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Superconvergence of Volterra-Urysohn integral equations with weakly singular kernels by iterated Jacobi spectral multi Galerkin method

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Abstract

We propose the iterated Jacobi spectral multi Galerkin method for weakly singular Volterra integral equations of Urysohn type and obtain the superconvergence results in uniform norm. The convergence analysis is discussed in two cases: when the solution is sufficiently smooth and when it is not. To back up our theoretical approach, we present numerical findings.

Keywords: Volterra integral equations, weakly singular kernels, Jacobi polynomials, Iterated Jacobi Spectral multi-Galerkin method, Superconvergence results
2020 MSC: 45D05, 45A05, 65R20

1 Introduction and convergence analysis

These forms of integral equations are commonly seen in situations involving gas absorption, heat conduction, and heat transfer [9, 7]. The goal of this research is to obtain approximate solutions to Volterra-Urysohn integral equations with weakly singular kernels. In [1, 2, 3], H. Brunner presented a polynomial spline collocation method based on both quasi uniform and graded meshes, demonstrating that it converges $\mathcal{O}(h^{1+\gamma})$, in the uniform mesh and with $\mathcal{O}(n^{-r})$, in graded mesh. K. Kant et al. [6] discussed the Galerkin and multi Galerkin methods for Volterra-Hammerstein weakly singular integral equations by using piecewise polynomials based on graded mesh and obtained the convergence analysis. In [8], R. Nigam et al. proposed the Galerkin and multi Galerkin methods based on piecewise polynomials for weakly singular Volterra-Urysohn integral equations and found the superconvergence results. When using piecewise polynomial-based projection methods, increasing the accuracy of the solution necessitates increasing the number of partitions, which raises the computational complexity of the approach. Therefore, Jacobi polynomials are used in place of piecewise polynomials to lower the computational complexity. In [5], K. Kant discussed the error analysis of Jacobi–Galerkin method for solving weakly singular Volterra–Hammerstein integral equations. In [10], for nonlinear Volterra integral equations with weakly singular kernels, the Jacobi spectral collocation approach is explained, and convergence results are obtained. Jacobi spectral approaches for Volterra-Urysohn integral equations with weakly singular kernels were studied by K. Kant et al. in [4]. The motivation is to consider the iterated Jacobi spectral Galerkin method is to incorporate the weakly singular kernel in the weight function and obtain the superconvergence results. Here, we develop the iterated Jacobi spectral multi Galerkin method and obtain the convergence analysis.

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Let $\mathbb{X} = \mathcal{C}[-1,1]$ be a Banach space. Consider the weakly singular Volterra integral equation of Urysohn type as follows:

$$y(x) = \int_0^x (x-\varrho)^{-\gamma} z(x,\varrho,y(\varrho))d\varrho + g(x), \quad x \in [0,1], \tag{1.1}$$

where z and g are known functions and y is the unknown function to be approximated. To use the Jacobi spectral methods, we must first define the transformations that will be used to alter the variables

$$x = \frac{1}{2}T(1+t), \quad t = \frac{2x}{T} - 1, \quad \varrho = \frac{1}{2}T(1+s), \quad s = \frac{2\varrho}{T} - 1, \tag{1.2}$$

and

$$s(t,\varphi) = \frac{1+t}{2}\varphi + \frac{t-1}{2}, \quad -1 \le \varphi \le 1. \tag{1.3}$$

Putting these transformations, the Volterra integral equation (1.1) transformed to the following

$$v(t) = \int_{-1}^1 (1-\varphi)^{-\gamma} \tilde{z}(t, s(t,\varphi), v(s(t,\varphi)))d\varphi + f(t), \tag{1.4}$$

where $\tilde{z}(t, s(t,\varphi), v(s(t,\varphi))) = \left(\frac{t+1}{2}\right)^{1-\gamma} l(t, s(t,\varphi), v(s(t,\varphi)))$. Note that $\tilde{z}(t, s(t,\varphi), v(s(t,\varphi)))$ is sufficiently differentiable with respect to variable φ and continuous with respect to variable t in the interval [-1, 1]. Let $\mathcal{Z} : \mathbb{X} \to \mathbb{X}$ be the integral operator defined by

$$\mathcal{Z}(v)(t) = \int_{-1}^1 (1-\varphi)^{-\gamma} \tilde{z}(t, s(t,\varphi), v(s(t,\varphi)))d\varphi. \tag{1.5}$$

At v, the Fréchet derivative of $\mathcal{Z}$ is defined as follows

$$\mathcal{Z}'(v)y(t) = \int_{-1}^1 (1-\varphi)^{-\gamma} \tilde{z}_v(t, s(t,\varphi), v(s(t,\varphi)))y(s(t,\varphi))d\varphi, \tag{1.6}$$

where $\tilde{z}_v(t, s(t,\varphi), v(s(t,\varphi))) = \frac{\partial}{\partial v}\tilde{z}(t, s(t,\varphi), v(s(t,\varphi)))$. We express the equation (1.4) as using the integral operator $\mathcal{Z}$,

$$v(t) - \mathcal{Z}(v)(t) = f(t), \quad t \in [-1,1]. \tag{1.7}$$

Now we define the operator $\mathcal{T}$ on $\mathbb{X}$ by

$$\mathcal{T}(v) = \mathcal{Z}(v) + f, \quad v \in \mathbb{X}. \tag{1.8}$$

The equation above can then be represented as

$$v = \mathcal{T}(v). \tag{1.9}$$

We make the following assumptions on $\tilde{z}(\cdot,\cdot,\cdot)$ throughout this article:

(i) $\tilde{z}(\cdot,\cdot,\cdot)$ is Lipschitz continuous w.r.t. third variable v i.e., for any $v_1, v_2 \in \mathbb{R}$, $\exists\ C_1 > 0$ such that

$$|\tilde{z}(t,s,v_1) - \tilde{z}(t,s,v_2)| \le C_1|v_1 - v_2|. \tag{1.10}$$

(ii) The derivative $\tilde{z}_v(\cdot,\cdot,\cdot)$ of $\tilde{z}(\cdot,\cdot,\cdot)$ exists and Lipschitz continuous w.r.t. third variable v, i.e., for any $v_1, v_2 \in \mathbb{R}$, $\exists\ C_2 > 0$ such that

$$|\tilde{z}_v(t,s,v_1) - \tilde{z}_v(t,s,v_2)| \le C_2|v_1 - v_2|. \tag{1.11}$$

If $\mathcal{M}C_1 < 1$, where $\mathcal{M} = \left(\frac{2^{1-\gamma}}{1-\gamma}\right)^{\frac{1}{2}}$, the equation (1.9) then has isolated solution, say $v_0 \in \mathbb{X}$.
Throughout this article, we assume that 1 is not an eigenvalue of the linear operator $\mathcal{T}'(v_0)$, i.e., $(\mathcal{I} - \mathcal{T}'(v_0))^{-1}$ exists and uniformly bounded in infinity and weighted L^2- norm.

Now we discuss the Jacobi spectral Galerkin method. Let $\mathbb{X}_N = \text{span}\{\psi_0, \psi_1, \psi_2, ..., \psi_N\}$, be the Jacobi polynomials of degree $\leq N$ on $\wedge = [-1, 1]$, where $\psi_j(x)$ is the j-th Jacobi polynomial corresponding to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \ -1 \leq \alpha, \beta \leq 1$.

We can generate the Jacobi polynomials by the following three-term recurrence relation:

$$J_{N+1}^{\alpha,\beta}(x) = (a_N^{\alpha,\beta} - b_N^{\alpha,\beta})J_N^{\alpha,\beta}(x) - c_N^{\alpha,\beta}J_{N-1}^{\alpha,\beta}(x), \quad N \geq 1, \tag{1.12}$$

$$J_0^{\alpha,\beta}(x) = 1, \ J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha - \beta), \quad -1 \leq \alpha, \beta \leq 1, \tag{1.13}$$

where

$$a_N^{\alpha,\beta} = \frac{(2N + \alpha + \beta + 1)(2N + \alpha + \beta + 2)}{2(N + 1)(N + \alpha + \beta + 1)},$$
$$b_N^{\alpha,\beta} = \frac{(\beta^2 - \alpha^2)(2N + \alpha + \beta + 1)}{2(N + 1)(N + \alpha + \beta + 1)(2N + \alpha + \beta)},$$
$$c_N^{\alpha,\beta} = \frac{(N + \alpha)(N + \beta)(2N + \alpha + \beta + 2)}{(N + 1)(N + \alpha + \beta + 1)(2N + \alpha + \beta)}.$$

Orthogonal projection : Let the orthogonal projection operator $\rho_N^{\alpha,\beta} : \mathbb{X} \to \mathbb{X}_N, \ -1 \leq \alpha, \beta \leq 1$, be defined by

$$(\rho_N^{\alpha,\beta}u, u_N)_{\omega^{\alpha,\beta}} = (u, u_N)_{w^{\alpha,\beta}}, \qquad \forall \ u \in L^2{}_{\omega^{\alpha,\beta}}, u_N \in \mathbb{X}_N, \tag{1.14}$$

where

$$(u_1, u_2)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u_1(\tau)u_2(\tau)\omega^{\alpha,\beta}(\tau)\, d\tau.$$

$$L^2_{w^{\alpha,\beta}} = \{u : \ u \text{ is measurable and } \|u\|_{w^{\alpha,\beta}} < \infty\},$$

$$\|u\|_{w^{\alpha,\beta}} = \Big(\int_{-1}^{1} u^2(\tau)\omega^{\alpha,\beta}(\tau)\, d\tau\Big)^{\frac{1}{2}}.$$

Further, define

$$H^r_{w^{\alpha,\beta}}(\Lambda) = \{v : D^k v \in L^2_{\omega^{\alpha,\beta}}(\Lambda), 0 \leq k \leq r\},$$

with

$$\|u\|_{H^r_{w^{\alpha,\beta}}} = \left(\sum_{k=0}^{r} \Big\|\frac{d^k u}{dx^k}\Big\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}$$

and the seminorm on $H^r_{w^{\alpha,\beta}}(\Lambda)$ is defined by

$$|u|_{H^{r;N}_{w^{\alpha,\beta}}} = \left(\sum_{k=\min(r,N+1)}^{r} \Big\|\frac{d^k u}{dx^k}\Big\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}. \tag{1.15}$$

The crucial properties of the orthogonal projection operator $\rho_N^{\alpha,\beta}$, which we need in our convergence analysis are ([11, 12]) the following, for any $u \in \mathcal{C}[-1, 1]$, we have

$$(i) \qquad \|\rho_N^{\alpha,\beta}\|_\infty \leq c(\log N), \tag{1.16}$$

$$(ii) \quad \|\rho_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leq p\|u\|_\infty, \tag{1.17}$$

$$(iii) \quad \|\rho_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leq \|u\|_{\omega^{\alpha,\beta}}, \tag{1.18}$$

and if $u \in \mathcal{C}^r[-1, 1]$ and $r \geq 1$, we have

$$(i) \quad \|u - \rho_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leq CN^{-r}|u|_{H^{r;N}_{w^{\alpha,\beta}}}, \tag{1.19}$$

$$(ii) \quad \|u - \rho_N^{\alpha,\beta}u\|_\infty \leq CN^{\frac{3}{4}-r}|u|_{H^{r;N}_{w^{\alpha,\beta}}}. \tag{1.20}$$

The main motivation behind using the Jacobi spectral method to incorporate the weakly singular kernel in the weight function. For the rest of the article, we restrict $\alpha = -\gamma,\ \beta = 0$. Define the multi projection operator on $\mathbb{X}$ by

$$(\mathcal{Z}_N^M)(v) = \rho_N^{-\gamma,0}\mathcal{Z}(v) + \mathcal{Z}(\rho_N^{-\gamma,0}u) - \rho_N^{-\gamma,0}\mathcal{Z}(\rho_N^{-\gamma,0}v). \tag{1.21}$$

For the equation (1.1), the multi Galerkin approach seeks an approximation $v_N^M \in \mathbb{X}$ such that

$$v_N^M - \mathcal{Z}_N^M(v_N^M) = f. \tag{1.22}$$

We define the iterated approximation solution by in order to obtain a more accurate approximate solution:

$$\tilde{v}_N^M = \mathcal{Z}(v_N^M) + f. \tag{1.23}$$

Lemma 1. Let v_0 be the non-smooth solution of the integral equation (1.9). Let the Jacobi orthogonal projection operator $\rho_N^{-\gamma,0} : \mathbb{X} \to \mathbb{X}_N$ be defined by (1.14) for $\alpha = -\gamma,\ \beta = 0$. Then there hold

$$\|(I - \rho_N^{-\gamma,0})v_0\|_{\omega^{-\gamma,0}} = \mathcal{O}(N^{-(1-\gamma)}), \tag{1.24}$$
$$\|(I - \rho_N^{-\gamma,0})v_0\|_\infty = \mathcal{O}(N^{-(1-\gamma)}\log N), \tag{1.25}$$
$$\|(I - \rho_N^{-\gamma,0})\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot)))\|_{\omega^{-\gamma,0}} = \mathcal{O}(N^{-(1-\gamma)}). \tag{1.26}$$

Proof . The proof of the above Lemma follows from Lemma 3.1 of [4]. □

Theorem 1.1. Let the orthogonal projection operator $\rho_N^{-\gamma,0} : \mathbb{X} \to \mathbb{X}_N$ be defined by (1.14) for $\alpha = -\gamma,\ \beta = 0$ and $v_0 \in \mathcal{C}[-1,1]$ be an isolated solution of the equation (1.9). For sufficiently large N, there exists $\mathcal{L}_2 > 0$ such that $\|(I - \mathcal{Z}_N^{M'}(v_0))^{-1}\|_\infty \le \mathcal{L}_2$.

Proof . First we show that $\mathcal{Z}_N^{M'}(v_0)$ is norm convergent to $\mathcal{Z}'(v_0)$ in infinity norm.
Consider for any $y \in \mathbb{X}$, we obtain

$$\begin{aligned}
||[\mathcal{Z}_N^{M'}(v_0) - \mathcal{Z}'(v_0)]y||_\infty &= ||[\rho_N^{-\gamma,0}\mathcal{Z}'(v_0) + (I - \rho_N^{-\gamma,0})\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)]y||_\infty \\
&= ||[(\rho_N^{-\gamma,0} - I)\mathcal{Z}'(v_0) - (\rho_N^{-\gamma,0} - I)\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}]y||_\infty \\
&= ||(\rho_N^{-\gamma,0} - I)[\mathcal{Z}'(v_0) - \mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}]y||_\infty \\
&\le (1 + ||\rho_N^{-\gamma,0}||_\infty)||[\mathcal{Z}'(v_0) - \mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}]y||_\infty \\
&\le (1 + C\log N)||[\mathcal{Z}'(v_0) - \mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}]y||_\infty.
\end{aligned} \tag{1.27}$$

Consider

$$\begin{aligned}
||\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)||_\infty &= ||\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)\rho_N^{-\gamma,0} + \mathcal{Z}'(v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)||_\infty \\
&\le ||\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)\rho_N^{-\gamma,0}||_\infty + ||\mathcal{Z}'(v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)||_\infty.
\end{aligned} \tag{1.28}$$

Now from the first term of the above estimate, we obtain

$$\begin{aligned}
&|[\mathcal{Z}'(\rho_N^{-\gamma,0}v_0) - \mathcal{Z}'(v_0)]\rho_N^{-\gamma,0}y(x)| \\
&= \Big|\int_{-1}^{1}(1-\theta)^{-\gamma}[\tilde{k}_v(x, s(x,\theta), \rho_N^{-\gamma,0}v_0(s(x,\theta))) - \tilde{k}_v(x, s(x,\theta), v_0(s(x,\theta)))]\rho_N^{-\gamma,0}y(s(x,\theta))d\theta\Big| \\
&\le ||[\tilde{k}_v(x, s(x,\theta), \rho_N^{-\gamma,0}v_0(s(x,\theta))) - \tilde{k}_v(x, s(x,\theta), v_0(s(x,\theta)))]||_{\omega^{-\gamma,0}}||\rho_N^{-\gamma,0}y(s(x,\theta))||_{\omega^{-\gamma,0}} \\
&\le C_2||[\rho_N^{-\gamma,0}v_0 - v_0](s(x,\cdot))||_{\omega^{-\gamma,0}}||\rho_N^{-\gamma,0}y||_{\omega^{-\gamma,0}} \\
&\le C_2||[\rho_N^{-\gamma,0}v_0 - v_0](s(x,\cdot))||_{\omega^{-\gamma,0}}||y||_{\omega^{-\gamma,0}}.
\end{aligned} \tag{1.29}$$

Now from the second term of the estimate (1.28) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |[\mathcal{Z}'(v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)]y(x)| &= |\mathcal{Z}'(v_0)[\rho_N^{-\gamma,0} - I]y(x)| \\ &= \Big| \int_{-1}^{1} (1-\theta)^{-\gamma} \tilde{k}_v(x, s(x,\theta), v_0(s(x,\theta)))(\rho_N^{-\gamma,0} - I)y(s(x,\theta))d\theta \Big| \\ &= |\langle \tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot))), (\rho_N^{-\gamma,0} - I)y(s(x,\cdot))\rangle_{\omega^{-\gamma,0}}| \\ &= |\langle (\rho_N^{-\gamma,0} - I)\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot))), y(s(x,\cdot))\rangle_{\omega^{-\gamma,0}}| \\ &\leq ||(\rho_N^{-\gamma,0} - I)\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot)))||_{\omega^{-\gamma,0}} ||y||_{\omega^{-\gamma,0}}. \end{aligned} \tag{1.30}$$

Combining the estimates (1.28), (1.29) and (1.30), we obtain

$$\begin{aligned} &||[\mathcal{Z}_N^{M'}(v_0) - \mathcal{Z}'(v_0)]y||_\infty \\ &= ||(I - \rho_N^{-\gamma,0})[\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)]y||_\infty \\ &\leq (1 + C\log N)[C_2||[\rho_N^{-\gamma,0}v_0 - v_0](s(x,\cdot))||_{\omega^{-\gamma,0}} + ||(\rho_N^{-\gamma,0} - I)\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot)))||_{\omega^{-\gamma,0}}]||y||_{\omega^{-\gamma,0}} \\ &\leq (1 + C\log N)[C_2||[\rho_N^{-\gamma,0}v_0 - v_0](s(x,\cdot))||_{\omega^{-\gamma,0}} \\ &+ ||(\rho_N^{-\gamma,0} - I)\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot)))||_{\omega^{-\gamma,0}}]||y||_\infty \to 0 \text{ as } N \to \infty. \end{aligned} \tag{1.31}$$

Now from Lemma 1 and estimate (1.31), we have

$$\begin{aligned} ||[\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0} - \mathcal{Z}'(v_0)]|| &\leq C_2(1 + C\log N)||[\rho_N^{-\gamma,0}v_0 - v_0](s(x,\cdot))||_{\omega^{-\gamma,0}} \\ &+ (1 + C\log N)||(\rho_N^{-\gamma,0} - I)\tilde{k}_v(x, s(x,\cdot), v_0(s(x,\cdot)))||_{\omega^{-\gamma,0}} \to 0 \text{ as } N \to \infty. \end{aligned} \tag{1.32}$$

From estimates (1.27) and (1.32), we can show that $||[\mathcal{Z}_N^{M'}(v_0) - \mathcal{Z}'(v_0)]||_\infty \to 0$ as $N \to \infty$. Since 1 is not an eigenvalue of the linear operator $\mathcal{Z}'(v_0)$. This implies that for sufficiently large N, there exists constant $\mathcal{L}_2 > 0$ such that $\|(I - \mathcal{Z}_N^{M'}(v_0))^{-1}\|_{\omega^{-\gamma,0}} \leq \mathcal{L}_2$.
This completes the proof. □

Theorem 1.2. Let $\rho_N^{-\gamma,0} : \mathbb{X} \to \mathbb{X}_N$ be defined by (1.14) be the orthogonal projection operator for $\alpha = -\gamma,\ \beta = 0$ and $v_0 \in \mathcal{C}[-1,1]$ be an isolated solution of the equation (1.9) and let v_N^M be the Jacobi spectral multi-Galerkin approximate solution defined by (1.22) and for sufficiently large N, $v_N^M \in B(v_0,\delta) = \{v \ :\ ||v - v_0||_{\omega^{-\gamma,0}} \leq \delta\}$, for some $\delta > 0$ and the following results holds

$$\|v_N^M - v_0\|_{\omega^{-\gamma,0}} = \begin{cases} \mathcal{O}(N^{-2r}), & \text{if } v_0 \text{ is sufficiently smooth}, \\ \mathcal{O}(N^{-2(1-\gamma)}), & \text{if } v_0 \text{ is nonsmooth}. \end{cases} \tag{1.33}$$

$$||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0) - \mathcal{Z}(v_0)]||_\infty = \begin{cases} \mathcal{O}(N^{-3r}), & \text{if } v_0 \text{ is sufficiently smooth}, \\ \mathcal{O}(N^{-3(1-\gamma)}), & \text{if } v_0 \text{ is nonsmooth}. \end{cases} \tag{1.34}$$

Proof . The proof follows from [4]. □

In the following theorem, we prove the error bound for iterated Jacobi spectral multi Galerkin approximate solution in uniform norm.

Theorem 1.3. Let the orthogonal projection operator $\rho_N^{-\gamma,0} : \mathbb{X} \to \mathbb{X}_N$ be defined by (1.14) for $\alpha = -\gamma,\ \beta = 0$ and $v_0 \in \mathcal{C}[-1,1]$ be an isolated solution of the equation (1.7) and let $\tilde{v}_N^M$ be the iterated Jacobi spectral multi-Galerkin approximate solution defined by (1.23). Then we have the following results

$$||\tilde{v}_N^M - v_0||_\infty = \begin{cases} \mathcal{O}(N^{-3r}\log N), & \text{if } v_0 \text{ is sufficiently smooth}, \\ \mathcal{O}(N^{-3(1-\gamma)}\log N), & \text{if } v_0 \text{ is nonsmooth}. \end{cases} \tag{1.35}$$

Proof . Consider

$$\begin{aligned}\|\mathcal{Z}'(v_0)y\|_\infty &= \sup_{t\in[-1,1]} |\mathcal{Z}'(v_0)y(t)| \\ &= \sup_{t\in[-1,1]} \Big| \int_{-1}^{1} (1-\varphi)^{-\gamma} \tilde{z}_v(t, s(t,\varphi), v_0(s(t,\varphi)))y(s(t,\varphi))\, d\varphi \Big| \\ &\le ||\tilde{z}_v(t, s(t,\cdot), v_0(s(t,\cdot)))||_{w^{-\gamma,0}} ||y(s(t,\cdot))||_{w^{-\gamma,0}} \\ &\le M||y(s(t,\cdot))||_{w^{-\gamma,0}}. \end{aligned} \tag{1.36}$$

Since $||\mathcal{Z}'(v_0)||_\infty \le M$ and $||(\mathcal{I} - \mathcal{Z}_N^{M'}(v_0))^{-1}||_\infty \le \mathcal{L}_2 < \infty$, it follows that

$$||\mathcal{Z}'(v_0)(\mathcal{I} - \mathcal{Z}_N^{M'}(v_0))^{-1}||_\infty \le M\mathcal{L}_2 = M_1 < \infty. \tag{1.37}$$

From equations (1.9) and (1.23), we have

$$\begin{aligned}\tilde{v}_N^M - v_0 &= \mathcal{Z}(v_N^M) - \mathcal{Z}(v_0) \\ &= \mathcal{Z}'(v_0 + \varphi_1(v_0 - v_N^M))(v_0 - v_N^M),\end{aligned}$$

in the above, we have used the Mean value theorem, where $0 < \varphi_1 < 1$. For simplicity, let $\xi_N = v_0 + \varphi_1(v_0 - v_N^M)$, then

$$\begin{aligned}||\tilde{v}_N^M - v_0||_\infty &\le ||\mathcal{Z}'(v_0 + \varphi_1(v_0 - v_N^M))(v_0 - v_N^M)||_\infty \\ &\le ||[\mathcal{Z}'(\xi_N) - \mathcal{Z}'(v_0)](v_0 - v_N^M)||_\infty + ||\mathcal{Z}'(v_0)(v_0 - v_N^M)||_\infty. \end{aligned} \tag{1.38}$$

Now consider the first term of the above estimate

$$\begin{aligned}&\big|[\mathcal{Z}'(\xi_N) - \mathcal{Z}'(v_0)](v_0 - v_N^M)(t)\big| \\ &= \Big| \int_{-1}^{1} (1-\varphi)^{-\gamma} [\tilde{z}_v(t, s(t,\varphi), \xi_N(s(t,\varphi))) - \tilde{z}_v(t, s(t,\varphi), v_0(s(t,\varphi))](v_0 - v_N^M)(s(t,\varphi))\, d\varphi \Big| \\ &\le M \int_{-1}^{1} (1-\varphi)^{-\gamma/2} |[\tilde{z}_v(t, s(t,\varphi), \xi_N(s(t,\varphi))) \\ &- \tilde{z}_v(t, s(t,\varphi), v_0(s(t,\varphi))]|(1-\varphi)^{-\gamma/2}|(v_0 - v_N^M)(s(t,\varphi))|\, d\varphi. \end{aligned} \tag{1.39}$$

Using the Cauchy Schwartz inequality, we obtain

$$\begin{aligned}&||[\mathcal{Z}'(\xi_N) - \mathcal{Z}'(v_0)](v_0 - v_N^M)(t)||_\infty \\ &\le M||[\tilde{z}_v(t, s(t,\cdot), \xi_N(s(t,\cdot))) - \tilde{z}_v(t, s(t,\cdot), v_0(s(t,\cdot))]||_{w^{-\gamma,0}} ||(v_0 - v_N^M)||_{w^{-\gamma,0}}. \end{aligned} \tag{1.40}$$

Now using the Lipschitz continuity of $\tilde{z}_v(t, s(t,\cdot), \xi_N(s(t,\cdot)))$, we obtain

$$\begin{aligned}&||\tilde{z}_v(t, s(t,\cdot), \xi_N(s(t,\cdot))) - \tilde{z}_v(t, s(t,\cdot), v_0(s(t,\cdot))||^2_{w^{-\gamma,0}} \\ &= \Big| \int_{-1}^{1} (1-\varphi)^{-\gamma} [\tilde{z}_v(t, s(t,\varphi), \xi_N(s(t,\varphi))) - \tilde{z}_v(t, s(t,\varphi), v_0(s(x,\varphi))]^2\, d\varphi \Big| \\ &\le \Big| \int_{-1}^{1} (1-\varphi)^{-\gamma} c_1^2 [\xi_N(s(t,\varphi)) - v_0(s(t,\varphi))]^2\, d\varphi \Big| \\ &= c_1^2 ||\xi_N - v_0||^2_{w^{-\gamma,0}} \le c_1^2 ||v_N^M - v_0||^2_{w^{-\gamma,0}}. \end{aligned} \tag{1.41}$$

Combining this estimate with (1.39), we obtain

$$||[\mathcal{Z}'(v_0 + \varphi_1(v_0 - v_N^M)) - \mathcal{Z}'(v_0)](v_0 - v_N^M)||_\infty \le Mc_1 ||v_N^M - v_0||^2_{w^{-\gamma,0}}. \tag{1.42}$$

Next for the second term of the estimate (1.38), consider

$$v_N^M - v_0 = \mathcal{Z}_N^M(v_N^M) - \mathcal{Z}(v_0) = \mathcal{Z}_N^M(v_N^M) - \mathcal{Z}_N^M(v_0) - \mathcal{Z}_N^{M'}(v_0)(v_N^M - v_0) + \mathcal{Z}_N^{M'}(v_0)(v_N^M - v_0) + \mathcal{Z}_N^M(v_0) - \mathcal{Z}(v_0).$$

This implies

$$(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))(v_N^M-v_0)=\mathcal{Z}_N^M(v_N^M)-\mathcal{Z}_N^M(v_0)-\mathcal{Z}_N^{M'}(v_0)(v_N^M-v_0)+\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0). \tag{1.43}$$

Hence, using the Mean value theorem, we have

$$\begin{aligned} v_N^M-v_0&=(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\mathcal{Z}_N^M(v_N^M)-\mathcal{Z}_N^M(v_0)-\mathcal{Z}_N^{M'}(v_0)(v_N^M-v_0)+\mathcal{Z}_N^M(v_0)-\mathcal{Z}(_0)]\\ &=(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\{\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)\}(v_N^M-v_0)]+(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]. \end{aligned} \tag{1.44}$$

Applying $\mathcal{Z}'(v_0)$ on both sides of the above equation, we obtain

$$\begin{aligned} ||\mathcal{Z}'(v_0)(v_N^M-v_0)||_\infty&\le||\mathcal{Z}'(v_0)(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}||_\infty||[\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)](v_N^M-v_0)||_\infty\\ &\quad+||\mathcal{Z}'(v_0)(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\\ &\le M_1||[\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)](v_N^M-v_0)||_\infty\\ &\quad+||\mathcal{Z}'(v_0)(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty \end{aligned} \tag{1.45}$$

Using the identity $(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}=\mathcal{I}+(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}\mathcal{Z}_N^{M'}(v_0)$, for the second term of the above estimate, we obtain

$$\begin{aligned} &||\mathcal{Z}'(v_0)(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\\ &\quad=||\mathcal{Z}'(v_0)\{\mathcal{I}+(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}\mathcal{Z}_N^{M'}(v_0)\}[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\\ &\quad\le||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty+||\mathcal{Z}'(v_0)(\mathcal{I}-\mathcal{Z}_N^{M'}(v_0))^{-1}\mathcal{Z}_N^{M'}(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\\ &\quad\le||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty+M_1||\mathcal{Z}_N^{M'}(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty \end{aligned} \tag{1.46}$$

Combining this with the estimate (1.45), we obtain

$$\begin{aligned} ||\mathcal{Z}'(v_0)(v_N^M-v_0)||_\infty&\le M_1||\{\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)\}(v_N^M-v_0)||_\infty+||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\\ &\quad+M_1||\mathcal{Z}_N^{M'}(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty. \end{aligned} \tag{1.47}$$

Note that

$$\begin{aligned} \mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)&=[\rho_N^{-\gamma,0}\mathcal{Z}'(\xi_N)+\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))\rho_N^{-\gamma,0}-\rho_N^{-\gamma,0}\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))\rho_N^{-\gamma,0}\\ &\quad-\rho_N^{-\gamma,0}\mathcal{Z}'(v_0)-\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}+\rho_N^{-\gamma,0}\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)\rho_N^{-\gamma,0}]\\ &=\rho_N^{-\gamma,0}[\mathcal{Z}'(\xi_N)-\mathcal{Z}'(v_0)]+(\mathcal{I}-\rho_N^{-\gamma,0})[\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))-\mathcal{Z}'(\rho_N^{-\gamma,0}u_0)]\rho_N^{-\gamma,0}. \end{aligned} \tag{1.48}$$

Therefore

$$\begin{aligned} ||\{\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)\}(v_N^M-v_0)||_\infty&\le||\rho_N^{-\gamma,0}[\mathcal{Z}'(\xi_N)-\mathcal{Z}'(v_0)](v_N^M-v_0)||_\infty\\ &\quad+||(\mathcal{I}-\rho_N^{-\gamma,0})[\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))-\mathcal{Z}'(\rho_N^{-\gamma,0}u_0)]\rho_N^{-\gamma,0}(v_N^M-v_0)||_\infty\\ &\le C\log N||[\mathcal{Z}'(\xi_N)-\mathcal{Z}'(v_0)](v_N^M-v_0)||_\infty\\ &\quad+(1+C\log N)||[\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))-\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)]\rho_N^{-\gamma,0}(v_N^M-v_0)||_\infty \end{aligned} \tag{1.49}$$

Now following the steps of (1.39) to (1.42) and using $||\rho_N^{-\gamma,0}v||_{w^{-\gamma,0}}\le||v||_{w^{-\gamma,0}}$, we can show that

$$||[\mathcal{Z}'(\rho_N^{-\gamma,0}(\xi_N))-\mathcal{Z}'(\rho_N^{-\gamma,0}v_0)]\rho_N^{-\gamma,0}(v_N^M-v_0)||_\infty\le Mc_1||v_N^M-v_0||^2_{w^{-\gamma,0}}. \tag{1.50}$$

Merging the estimates (1.49) and (1.50), we have

$$||\{\mathcal{Z}_N^{M'}(\xi_N)-\mathcal{Z}_N^{M'}(v_0)\}(v_N^M-v_0)||_\infty\le Mc_1(\log N+(1+C\log N))||v_N^M-v_0||^2_{w^{-\gamma,0}}. \tag{1.51}$$

Also note that

$$||\mathcal{Z}_N^{M'}(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty=||\mathcal{P}_N^{-\gamma,0}\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty\le\log N||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0)-\mathcal{Z}(v_0)]||_\infty. \tag{1.52}$$

Now using the estimates (1.51) and (1.52) in the estimate (1.47), we have

$$||\mathcal{Z}'(v_0)(v_N^M - v_0)||_\infty \le M_1 M_2 \log N ||v_N^M - v_0||^2_{w^{-\gamma,0}} + ||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0) - \mathcal{Z}(v_0)]||_\infty$$
$$+ M_1 \log N ||\mathcal{Z}'(v_0)[\mathcal{Z}_N^M(v_0) - \mathcal{Z}(v_0)]||_\infty. \tag{1.53}$$

Now combining this with the estimate (1.38), we have

$$||\tilde{v}_N^M - v_0||_\infty \le (Mc_1 + M_1 M_2 \log N)||v_N^M - v_0||^2_{w^{-\gamma,0}}$$
$$+ (1 + M_1 \log N)||\mathcal{K}'(v_0)[\mathcal{K}_N^M(v_0) - \mathcal{K}(v_0)]||_\infty. \tag{1.54}$$

Now from Theorem 1.2, we have

$$||\tilde{v}_N^M - v_0||_\infty = \begin{cases} \mathcal{O}(N^{-3r}\log N), & \text{if } v_0 \text{ is sufficiently smooth,} \\ \mathcal{O}(N^{-3(1-\gamma)}\log N), & \text{if } v_0 \text{ is nonsmooth.} \end{cases} \tag{1.55}$$

Hence proved. □

Remark 1.1. From Theorem 1.3, we have seen that the error bound in iterated Jacobi spectral multi Galerkin solution in uniform norm is $\mathcal{O}(N^{-3r}\log N)$, when solution is sufficient smooth and $\mathcal{O}(N^{-3(1-\gamma)}\log N)$, when solution is nonsmooth.

2 Numerical Illustration

Numerical examples are offered in this section to support our theoretical conclusions. Jacobi polynomials are used in this case as basis functions of the subspace $\mathbb{X}_N$, which are created by the recurrence relations established in (1.12) and (1.13). In uniform norm, we discuss the errors in iterated Jacobi spectral multi-Galerkin methods. The approximate solutions in Jacobi spectral multi-Galerkin methods are denoted by $\tilde{v}_N^M$ in the following Tables.

Example 2.1. Consider the following weakly singular Volterra Urysohn integral equation of second kind

$$y(x) + \int_0^x (x-\varrho)^{-\gamma} z(x,\varrho,y(\varrho))d\varrho = g(x), \quad x \in [0,1], \quad 0 < \gamma < 1,$$

with $z(x,\varrho,y(\varrho)) = y(\varrho)^2$, $\gamma = \frac{1}{2}$ and $g(x) = \frac{4}{3}x^{\frac{3}{2}} + x^{\frac{1}{2}}$ and the exact solution is given by $y(x) = x^{\frac{1}{2}}$. We use the following formula for obtaining the order of convergence a.

$$a = -\frac{\log \|v - \tilde{v}_N^M\|_\infty}{\log n} \tag{2.1}$$

Table 1: Iterated Jacobi spectral-multi Galerkin methods

n	$\|v - \tilde{v}_N^M\|_\infty$	a
2	1.04346 $\times 10^{-4}$	13.22
3	1.84747 $\times 10^{-6}$	12.01
4	4.53015 $\times 10^{-8}$	12.91
5	2.78615 $\times 10^{-10}$	13.67
6	1.15702 $\times 10^{-11}$	14.05

Example 2.2. Consider the following weakly singular Volterra Urysohn integral equation of second kind

$$y(x) - \int_0^x (x-\varrho)^{-\gamma} z(x,\varrho,y(\varrho))d\varrho = g(x), \quad x \in [0,1], \quad 0 < \gamma < 1,$$

with $z(x,\varrho,y(\varrho)) = y(\varrho)^2$, $\gamma = \frac{1}{2}$ and $g(x) = x^4 - \frac{65536}{109395}x^{17/2}$ and the solution is $y(x) = x^4$.

Remark 2.1. Here we have developed the iterated Jacobi spectral multi Galerkin method and obtained the convergence analysis in uniform norm and verified the convergence snslydianalysis numerically.

Table 2: Iterated Jacobi spectral multi-Galerkin methods

n	$\|v - \tilde{v}_N^M\|_\infty$	a
5	1.9447×10^{-11}	15.32
6	3.222×10^{-13}	16.05
7	1.6142×10^{-14}	16.32
8	2.8770×10^{-16}	17.20
9	1.7640×10^{-17}	17.55

3 Declarations

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