \frac{1}{9}$, $S_3 x = \frac{1}{4}$ and $S_4 y = \frac{1}{5}$. Therefore,

$$\int_0^{\mathcal{G}(S_1 x, S_2 y, S_2 y)} \mu(s) ds = \int_0^{2 \times \frac{1}{72}} 4s ds = 2s^2 \Big|_0^{2 \times \frac{1}{72}} = 2 \times \frac{1}{1296} = \frac{1}{648} = 0.001543.$$

Also, $\nu(x, y, z) = \max\{\frac{1}{10}, \frac{10}{36}, \frac{8}{45}, \frac{1}{4}\} = \frac{1}{4}$. Thus,

$$\sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right) = \sigma \left(\int_0^{\frac{1}{4}} 4s ds \right) = \frac{1}{4} \times \left(\int_0^{\frac{1}{4}} 4s ds \right) = \frac{1}{4} \times 2 \times \frac{1}{16} = 0.03125.$$

From above two equations, it follows that

$$\int_0^{\mathcal{G}(S_1 x, S_2 y, S_2 y)} \mu(s) ds \leq \sigma \left(\int_0^{\nu(x, y, z)} \mu(s) ds \right).$$

Now, hypothesis of Theorem 2.1 is fulfilled. Therefore, S_1, S_2, S_3 and S_4 have a unique common fixed point in X which is $x = 1$.

3 Conclusion

In this paper, the fixed point results are investigated with the aid of CLR property for two pairs of functions in the context of \mathcal{G} -metric space. Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Our results can be utilized to find solution of fractional non-linear differential and integral equations.

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