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Common fixed point theorems of integral type in $\mathcal{G}$-metric space via control function

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Abstract

In this paper, we establish fixed point results for two pairs of functions with the assistance of CLR property in the context of $\mathcal{G}$-metric space. Our sequel generalizes various existing fixed-point results that are given in the literature. An illustrative example is likewise given to demonstrate that our speculation from metric space to $\mathcal{G}$-metric spaces is genuine.

Keywords: CLR property, G-metric space, common fixed point, weakly compatible map
2020 MSC: 47H10, 54H25

1 Introduction

In 2002, Branciari [4] gave the perception of new kind of contraction known as integral type contraction in the framework of complete metric space. Afterwards many authors established common fixed point theorems for integral type contraction in metric space, fuzzy metric space and cone metric spaces (see [6]-[7]). In 2006, Mustafa and Sims [14] proposed the perception of G-metric spaces as a generalization of metric space. Afterwards Mustafa et al. [15] proved fixed point results for one map satisfying condition in the frame of complete G-metric space. In 2015, Sarwar et al. [19] established fixed point results with the aid of CLR propery in the frame of metric spaces. In 2016, Rahman et al.[17] gave common fixed point results of Altman integral type for four self-maps in the setting of S-metric space. In 2018, Panda et al.[16] generalized results of [4] with the assistance of a continuous cyclic map and established fixed point results for weaker integral contraction. In 2019, Kumar et al.[8] established common fixed point theorems in symmetrical G-metric space. In 2020, Arora et al. conferred common fixed point results for modified β-admissible contraction and almost Z-contraction in the edge of metric space and G-metric space (see [1, 9]). Afterwards, Debnath et al. [5] studied the existence and uniqueness of common fixed point theorems for Kannan, Reich and Chatterjea type pairs of self-maps in the context of complete metric space. In 2021, Arora [2] established some common fixed point results for four self maps in the context of Gs-metric space via CLR property. Recently, Murthy et al. [13] investigated common fixed point results for two covariant functions with the assistance of upper semi-continuous function in the frame of bipolar metric space.

Now, we present the significant definitions and theorems which are favourable in the proof of our sequel.

Theorem 1.1. [4] Let (X, σ) be a metric space, $m < 1$ and S_1 be self-mapping such that for every $x, y \in X$,

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$$\int_0^{\sigma(S_1x,S_1y)} \mu(s)ds \leq m \int_0^{\sigma(x,y)} \mu(s)ds$$

where $\mu : R^+ \to R$ is Lebesgue integrable mapping which is summable and non- negative, satisfying $\int_0^{\ell} \mu(s)ds > 0$, for each $\ell > 0$, then S_1 has a unique fixed point.

Definition 1.2. [14] Let X be a non empty set and $\mathcal{G} : X^3 \to [0, \infty)$ be a map which fulfils the following conditions:
(i) $\mathcal{G}(x_1, y_1, z_1) = 0$ if $x_1 = y_1 = z_1$;
(ii) $0 < \mathcal{G}(x_1, x_1, y_1)$ whenever $x_1 \neq y_1$, for all $x_1, y_1 \in X$;
(iii) $\mathcal{G}(x_1, x_1, y_1) \leq \mathcal{G}(x_1, y_1, z_1)$, $y_1 \neq z_1$;
(iv) $\mathcal{G}(x_1, y_1, z_1) = \mathcal{G}(x_1, z_1, y_1) = \mathcal{G}(y_1, x_1, z_1) = \mathcal{G}(z_1, x_1, y_1) = \mathcal{G}(y_1, z_1, x_1) = \mathcal{G}(z_1, y_1, x_1)$;
(v) $\mathcal{G}(x_1, y_1, z_1) \leq \mathcal{G}(x_1, a_1, a_1) + \mathcal{G}(a_1, y_1, z_1)$;
for every $x_1, y_1, z_1, a_1 \in X$, then the function $\mathcal{G}$ is said to be $\mathcal{G}$-metric on X and $(X, \mathcal{G})$ is known as $\mathcal{G}$-metric space.

Definition 1.3. [18] Let Ψ be a family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i)ψ is upper semi-continuous, strictly increasing;
(ii)$\{\psi^n(t)\}_{n\in N}$ converges to 0 as $n \to \infty$, for all $t > 0$;
(iii)$\psi(t) < t$, for every $t > 0$.
These functions are known as comparison functions.

Definition 1.4. [3] Two self mappings A and B on $(X, \mathcal{G})$ are said to be weakly compatible if they commute at coincident points.

Definition 1.5. Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space and f, g, P, Q be self maps on X.The pairs (P, Q) and (R, S) are satisfy the joint common limit in the range of mappings (JCLR) property if there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Sx_n = Qv = Rv$
for some v $\in X$.

Definition 1.6. [11] Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space and f, g, h and j be four self maps. The pairs (f, j) and (g, h) satisfy common limit range property with respect to j and h, denoted by CLR_{jh} if there exists two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} jx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hy_n = \ell \in j(X) \cap h(X).$$

The main aim of our paper is to establish fixed point theorems with the help of control function and CLR property in the context of generalized metric space. Our results enhance and unify the results established by [16] and various other results in the literature.

2 Main Results

Theorem 2.1. Let $(X, \mathcal{G})$ be $\mathcal{G}$-metric space and S_1, S_2, S_3, S_4 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x,S_2y,S_2y)} \mu(s)ds \leq \sigma\left(\int_0^{\nu(x,y,z)} \mu(s)ds\right)$$

where $\mu : R^+ \to R$ is Lebesgue integrable mapping which is summable and non-negative,

$$\sigma = \{\sigma/\sigma : \mathbb{R} \to \mathbb{R} \text{ is upper semi continuous, } \sigma(0) = 0 \text{ and } \sigma(s) < s \text{ for each } s > 0\}$$

and

$$\nu(x, y, z) = \max\{\mathcal{G}(S_3x, S_4y, S_4y), \mathcal{G}(S_3x, S_2y, S_2y), \mathcal{G}(S_4y, S_2y, S_2y), \mathcal{G}(S_1x, S_3x, S_3x)\}.$$

Also, satisfying the following condition that the pairs (S_1, S_3) and (S_2, S_4) satisfy CLR property with respect to maps S_3 and S_4. Further if both the pairs (S_1, S_3) and (S_2, S_4) are weakly compatible, then S_1, S_2, S_3, S_4 have a unique common fixed point in X.

Proof . Let (S_1, S_3) and (S_2, S_4) satisfy $CLR_{S_3S_4}$ property, therefore there exist two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n\to\infty} S_1u_n = \lim_{n\to\infty} S_3u_n = \lim_{n\to\infty} S_2v_n = \lim_{n\to\infty} S_4v_n = w. \tag{2.1}$$

for some $w \in S_4(X) \cap S_3(X)$. Since $w \in S_3(X)$, so there exists a point $z \in X$ such that $S_3z = w$. Taking (2.1) into account, we obtain

$$\lim_{n\to\infty} S_1u_n = \lim_{n\to\infty} S_3u_n = \lim_{n\to\infty} S_2v_n = \lim_{n\to\infty} S_4v_n = w = S_3z. \tag{2.2}$$

Next, we claim that $S_1z = S_3z$. Suppose on the contrary that $S_1z \neq S_3z$. Substituting $x = z$ and $y = v_n$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1z,S_2v_n,S_2v_n)} \mu(s)ds \leq \sigma(\int_0^{\nu(z,v_n,v_n)} \mu(s)ds),$$

where

$$\nu(z, v_n, v_n) = \max\{\mathcal{G}(S_3z, S_4v_n, S_4v_n), \mathcal{G}(S_3z, S_2v_n, S_2v_n), \mathcal{G}(S_4v_n, S_2v_n, S_2v_n), \mathcal{G}(S_1z, S_3z, S_3z)\}.$$

Letting $n \to \infty$ in the above inequalities and using (2.2), we obtain

$$\begin{aligned}\lim_{n\to\infty} \nu(z, v_n, v_n) &= \max\{\mathcal{G}(w, w, w), \mathcal{G}(w, w, w)), \mathcal{G}(w, w, w), \mathcal{G}(S_1z, w, w)\}\\ &= \mathcal{G}(S_1z, w, w)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathcal{G}(S_1z,w,w)} \mu(s)ds &= \lim_{n\to\infty} sup \int_0^{\mathcal{G}(S_1z,S_2v_n,S_2v_n)} \mu(s)ds\\ &\leq \lim_{n\to\infty} sup\sigma(\int_0^{\lambda(z,v_n,v_n)} \mu(s)ds)\\ &\leq \sigma(\lim_{n\to\infty} sup \int_0^{\lambda(z,v_n,v_n)} \mu(s)ds)\\ &= \sigma(\int_0^{\mathcal{G}(S_1z,w,w)} \mu(s)ds)\\ &< \int_0^{\mathcal{G}(S_1z,w,w)} \mu(s)ds,\end{aligned}$$

which is a contradiction. Therefore, $S_1z = S_3z$. Hence,

$$S_1z = S_3z = w. \tag{2.3}$$

Since $w \in j(X)$, there exists $\rho \in X$ such that $j\rho = w$. Substituting, $j\rho = w$ in (2.1), we obtain

$$\lim_{n\to\infty} S_1u_n = \lim_{n\to\infty} S_3u_n = \lim_{n\to\infty} S_2v_n = \lim_{n\to\infty} S_4v_n = w = j\rho. \tag{2.4}$$

Now, we want to prove that $S_2\rho = S_4\rho$. Suppose that $S_2\rho \neq S_4\rho$. Substituting $x = u_n$ and $y = \rho$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1u_n,S_2\rho,S_2\rho)} \mu(s)ds \leq \sigma(\int_0^{\nu(u_n,\rho,\rho)} \mu(s)ds),$$

where

$$\nu(u_n, \rho, \rho) = \max\{\mathcal{G}(S_3u_n, S_4\rho, S_4\rho), \mathcal{G}(S_3u_n, S_2\rho, S_2\rho), \mathcal{G}(S_4\rho, S_2\rho, S_2\rho), \mathcal{G}(S_1u_n, S_3u_n, S_3u_n)\}.$$

Letting $n \to \infty$ and using (2.4), we obtain

$$\begin{aligned}\lim_{n\to\infty} \nu(u_n, \rho, \rho) &= \max\{\mathcal{G}(w,w,w), \mathcal{G}(w,w,w)), \mathcal{G}(w,w,w), \mathcal{G}(S_1u_n, w, w)\}\\ &= \mathcal{G}(S_1u_n, w, w)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathcal{G}(S_1u_n,w,w)} \mu(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathcal{G}(S_1u_n,S_2\rho,S_2\rho)} \mu(s)ds\\ &\leq \lim_{n\to\infty}\sup \sigma(\int_0^{\lambda(u_n,\rho,\rho)} \mu(s)ds)\\ &\leq \sigma(\lim_{n\to\infty}\sup \int_0^{\lambda(u_n,\rho,\rho)} \mu(s)ds)\\ &= \sigma(\int_0^{\mathcal{G}(S_1u_n,w,w)} \mu(s)ds)\\ &< \int_0^{\mathcal{G}(S_1u_n,w,w)} \mu(s)ds,\end{aligned}$$

which is a contradiction. Therefore, $S_2\rho = S_4\rho$. Hence,

$$S_2\rho = S_4\rho = w. \tag{2.5}$$

From (2.3) and (2.5), we obtain

$$S_1z = S_3z = S_2\rho = S_4\rho = w. \tag{2.6}$$

Next, we show that w is common fixed point of S_1, S_2, S_3, S_4.
Since (S_1, S_3) and (S_2, S_4) are weakly compatible, $S_1z = S_3z$ implies that $S_3S_1z = S_1S_3z$. Now, using (2.6), we obtain

$$S_1w = S_3w. \tag{2.7}$$

Also, $S_2\rho = S_4\rho$ implies $S_4S_2\rho = S_2S_4\rho$. Now, using (2.6), we obtain

$$S_2w = S_4w. \tag{2.8}$$

Next, we claim that $S_1w = w$. Let us suppose contrary that $S_1w \neq w$. Substituting $x = w$ and $y = \rho$ in the assumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1w,S_2\rho,S_2\rho)} \mu(s)ds \leq \sigma(\int_0^{\nu(w,\rho,\rho)} \mu(s)ds)$$

where

$$\nu(w, \rho, \rho) = \max\{\mathcal{G}(S_3w, S_4\rho, S_4\rho), \mathcal{G}(S_3w, S_2\rho, S_2\rho), \mathcal{G}(S_4\rho, S_2\rho, S_2\rho), \mathcal{G}(S_1w, S_3w, S_3w)\}.$$

Letting $n \to \infty$ and using (2.6), (2.7), (2.8), we obtain

$$\begin{aligned}\lim_{n\to\infty} \nu(w, \rho, \rho) &= \max\{\mathcal{G}(S_1w, w, w), \mathcal{G}(S_1w, w, w)), \mathcal{G}(w, w, w), \mathcal{G}(S_1w, S_1w, S_1w)\}\\ &= \mathcal{G}(S_1w, w, w)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathcal{G}(S_1w,w,w)}\mu(s)ds &= \lim_{n\to\infty}\sup\int_0^{\mathcal{G}(S_1w,S_2\rho,S_2\rho)}\mu(s)ds\\ &\le \lim_{n\to\infty}\sup\sigma(\int_0^{\lambda(w,\rho,\rho)}\mu(s)ds)\\ &\le \sigma(\lim_{n\to\infty}\sup\int_0^{\lambda(w,\rho,\rho)}\mu(s)ds)\\ &= \sigma(\int_0^{\mathcal{G}(S_1w,w,w)}\mu(s)ds)\\ &< \int_0^{\mathcal{G}(S_1w,w,w)}\mu(s)ds,\end{aligned}$$

which is a contradiction. Therefore, $S_1w=w$. From (2.7), we obtain

$$S_1w=S_3w=w. \tag{2.9}$$

In the similar way by substituting $x=z, y=w$ in assumption of Theorem 2.1 and using (2.6), (2.8), we obtain

$$S_2w=S_4w=w. \tag{2.10}$$

Combining (2.9) and (2.10), we obtain

$$S_1w=S_3w=w=S_2w=S_4w.$$

Consequently, w is a common fixed point of S_1,S_2,S_3 and S_4. Lastly, we shall examine the uniqueness of common fixed point of S_1,S_2,S_3 and S_4. Let us assume that ρ_1 and ρ_2 are two common fixed points of S_1,S_2,S_3 and S_4. Substituting $x=\rho_1$ and $y=\rho_2$ in the presumption of Theorem 2.1, we obtain

$$\int_0^{\mathcal{G}(S_1\rho_1,S_2\rho_2,S_2\rho_2)}\mu(s)ds\le\sigma\left(\int_0^{\nu(\rho_1,\rho_2,\rho_2)}\mu(s)ds\right),$$

where

$$\begin{aligned}\nu(\rho_1,\rho_2,\rho_2)&=\max\{\mathcal{G}(S_3\rho_1,S_4\rho_2,S_4\rho_2),\mathcal{G}(S_3\rho_1,S_2\rho_2,S_2\rho_2),\mathcal{G}(S_4\rho_2,S_2\rho_2,S_2\rho_2),\mathcal{G}(S_1\rho_1,S_3\rho_1,S_3\rho_1)\}\\ &=\max\{\mathcal{G}(\rho_1,\rho_2,\rho_2),\mathcal{G}(\rho_1,\rho_2,\rho_2),\mathcal{G}(\rho_2,\rho_2,\rho_2),\mathcal{G}(\rho_1,\rho_1,\rho_1)\}\\ &=\mathcal{G}(\rho_1,\rho_2,\rho_2).\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathcal{G}(\rho_1,\rho_2,\rho_2)}\mu(s)ds &\le \sigma\left(\int_0^{\mathcal{G}(\rho_1,\rho_2,\rho_2)}\mu(s)ds\right)\\ &< \int_0^{\mathcal{G}(\rho_1,\rho_2,\rho_2)}\mu(s)ds,\end{aligned}$$

which is a logical inconsistency. Therefore, $\rho_1=\rho_2$. Consequently, w is a unique common fixed point of S_1,S_2,S_3 and S_4. □

Theorem 2.2. Let $(X,\mathcal{G})$ be $\mathcal{G}$-metric space and S_1,S_2,S_3,S_4 be self-mappings such that for every $x,y\in X$,

$$\int_0^{\mathcal{G}(S_1x,S_2y,S_2y)}\mu(s)ds\le\sigma\left(\int_0^{\nu(x,y,z)}\mu(s)ds\right)$$

where $\mu : R^+ \to R$ is Lebesgue integrable mapping which is summable and non-negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \to \mathbb{R}$ is upper semi continuous, $\sigma(0) = 0$ and $\sigma(s) < s$ for each $s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}(S_3x, S_4y, S_4y), \mathcal{G}(S_3x, S_2y, S_2y), \mathcal{G}(S_4y, S_2y, S_2y), \mathcal{G}(S_1x, S_3x, S_3x)\}.$$

Also satisfying the following condition that the pairs (S_1, S_3) and (S_2, S_4) satisfy E.A property such that S_3 or S_4 is closed subspace of X. Further if both the pairs (S_1, S_3) and (S_2, S_4) are weakly compatible, then S_1, S_2, S_3, S_4 have a unique common fixed point in X.

Corollary 2.3. Let $(X, \mathcal{G})$ be $\mathcal{G}$-metric space and S_1, S_2 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x, S_1y, S_1z)} \mu(s)ds \le \sigma\left(\int_0^{\nu(x,y,z)} \mu(s)ds\right)$$

where $\mu : R^+ \to R$ is Lebesgue integrable mapping which is summable and non- negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \to \mathbb{R}$ is upper semi continuous, $\sigma(0) = 0$ and $\sigma(s) < s$ for each $s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}((S_2x, S_2y, S_2z), \mathcal{G}((S_2x, S_1x, S_1x), \mathcal{G}(S_2y, S_1y, S_1y), \mathcal{G}(S_2z, S_1z, S_1z)\}.$$

Also satisfying the following condition that the pair (S_1, S_2) satisfy CLR property with respect to map S_2, then S_1 and S_2 have a coincidence point in X. Further, if the pair (S_1, S_2) is weakly compatible, then S_1, S_2 have a unique common fixed point in X.

Corollary 2.4. Let $(X, \mathcal{G})$ be $\mathcal{G}$-metric space and S_1, S_2, S_3 be self-mappings such that for every $x, y \in X$,

$$\int_0^{\mathcal{G}(S_1x, S_2y, S_2y)} \mu(s)ds \le \sigma\left(\int_0^{\nu(x,y,z)} \mu(s)ds\right)$$

where $\mu : R^+ \to R$ is Lebesgue integrable mapping which is summable and non-negative, $\sigma = \{\sigma/\sigma : \mathbb{R} \to \mathbb{R}$ is upper semi continuous, $\sigma(0) = 0$ and $\sigma(s) < s$ for each $s > 0\}$,

$$\nu(x, y, z) = \max\{\mathcal{G}((S_3x, S_2y, S_2y), \mathcal{G}((S_3y, S_1x, S_1x), \mathcal{G}(S_3x, S_1y, S_1y), \mathcal{G}(S_3y, S_2x, S_2x)\}.$$

Also satisfying the following condition that the pairs (S_1, S_2) and (S_1, S_3) satisfy CLR property with respect to maps S_2 and S_3. Further, if the pair (S_1, S_2) and (S_1, S_3) are weakly compatible, then S_1, S_2 and S_3 have a unique common fixed point in X.

Example 2.5. Let $(X, \mathcal{G})$ be $\mathcal{G}$-metric space with the metric $\mathcal{G}(x, y, z) = |x-y|+|y-z|+|z-x|\}$, for all x, y, z in $X = (0, 5)$. Let S_1, S_2, S_3 and S_4 be four self-mappings such that for every $x, y \in X$, $t > 0$,

$$S_1x = \begin{cases} 1 & \text{if } x \in (0,4] \\ \frac{1}{8} & \text{otherwise} \end{cases}; \qquad S_2x = \begin{cases} 1 & \text{if } x \in (0,4] \\ \frac{1}{9} & \text{otherwise} \end{cases}$$

$$S_3x = \begin{cases} 1 & \text{if } x \in (0,4] \\ \frac{1}{5} & \text{otherwise} \end{cases} \quad \text{and} \quad S_4x = \begin{cases} 1 & \text{if } x \in (0,4] \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Let $\{u_n\} = \frac{1}{n}$ and $\{v_n\} = \frac{1}{n+1}$ be two sequences in X. Then,

$$\lim_{n\to\infty} S_1u_n = \lim_{n\to\infty} S_1(\frac{1}{n}) = 1;$$
$$\lim_{n\to\infty} S_2v_n = \lim_{n\to\infty} S_2(\frac{1}{n+1}) = 1;$$
$$\lim_{n\to\infty} S_3v_n = \lim_{n\to\infty} S_3(\frac{1}{n+1}) = 1;$$
$$\lim_{n\to\infty} S_4u_n = \lim_{n\to\infty} S_4(\frac{1}{n}) = 1.$$

Therefore,

$$\lim_{n\to\infty} S_1u_n = \lim_{n\to\infty} S_2v_n = \lim_{n\to\infty} S_3v_n = \lim_{n\to\infty} S_4u_n = 1.$$

So, (S_1, S_3) and (S_2, S_4) enjoys the $CLR_{S_3S_4}$ property. Let us define $\mu(s) = 4s$, and $\sigma(s) = \frac{s}{4}$. Whenever, $x, y \in [0, 4]$, then, $S_1x = S_2y = S_3y = S_4x = 1$, which implies that

$$\nu(x, y, z) = 0 = \max\{\mathcal{G}(S_3x, S_4y, S_4y), \mathcal{G}(S_3x, S_2y, S_2y), \mathcal{G}(S_4y, S_2y, S_2y), \mathcal{G}(S_1x, S_3x, S_3x)\}.$$

Also, $\mathcal{G}(S_1x, S_2y, S_2y) = 0$. Therefore,

$$\int_0^{\mathcal{G}(S_1x,S_2y,S_2y)} \mu(s)ds = 0 = \sigma(\int_0^{\nu(x,y,z)} \mu(s)ds)$$

Whenever, $x, y \in [4, 5]$, then $S_1x = \frac{1}{8}, S_2y = \frac{1}{9}, S_3x = \frac{1}{4}$ and $S_4y = \frac{1}{5}$. Therefore,

$$\int_0^{\mathcal{G}(S_1x,S_2y,S_2y)} \mu(s)ds = \int_0^{2\times\frac{1}{72}} 4sds = 2s^2\Big|_0^{2\times\frac{1}{72}} = 2 \times \frac{1}{1296} = \frac{1}{648} = 0.001543.$$

Also, $\nu(x, y, z) = \max\{\frac{1}{10}, \frac{10}{36}, \frac{8}{45}, \frac{1}{4}\} = \frac{1}{4}$. Thus,

$$\sigma(\int_0^{\nu(x,y,z)} \mu(s)ds) = \sigma(\int_0^{\frac{1}{4}} 4sds) = \frac{1}{4} \times (\int_0^{\frac{1}{4}} 4sds) = \frac{1}{4} \times 2 \times \frac{1}{16} = 0.03125.$$

From above two equations, it follows that

$$\int_0^{\mathcal{G}(S_1x,S_2y,S_2y)} \mu(s)ds \le \sigma(\int_0^{\nu(x,y,z)} \mu(s)ds).$$

Now, hypothesis of Theorem 2.1 is fulfilled. Therefore, S_1, S_2, S_3 and S_4 have a unique common fixed point in X which is $x = 1$.

3 Conclusion

In this paper, the fixed point results are investigated with the aid of CLR property for two pairs of functions in the context of $\mathcal{G}-$metric space. Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Our results can be utilized to find solution of fractional non-linear differential and integral equations.

References

[1] S. Arora, M. Kumar and S. Mishra, *A new type of coincidence and common fixed-point theorems for modified α-admissible Z-contraction via simulation function*, J. Math. Fund. Sci. **52** (2020), no. 1, 27–42.

[2] S. Arora, *Common fixed point theorems satisfying common limit range property in the frame of Gs metric spaces*, Math. Sci. Lett. **10** (2021), no. 2, 1–5.

[3] H. Aydi, S. Chauhan and S. Radenovi, *Fixed point of weakly compatible mappings in G-metric spaces satisfying common limit range property*, Ser. Math. Inf. **28** (2013), no. 2, 197–210.

[4] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (2002), 531–536.

[5] P. Debnath, Z.D. Mitrović and S.Y. Cho, *Common fixed points of Kannan, Chatterjea and Reich type pairs of self-maps in a complete metric space*, Sao Paulo J. Math. Sci. **15** (2021), 383–391.

[6] A. Djoudi and F. Merghadi, *Common fixed point theorems for maps under a contractive condition of integral type*, J. Math. Anal. Appl. **341** (2012), 953–960.

[7] F. Khojasteh, Z. Goodarzi and A. Razani, *Some fixed point theorems of integral type contraction in cone metric spaces*, Fixed Point Theory Appl. **2010** (2010), Article ID 189684, 1–10.

[8] M. Kumar, S. Arora, M. Imdad and W.M. Alfaqih, *Coincidence and common fixed point results via simulation-functions in G-metric spaces*, J. Math. Comput. Sci. **19** (2019), 288–300.

[9] M. Kumar, S. Arora and S. Mishra, *On the power of simulation map for almost Z-contraction in G-metric space with applications to the solution of the integral equation*, Ital. J. Pure Appl. Math. **44** (2020), 639–648.

[10] Z. Liu, X. Li, S. Kang and S. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory Appl. **2011** (2011), Article ID 64, 1-9.

[11] S. Manro, S.S. Bhatia, S. Kumar and C. Vetro, *A common fixed point theorem for two weakly compatible pairs in G-metric spaces using the property E.A*, Fixed Point Theory Appl. **41** (2013), no. 2, 1–9.

[12] P. Murthy, S. Kumar and K. Tas, *Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces*, Math. Commun. **15** (2010), 521–537.

[13] P.P. Murthy, Z. Mitrovic, C.P. Dhuri and S. Radenovic, *The common fixed points in a bipolar metric space*, Gulf J. Math. **12** (2022), no. 2 31–38.

[14] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 289–297.

[15] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory Appl. **2008** (2008), Article ID 189870, 1–12.

[16] S.K. Panda, B. Alamri, N. Hussain and S. Chandok, *Unification of the fixed point in integral type metric spaces*, Symmetry **732** (2018), no. 10, 1–21.

[17] M. Rahman, M. Sarwar and M. Rahman, *Fixed point results of Altman integral type mappings in S-metric spaces*, Int. J. Anal. Appl. **10** (2016), no. 1, 58–63.

[18] B. Samet, C. Vetro and P. Vetro, *Fixed point theorem for α-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[19] M. Sarwar, M.B. Zada and I.M. Erhan, *Common fixed point theorems of integral type contraction on metric spaces and its applications to system of functional equations*, Fixed Point Theory Appl. **217** (2015), 1–15.

[20] W. Sintunavarat and P. Kumam, *Gregus-type common fixed point theorems for tangential multi-valued mappings of integral type in metric spaces*, Int. J. Math. Math. Sci. **2011** (2011), Article ID 923458, 1–9.