

# A new subclass of univalent holomorphic functions based on $q$ -analogue of Noor operator

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## Abstract

In this article, we introduce another new subclass by using  $q$ -analogue of the Noor operator and based on it we investigate a subclass with fixed finitely many coefficients for the univalent holomorphic functions. We obtain a number of useful properties such as coefficient estimates, extreme points, convexity and convolution-preserving properties.

Keywords:  $q$ -analogue of Noor Operator, Coefficient bound, Extreme points, Convex set, Convolution  
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## 1 Introduction

These days, if mathematics is expressed in a combination with other sciences, the motivation for research and study in the basic sciences will increase, and the geometric and intuitive fields of mathematics will be displayed. The theory of geometric functions and integration of mathematics and differential has obtained new and useful achievements, including  $q$ -calculus and  $q$ -differential equations [13, 22]. Collaborative researchers between mathematics, physics, geometry and mechanics have called for the study of linear operators in the field of geometric function theory. Because  $q$ -analogue linear operators have brought very effective applications to this group of authors [1, 10, 12]. At the beginning of the way, we start with the  $q$ -analogue of the derivative and integral operator of Ruscheweyh [4, 9], and then the  $q$ -analogue of the Noor integral operator [2, 6] and  $q$ -Bernardi differential operators were introduced [21]. The rest of the researchers did not sit idle and introduced new achievements. Among other complex and important operators are  $q$ -Picard and  $q$ -Gauss-Weierstrass [7]. But no operator has been given as much importance and attention as the  $q$ -analogue of the Noor integral operator in the field of the theory of geometric functions [3, 16]. In this article, with the help of this operator, we introduce a new interesting subclass of univalent holomorphic functions, and for this subclass, we examine and present the estimation of coefficients and some related properties and results, see [15, 20] and also [11].

Let  $\mathcal{A}$  indicate the family of analytic functions having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  that are equal to 0 at  $z = 0$  and the derivative of these functions is equal to one at  $z = 0$ . This property is often called normalized property. Furthermore,  $\mathcal{N}$  as a subclass of  $\mathcal{A}$  by changing with negative coefficients is of the type

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0). \quad (1.1)$$

For functions  $f$  and  $g$  which are analytic in  $\mathbb{D}$  and have the form (1.1), we define the Hadamard product (convolution) of  $f$  and  $g$  by setting

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \mathbb{D}).$$

For more details see [10, 19]. We first review some basic and practical definitions from [18].

**Definition 1.1.** For  $0 < q < 1$  the  $q$ -derivative of function  $f \in \mathcal{A}$  is defined by the equation

$$\partial_q f(z) = \frac{f(zq) - f(z)}{z(q-1)} \quad (z \neq 0) \quad (1.2)$$

and  $\partial_q f(z)$  in  $z = 0$  is equal to  $f'(0)$ . According to the above definition for  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$  we have

$$\partial_q \left( z + \sum_{k=2}^{\infty} a_k z^k \right) = 1 + \sum_{k=2}^{\infty} [k, q] a_k z^k \quad (k \in \mathbb{N}, z \in \mathbb{D}),$$

where

$$[k, q] = \frac{1 - q^k}{1 - q} = 1 + \sum_{t=1}^{k-1} q^t \quad ([0, q] = 0) \quad (1.3)$$

and the  $q$ -generalized Pochhammer symbol for  $y \geq 0$  is defined by

$$[y, q]_k = \begin{cases} [y, q][y+1, q] \cdots [y+k-1, q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}$$

for  $h(z) = z^k$ , if  $q \rightarrow 1$  we have

$$\partial_q h(z) = [k, q] z^{k-1} = h'(z),$$

here  $h'$  follows from that  $q \rightarrow 0$  and then  $[k, q] \rightarrow k$ .

We finally want to use the function  $\mathcal{T}_{q, \mu+1}^{-1}(z)$  which has been defined by Arif et. al [8] and define  $\mathcal{N}_q^\mu f(z)$  as a subclass of functions with negative and fixed finitely many coefficient. We have

$$\mathcal{T}_{q, \mu+1}^{-1} * \mathcal{T}_{q, \mu+1}(z) = z \partial_q f(z) \quad (\mu > -1),$$

where

$$\mathcal{T}_{q, \mu+1}(z) = z - \sum_{k=2}^{\infty} \frac{[\mu+1, q]_{k-1}}{[k, q]!} z^k.$$

The right-hand side of the above equality is absolutely convergent in  $\mathbb{D}$ . We now define the Noor integral operator  $\mathcal{N}_q^\mu f(z)$  by using the definition of  $q$ -derivatives and Hadamard product as follows

$$\begin{aligned} \mathcal{N}_q^\mu f(z) &= \mathcal{T}_{q, \mu+1}^{-1}(z) * f(z) \\ &= z - \sum_{k=2}^{\infty} \Psi_{k-1} a_k z^k \quad (z \in \mathbb{D}), \end{aligned} \quad (1.4)$$

where

$$\Psi_{k-1} = \frac{[k, q]!}{[\mu+1, q]_{k-1}} \quad (1.5)$$

also

$$[k, q]! = \begin{cases} 1, & k = 0, \\ [1, q][2, q] \cdots [k, q], & k \in \mathbb{N}. \end{cases}$$

It can be easily checked that

$$\mathcal{N}_q^0 f(z) = z \partial_q f(z), \quad \mathcal{N}_q^1 f(z) = f(z)$$

and

$$\lim_{q \rightarrow 1^-} \mathcal{N}_q^\mu f(z) = z - \sum_{k=2}^{\infty} \frac{k!}{(\mu + 1)_{k-1}} a_k z^k,$$

which is the familiar Noor integral operator, see [17, 18]. For  $0 \leq \alpha \leq 1$  and  $0 \leq \beta < 1$ , the function  $f \in \mathcal{N}$  is in the class  $\mathcal{N}_q^\mu(\alpha, \beta)$  if it satisfies

$$\operatorname{Re} \left\{ \frac{z \partial_q (\mathcal{N}_q^\mu f(x)) + \alpha z^2 \partial_q^2 (\mathcal{N}_q^\mu f(x))}{\alpha z \partial_q (\mathcal{N}_q^\mu f(x)) + (1 - \alpha) \mathcal{N}_q^\mu f(x)} \right\} > \beta, \tag{1.6}$$

where  $\partial_q$  and  $\mathcal{N}_q^\mu$  are defined in (1.2) and (1.4) respectively. Also  $\partial_q^2 (\mathcal{N}_q^\mu f(z))$  means  $\partial_q (\partial_q (\mathcal{N}_q^\mu f(z)))$ . Now, we consider the class  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  consisting of functions with negative and fixed finitely many coefficient of the following form

$$\begin{aligned} f(z) &= z - \sum_{m=2}^n \frac{1 - \beta}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \\ &\quad - \sum_{k=n+1}^{\infty} a_k z^k, \end{aligned} \tag{1.7}$$

where satisfies (1.6). We need the following Lemma which has been proved in a general case in [15].

**Lemma 1.2.**  $f(z) \in \mathcal{N}$  is in the class  $\mathcal{N}_q^\mu(\alpha, \beta)$  if and only if

$$\sum_{k=n+1}^{\infty} \Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)) a_k < 1 - \beta,$$

where  $\Psi_{k-1}$  and  $[k, q]$  are given by (1.5) and (1.3), respectively.

## 2 Main results

In this section, we obtain a sharp coefficient bound for functions in the class  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$ . We also investigate the convexity of  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$ .

**Theorem 2.1.** The function  $f(z)$  of the form (1.7) is in the class  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  if and only if

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} a_k < 1 - \sum_{m=2}^n d_m. \tag{2.1}$$

**Proof .** Consider

$$a_m = \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m.$$

Since  $\mathcal{N}_q^\mu(\alpha, \beta, d_m) \subset \mathcal{N}_q^\mu(\alpha, \beta)$ , so  $f \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$  if and only if

$$\sum_{m=2}^n \frac{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} a_m$$

$$+ \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} a_k < 1$$

or

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} a_k < 1 - \sum_{m=2}^n d_m$$

and this gives the desired result.  $\square$

**Remark 2.2.** By (2.1) we conclude that for  $k \geq n + 1$  the following inequality holds

$$a_k \leq \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left(1 - \sum_{m=2}^n d_m\right).$$

**Remark 2.3.** Define  $F(z)$  by setting

$$F(z) = z - \sum_{m=2}^n \frac{1 - \beta}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m - \frac{1 - \beta}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left(1 - \sum_{m=2}^n d_m\right) z^k.$$

The inequality (2.1) is sharp for  $F(z)$ .

**Theorem 2.4.** The class  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  is a convex set.

**Proof .** We have to show that if

$$f_j(z) = z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m - \sum_{k=n+1}^{\infty} a_{k,j} z^k$$

is in  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  for  $j = 1, 2, \dots, t$ , then the function  $F(z) = \sum_{j=1}^t \lambda_j f_j(z)$  is also in  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  where

$$\sum_{j=1}^t \lambda_j = 1, \quad 0 \leq \sum_{m=2}^n d_m \leq 1$$

and  $0 \leq d_m \leq 1$ . By Theorem 2.1 we have

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} a_{k,j} < 1 - \sum_{m=2}^n d_m$$

for every  $j = 1, 2, \dots, t$ . Since

$$\begin{aligned} F(z) &= \sum_{j=1}^t \lambda_j f_j(z) \\ &= z - \sum_{m=2}^n \frac{(1 - \beta)d_m}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} - \sum_{k=n+1}^{\infty} \left( \sum_{j=1}^t \lambda_j a_{k,j} \right) z^k \end{aligned}$$

and

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} \left( \sum_{j=1}^t \lambda_j a_{k,j} \right)$$

$$\begin{aligned}
 &= \sum_{j=1}^t \left( \sum_{k=n+1}^{\infty} \left[ \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta} \right] \lambda_j \right) \\
 &< \sum_{j=1}^t \left( 1 - \sum_{m=2}^n d_m \right) \lambda_j \\
 &= 1 - \sum_{m=2}^n d_m,
 \end{aligned}$$

so by Theorem 2.1 we get  $F(z) \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$ .  $\square$

### 3 Geometric properties of $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$

In this section, we introduce the extreme points of  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$ . The special geometric property and convolution-preserving concept are also investigated.

**Theorem 3.1.** The extreme points of the class  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$  are the functions  $f_n(z)$  and  $f_k(z)$  ( $k \geq n + 1$ ) defined by setting

$$\begin{aligned}
 f_n(z) &= z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m, \\
 f_k(z) &= z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \\
 &\quad - \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left( 1 - \sum_{m=2}^n d_m \right) z^k \quad (k \geq n + 1).
 \end{aligned}$$

**Proof .** We show that  $F(z) \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$  if and only if it can be expressed in the following form

$$F(z) = \sum_{k=n}^{\infty} \lambda_k f_k(z)$$

where  $\lambda_k \geq 0$  ( $k \geq n$ ) and  $\sum_{k=n}^{\infty} \lambda_k = 1$ . Let  $F(z) = \sum_{k=n}^{\infty} \lambda_k f_k(z)$ . Then

$$\begin{aligned}
 F(z) &= \lambda_n f_n(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z) \\
 &= \lambda_n z - \lambda_n \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \\
 &\quad + \sum_{k=n+1}^{\infty} \lambda_k z - \sum_{k=n+1}^{\infty} \lambda_k \left( \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \right) \\
 &\quad - \sum_{k=n+1}^{\infty} \lambda_k \left( \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left( 1 - \sum_{m=2}^n d_m \right) z^k \right) \\
 &= z - \left( \lambda_n + \sum_{k=n+1}^{\infty} \lambda_k \right) \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m \\
 &\quad - \sum_{k=n+1}^{\infty} \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left( 1 - \sum_{m=2}^n d_m \right) \lambda_k z^k \\
 &= z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \\
 &\quad - \sum_{k=n+1}^{\infty} \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left( 1 - \sum_{m=2}^n d_m \right) \lambda_k z^k.
 \end{aligned}$$

Since

$$\begin{aligned}
& \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))(1 - \beta)}{(1 - \beta)\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left(1 - \sum_{m=2}^n d_m\right) \lambda_k \\
&= \left(1 - \sum_{m=2}^n d_m\right) \sum_{k=n+1}^{\infty} \lambda_k \\
&= \left(1 - \sum_{m=2}^n d_m\right) (1 - \lambda_n) \\
&< 1 - \sum_{m=2}^n d_m,
\end{aligned}$$

so by Theorem 2.1 we deduce  $F(z) \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$ .

Conversely, suppose  $F(z) \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$ . By putting

$$\lambda_k = \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n d_m)} a_k \quad (k \geq n + 1)$$

we have  $\lambda_k \geq 0$  and if we set  $\lambda_n = 1 - \sum_{k=n+1}^{\infty} \lambda_k$ , we reach

$$\begin{aligned}
F(z) &= z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m z^m \\
&\quad - \sum_{k=n+1}^{\infty} \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left(1 - \sum_{m=2}^n d_m\right) \lambda_k z^k \\
&= f_n(z) - \sum_{k=n+1}^{\infty} \left( z - \sum_{m=2}^n \frac{(1 - \beta)d_m}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} z^m - f_k(z) \right) \lambda_k \\
&= f_n(z) - \sum_{k=n+1}^{\infty} (f_n(z) - f_k(z)) \lambda_k \\
&= \left(1 - \sum_{k=n+1}^{\infty} \lambda_k\right) f_n(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z) \\
&= \sum_{k=n}^{\infty} \lambda_k f_k(z).
\end{aligned}$$

Hence, the proof is complete.  $\square$

**Theorem 3.2.** Let  $f(z) \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$ . If

$$c_m = \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m^2 \quad (2 \leq m \leq n), \quad (3.1)$$

then the function  $G$  defined by

$$G(z) = z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} c_m z^m - \sum_{k=n+1}^{\infty} a_k z^k$$

is also in  $\mathcal{N}_q^\mu(\alpha, \beta, d_m)$ .

**Proof .** Since  $\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha)) > 1$ , we get

$$c_m = \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} d_m^2$$

$$\begin{aligned} &< d_m \\ &\leq 1. \end{aligned}$$

So,  $0 \leq \sum_{m=2}^n c_m < \sum_{m=2}^n d_m \leq 1$  and hence

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} a_k \\ &< \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n d_m)} a_k \\ &< 1. \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.3.** Let  $f, g \in \mathcal{N}_q^\mu(\alpha, \beta, d_m)$ . Then

$$\begin{aligned} (f * g)(z) &= z - \sum_{m=2}^n \frac{(1 - \beta)^2}{[\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))]^2} d_m^2 z^m \\ &\quad - \sum_{k=n+1}^{\infty} a_k b_k z^k, \end{aligned}$$

is in  $\mathcal{N}_q^\mu(\alpha, \beta_0, c_m)$ , where  $c_m$  ( $2 \leq m < n$ ) is defined by (3.1) and

$$\beta_0 \leq \frac{M - [k, q] - \alpha([k, q])^2}{M + \alpha([k, q] - 1) + 1},$$

$$M = \frac{\Psi_{k-1}}{1 - \sum_{m=2}^n c_m} \left( \frac{[k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)}{1 - \beta} \right)^2. \tag{3.2}$$

**Proof .** By (3.1) we have

$$\begin{aligned} (f * g)(z) &= z - \sum_{m=2}^n \frac{(1 - \beta)}{\Psi_{m-1}([m, q](1 + \alpha[m, q] - \alpha\beta) + \beta(1 - \alpha))} c_m z^m \\ &\quad - \sum_{k=n+1}^{\infty} a_k b_k z^k. \end{aligned}$$

By applying Theorem 3.2 we get

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} a_k < 1$$

and

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} b_k < 1.$$

It is now sufficient to show that

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} a_k b_k \\ &\leq \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} \sqrt{a_k b_k} \\ &\leq 1. \end{aligned}$$

We use the Cauchy-Schwartz inequality for this purpose and find the largest  $\beta_0$  such that

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta_0) + \beta_0(1 - \alpha))}{(1 - \beta_0)(1 - \sum_{m=2}^n c_m)} a_k b_k \\ & \leq \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)(1 - \sum_{m=2}^n c_m)} \sqrt{a_k b_k} \\ & \leq 1 \end{aligned}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{(1 - \beta_0)([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)([k, q](1 + \alpha[k, q] - \alpha\beta_0) + \beta_0(1 - \alpha))}, \quad (k \geq n + 1).$$

This inequality holds when

$$\begin{aligned} & \frac{(1 - \beta)}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} \left(1 - \sum_{m=2}^n c_m\right) \\ & \leq \frac{(1 - \beta_0)([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{(1 - \beta)([k, q](1 + \alpha[k, q] - \alpha\beta_0) + \beta_0(1 - \alpha))} \end{aligned}$$

or equivalently

$$\beta_0 \leq \frac{M - [k, q] - \alpha([k, q])^2}{M + \alpha([k, q] - 1) + 1},$$

where  $M$  is given by (3.2). This completes the proof.  $\square$

In the forthcoming article, we verify the connection between the class defined in this article and bi-univalent functions. We consider Lucas polynomials [6] and Faber polynomial [14] and we discuss their characteristics through the  $q$ -analogue of the Noor integral operator.

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