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The numerical solution of the second kind of Abel equations by the modified matrix-exponential method

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Abstract

In this paper, the modified matrix exponential method (MME), under the zero-order hold (ZOH) assumption, is applied to solve the Abel equation of the second kind. The modified exponential matrix method is iterative, and by increasing the iteration, we can get a better approximation with fewer errors. We use the MME to turn an Abel differential equation into a system of nonlinear equations and determine the solution. By using the MME, the Abel differential equations approximate well. Using the numerical results, we can conclude that this method is effective, and in comparison with well-known techniques, the MME is highly accurate.

Keywords: Modified matrix exponential, Matrix exponential, Abel equation of the second kind, nonlinear differential equations, Jacobian matrix
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1 Introduction

Dynamic Abel equations of the second kind have the following general form:

$$[f_1(x)V + f_0(x)]VX' = g_2(x)V^2 + g_1(x)V + g_0(x) \tag{1.1}$$

for $g_i, f_k : R \longrightarrow R$, $i = 0, 1, 2$, and $k = 0, 1$. The Abel equations are prevalent in some fields, such as the model equation for western boundary outflow in the Stommel model of the large scale ocean circulation [3], on two coupled Abel-type differential equations arising in a magnetostatic problem [9], stable inversion of Abel equations in application to tracking control [11] and solving relativistic dissipative cosmological models by converting to Abel differential equation [8].

In the last several decades, some of the numerical methods for Abel equations have been studied, such as the shifted Chebyshev polynomials [6], Adomian decomposition [1], Transform Method [10], the Gauss–Jacobi quadra-ture rule [4], the integral transformations [13], wavelet method [7] and so on.

Our paper uses the modified matrix exponential method to solve the second kind of Abel equation and is organized as follows: The MME method is briefly described in section 2; section 4 illustrates the method's accuracy with an example, and the conclusion is described in section 5.

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2 A brief description of the MME

Our objective is to present a time discretization of non-linear systems using modified matrix exponential methods as follows [12, 14]:

$$\frac{dY(t)}{dt} = f(Y(t)) + g(Y(t)) \odot v(t) \tag{2.1}$$

where the vector $Y(t) = [y_1(t), y_2(t), y_3(t), ..., y_n(t)]^t \in X \subset R^{n\times 1}$ represents a set of open and connected states, and $v(t) = [v_1(t), v_2(t), ..., v_n(t)] \in R^{n\times 1}$ are the input variables and $\odot$ is a scalar product. Suppose that $f(x)$ and $g(x)$ are real analytic vector fields on X. In general, the mesh $T = t_{k+1} - t_k > 0$ represents an equidistant grid of points on the time axis, $[t_k, t_{k+1}) = [kT, (k+1)T)$ and T are the sampling interval and the sampling period, respectively.

It is also assumed that (2.1) is driven by an input, $v_i(t)$, that is piecewise constant over the sampling interval, i.e, the ZOH is true. For the ZOH assumption,

$$v_i(t) = v_i(t_k) = constant \tag{2.2}$$

For $a \le t_k \le b$. For $i, j = 1, 2, 3, ..., n$, we consider a time interval $t \in [t_k, t_{k+1})$ with the ZOH assumption, we have

$$\zeta_i(t) = Y_j(t) - Y_j(t_k) \tag{2.3}$$

and the following second-order approximation can be obtained:

$$f_i(Y(t)) \approx f_i(Y(t_k)) + \frac{\partial f_i(Y(t_k))}{\partial y_j(t)}\zeta_j(t) + \frac{\partial^2 f_i(Y(t_k))}{\partial y_j(t)^2}\frac{\zeta_j^2(t)}{2} \tag{2.4}$$

$$g_i(Y(t)) \approx g_i(Y(t_k)) + \frac{\partial g_i(Y(t_k))}{\partial y_j(t)}\zeta_j(t) + \frac{\partial^2 g_i(Y(t_k))}{\partial y_j(t)^2}\frac{\zeta_j^2(t)}{2} \tag{2.5}$$

From (2.3), we have

$$\dot{\zeta}_j(t) = \dot{Y}_j(t) \tag{2.6}$$

Thus, (2.1) can be approximated as follows:

$$\begin{aligned}
\dot{\zeta}_j(t) &\approx f_i(Y(t_k)) + \frac{\partial f_i(Y(t_k))}{\partial y_j(t)}\zeta_j(t) + \frac{\partial^2 f_i(Y(t_k))}{\partial y_j(t)^2}\frac{\zeta_j^2(t)}{2} + \left(g_i(Y(t_k)) + \frac{\partial g_i(Y(t_k))}{\partial y_j(t)}\zeta_j(t) + \frac{\partial^2 g_i(Y(t_k))}{\partial y_j(t)^2}\frac{\zeta_j^2(t)}{2}\right) v_i \\
&= (f_i(Y(t_k)) + g_i(Y(t_k))v_i) + \left(\frac{\partial f_i(Y(t_k))}{\partial y_j(t)} + \frac{\partial g_i(Y(t_k))}{\partial y_j(t)}v_i\right)\zeta_j(t) + \left(\frac{\partial^2 f_i(Y(t_k))}{\partial y_j(t)^2} + \frac{\partial^2 g_i(Y(t_k))}{\partial y_j(t)^2}\right)\frac{\zeta_j^2(t)}{2} \\
&= \tilde{f}_{ik} + J_{ik}\zeta_j(t) + J'_{ik}\frac{\zeta_j^2(t)}{2}
\end{aligned} \tag{2.7}$$

where

$$\tilde{f}_{ik} = \tilde{f}_i(Y(t_k), v_i) = f_i(Y(t_k)) + g_i(Y(t_k))v_i \tag{2.8}$$

$$J_{ik} = J_i(Y(t_k), v_i) = \frac{\partial f_i(Y(t_k))}{\partial y_j(t)} + \frac{\partial g_i(Y(t_k))}{\partial y_j(t)}v_i \tag{2.9}$$

$$J'_{ik} = J'_i(Y(t_k), v_i) = \frac{\partial^2 f_i(Y(t_k))}{\partial y_j(t)^2} + \frac{\partial^2 g_i(Y(t_k))}{\partial y_j(t)^2}v_i \tag{2.10}$$

Rewriting (2.7), we get:

$$\dot{\zeta}_j(t) = \tilde{f}_{ik} + J_{ik}\zeta_j(t) + J'_{ik}\frac{\zeta_j^2(t)}{2}, \quad \zeta_j(t_k) = 0 \tag{2.11}$$

Let $N > 0$ be an integer number, the step length is as follows:

$$h_k = \frac{t_{k+1} - t_k}{N} \tag{2.12}$$

An expand vector is considered:

$$\eta_j(t) = \begin{pmatrix} \zeta_j(t) \\ \frac{\zeta_j^2(t)}{2} \\ 1 \end{pmatrix} \tag{2.13}$$

(2.11) Can be written as follows:

$$\begin{pmatrix} \dot{\zeta}_j(t) \\ \dot{\zeta}_j(t)\zeta_j(t) \\ 0 \end{pmatrix}_{(i+1)\times 1} = \begin{pmatrix} J_{ik} & J'_{ik} & \tilde{f}_{ik} \\ \dot{\zeta}_j(t) & 0 & 0 \\ \bar{0}^T & 0 & 0 \end{pmatrix}_{(i+1)\times(i+1)} \begin{pmatrix} \zeta_j(t) \\ \frac{\zeta_j^2(t)}{2} \\ 0 \end{pmatrix}_{(i+1)\times 1} \tag{2.14}$$

Rewriting (2.14), we get:

$$\dot{\eta}_j(t) = C_{ik}\eta_j(t) \tag{2.15}$$

where

$$\dot{\zeta}_j(t_k) = \tilde{f}_{ik}(t_k), \quad \eta_j(t_k) = \begin{pmatrix} 0 \\ \bar{0} \\ 1 \end{pmatrix} = \eta_{j0} \tag{2.16}$$

$$C_{ik}(t_k) = \begin{pmatrix} J_{ik}(t_k) & J'_{ik}(t_k) & \tilde{f}_{ik}(t_k) \\ \tilde{f}_{ik}(t_k) & 0 & 0 \\ \bar{0}^T & 0 & 0 \end{pmatrix} \in R^{(n+1)\times(n+1)} \tag{2.17}$$

and $\bar{\bar{0}}$ is an n-dimensional zero column vector and J_{ik} is the first-order derivative and J'_{ik} is the second-order derivative of the Jacobian matrix and $\tilde{\tilde{f}}_{ik}$ is the values of equations in $y_i(t_k)$. The solution of (2.15) within the time interval $[t_k, t_{k+1})$ is as follows:

$$\eta_j(t_{k+1}) = e^{C_{ik}(t_k)(t_{k+1}-t_k)}\eta_{j0}. \tag{2.18}$$

An exponential matrix is calculated by taking Z as the square matrix and I as the identity matrix. Its exact formula would be as follows:

$$e^z = \lim_{N\to\infty}\left(1 + \frac{Z}{N}\right)^N \tag{2.19}$$

The following truncated approximation is applicable for a appropriate value of N:

$$e^z \approx \left(1 + \frac{Z}{N}\right)^N. \tag{2.20}$$

Using (2.18) and (2.20) we get:

$$e^{C_{ik}(t_k)(t_{k+1}-t_k)} \approx (I_{(i+1)\times(i+1)} + C_{ik}(t_k)h_k)^N. \tag{2.21}$$

From (2.18) and (2.21), we can obtain:

$$\eta_j(t_{k+1}) = (I_{(2i+1)\times(2i+1)} + C_{ik}(t_k)h_k)^N\eta_{j0}. \tag{2.22}$$

By multiplying the vector $(I_{i\times i}\ \bar{0}\ \bar{0})$ on the sides of (2.22):

$$\zeta_j(t_{k+1}) = (I_{i\times i}\ \bar{0}\ \bar{0})(I_{(2i+1)\times(2i+1)} + C_{ik}(t_k)h_k)^N \begin{pmatrix} \bar{0} \\ \bar{0} \\ 1 \end{pmatrix} \tag{2.23}$$

where $(I_{i\times i}\ \bar{0}\ \bar{0}) \in R^{(n)\times(n+1)}$. So, the final equation can be obtained as follows:

$$y(t_{k+1}) = y(t_k) + (I_{i\times i}\ \bar{0}\ \bar{0})(I_{(2i+1)\times(2i+1)} + C_{ik}(t_k)h_k)^N \begin{pmatrix} \bar{0} \\ \bar{0} \\ 1 \end{pmatrix} \tag{2.24}$$

We used the extended vector to apply the modified matrix exponential method. The (2.24) can be written in extended form as follows:

$$\begin{pmatrix} y_1(t_{k+1}) \\ y_2(t_{k+1}) \\ \vdots \\ y_i(t_{k+1}) \end{pmatrix}_{i\times 1} = \begin{pmatrix} y_1(t_k) \\ y_2(t_k) \\ \vdots \\ y_i(t_k) \end{pmatrix}_{i\times 1} + H(t_k, y_i(t_k)) \tag{2.25}$$

where

$$H(t_k, y_i(t_k)) = (I_{i\times i}\ \bar{0}\ \bar{0})(I_{(2i+1)\times(2i+1)} + C_{ik}(t_k)h_k)^N \begin{pmatrix} \bar{0} \\ \bar{0} \\ 1 \end{pmatrix}.$$

For $i = 1, 2, 3, ..., n$. If $J'_{ik}(t_k) = 0$, then modified matrix exponential (MME) and matrix exponential (ME) methods are equivalent to each other.

3 Finding the appropriate value of N in MME

A proper value for N is essential [2]. An improved form for (2.20) is $e^z \cong \left(1 + \frac{Z}{2^b}\right)^{2^b}$, for an appropriate value of b. We can show that analytical relative matrix error E_t defined by $\left(1 + \frac{Z}{2^b}\right)^{2^b} \equiv e^z(1 + E_t)$ is given approximately by $E_t \approx -\frac{1}{2}.\frac{Z^2}{2^b}$, and therefore, for any matrix form, $\|E_t\| \approx \frac{1}{2}.\frac{\|Z\|^2}{2^b} \leq \frac{1}{2}.\frac{\|Z\|^2}{2^b}$. Estimating the value of required to have $E_t < \epsilon$, as follows $b^* \equiv int\left(\log_2\left(\frac{\|Z\|^2}{2\varepsilon}\right)\right)$, where $int(x)$ is the lowest integer greater than or equal to x and $E_t < \epsilon$ is a preassigned tolerance (maximum tolerable value) for E_t and $Z = (t_{k+1} - t_k)C_{ik}$. For the sake of safety, it's recommended to choose $b = b^* + 3$.

4 Numerical illustration and discussion

Example 4.1. Let us consider the Abel equations of the second kind in the following manner [5]:

$$yy' + ty + y^2 + t^2y^3 = te^{-t} + t^2e^{-3t} \tag{4.1}$$

with initial value $y(0) = 1$. The exact solutions is $y(0) = e^{-t}$. Now we construct as follows:

$$\begin{aligned} y'(t) &= (-t - y(t) - t^2y^2(t) + te^{-t}y^{-1}(t) + t^2e^{-3t}y^{-1}(t))_{1\times 1} \\ J_{ik} &= (-1 - 2t_k^2y(t_k) - t_ke^{-t_k}y^{-2}(t_k) - t_k^2e^{-3t_k}y^{-2}(t_k))_{1\times 1} \\ J'_{ik} &= (-2t_k^2 + 2t_ke^{-t_k}y^{-3}(t_k) + 2t_k^2e^{-3t_k}y^{-3}(t_k))_{1\times 1} \\ \tilde{f}_{ik} &= (-t_k - y(t_k) - t_k^2y^2(t_k) + t_ke^{-t_k}y^{-1}(t_k) + t_k^2e^{-3t_k}y^{-1}(t_k))_{1\times 1} \end{aligned}$$

where $i = 1$ To use the modified matrix exponential method, we used the extended vector as (2.25). The solutions are listed in table 1 and plotted in figure 1 for iteration= 2×10^7 and $N = 5$ and the method error values presented in figure 2 and the average of absolute error of example is listed in table 2. The average of absolute error of example for MME and matrix exponential method (ME) is listed in table 3.

Table 1: The numerical solutions by different methods.

t	Taylor [5]	Pade [5]	Exact	Chebyshev	MME
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.90483741	0.90483741	0.90483741	0.90483741	0.90483741
0.2	0.81873066	0.81873074	0.81873075	0.81873074	0.81873075
0.3	0.74081725	0.74081814	0.74081822	0.74081814	0.74081821
0.4	0.67031466	0.67031963	0.67032004	0.67031963	0.67032004
0.5	0.60651041	0.60652920	0.60653065	0.60652920	0.60653065
0.6	0.54875200	0.54880763	0.54881163	0.54880763	0.54881163
0.7	0.49643691	0.49657595	0.49658530	0.49657595	0.49658529
0.8	0.44900266	0.44930966	0.44932896	0.44930965	0.44932895
0.9	0.40591675	0.40653338	0.40656965	0.40653337	0.40656964
1.0	0.36666666	0.36781609	0.36787944	0.36781609	0.36787942

Table 2: The average of absolute errors of methods.

methods	Taylor	Pade	Chebyshev	MME
error	2.20607e-04	1.22021e-05	1.22049e-05	4.03048e-09

Table 3: The average of absolute error of example for MME and ME

methods	it=50 and N=5	it=100 and N=5	it=10^5 and N=5
MME	7.363806e-04	1.987538e-04	3.789751e-07
ME	7.424016e-04	1.994581e-04	3.789766e-07

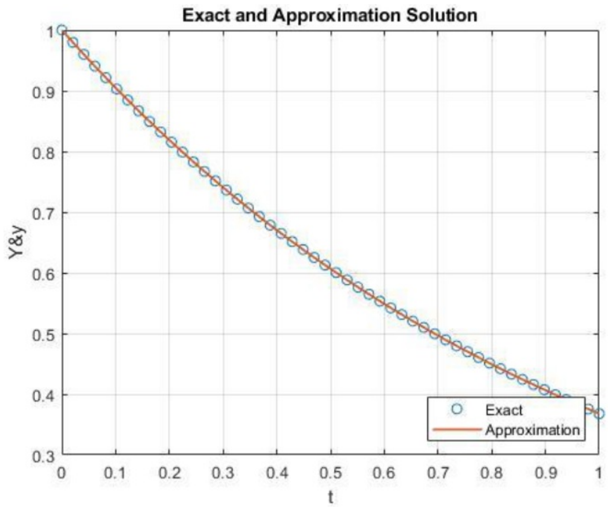

Figure 1: The exact and MME solution of example.

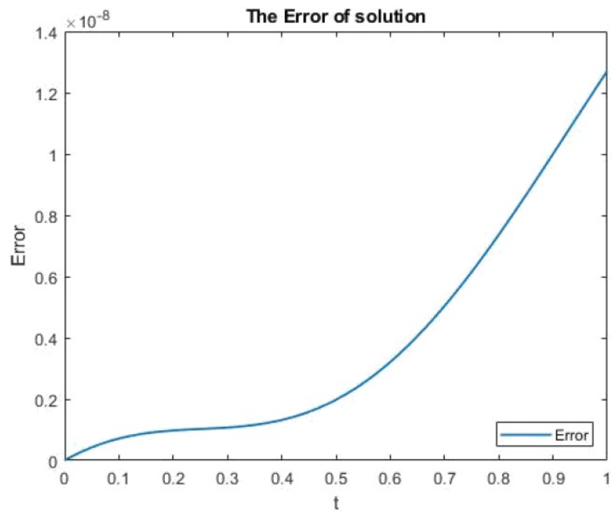

Figure 2: The errors of the MME method of example.

5 Conclusion

In this paper, we studied the MME for solving the second kind of the Abel equation. For using the MME, we used the extended vector as (2.25). In table 1, by comparing the methods, we can see the accuracy of the MME method. It is clear that our numerical solutions are in good accordance with the exact one. As shown in the numerical example, the MME method is a perfect method for solving the Abel equations. We can get a better approximation with fewer errors by increasing the iteration of the method, or the value of N.

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