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A new subclass of univalent holomorphic functions based on q-analogue of Noor operator

Seyed Hadi Sayedain Boroujeni*, Shahram Najafzadeh, Ismail Nikoufar

Department of Mathematics, Payame Noor University, Tehran, Iran

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Abstract

In this article, we introduce another new subclass by using q-analogue of the Noor operator and based on it we investigate a subclass with fixed finitely many coefficients for the univalent holomorphic functions. We obtain a number of useful properties such as coefficient estimates, extreme points, convexity and convolution-preserving properties.

Keywords: q-analogue of Noor Operator, Coefficient bound, Extreme points, Convex set, Convolution
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1 Introduction

These days, if mathematics is expressed in a combination with other sciences, the motivation for research and study in the basic sciences will increase, and the geometric and intuitive fields of mathematics will be displayed. The theory of geometric functions and integration of mathematics and differential has obtained new and useful achievements, including q-calculus and q-differential equations [13, 22]. Collaborative researchers between mathematics, physics, geometry and mechanics have called for the study of linear operators in the field of geometric function theory. Because q-analogue linear operators have brought very effective applications to this group of authors [1, 10, 12]. At the beginning of the way, we start with the q-analogue of the derivative and integral operator of Ruscheweyh [4, 9], and then the q-analogue of the Noor integral operator [2, 6] and q-Bernardi differential operators were introduced [21]. The rest of the researchers did not sit idle and introduced new achievements. Among other complex and important operators are q-Picard and q-Gauss-Weierstrass [7]. But no operator has been given as much importance and attention as the q-analogue of the Noor integral operator in the field of the theory of geometric functions [3, 16]. In this article, with the help of this operator, we introduce a new interesting subclass of univalent holomorphic functions, and for this subclass, we examine and present the estimation of coefficients and some related properties and results, see [15, 20] and also [11].

Let $\mathcal{A}$ indicate the family of analytic functions having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

*Corresponding author
Email addresses: h.sayedain@pnu.ac.ir (Seyed Hadi Sayedain Boroujeni), shnajafzadeh44@pnu.ac.ir (Shahram Najafzadeh), nikoufar@pnu.ac.ir (Ismail Nikoufar)

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in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that are equal to 0 at $z = 0$ and the derivative of these functions is equal to one at $z = 0$. This property is often called normalized property. Furthermore, $\mathcal{N}$ as a subclass of $\mathcal{A}$ by changing with negative coefficients is of the type

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geqslant 0). \tag{1.1}$$

For functions f and g which are analytic in $\mathbb{D}$ and have the form (1.1), we define the Hadamard product (convolution) of f and g by setting

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \qquad (z \in \mathbb{D}).$$

For more details see [10, 19]. We first review some basic and practical definitions from [18].

Definition 1.1. For $0 < q < 1$ the q-derivative of function $f \in \mathcal{A}$ is defined by the equation

$$\partial_q f(z) = \frac{f(zq) - f(z)}{z(q-1)} \qquad (z \neq 0) \tag{1.2}$$

and $\partial_q f(z)$ in $z = 0$ is equal to $f'(0)$. According to the above definition for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ we have

$$\partial_q \Big(z + \sum_{k=2}^{\infty} a_k z^k \Big) = 1 + \sum_{k=2}^{\infty} [k, q] a_k z^k \qquad (k \in \mathbb{N}, z \in \mathbb{D}),$$

where

$$[k, q] = \frac{1 - q^k}{1 - q} = 1 + \sum_{t=1}^{k-1} q^t \qquad ([0, q] = 0) \tag{1.3}$$

and the q-generalized Pochhammer symbol for $y \geq 0$ is defined by

$$[y, q]_k = \begin{cases} [y, q][y+1, q] \cdots [y+k-1, q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}$$

for $h(z) = z^k$, if $q \to 1$ we have

$$\partial_q h(z) = [k, q] z^{k-1} = h^{'}(z),$$

here $h^{'}$ follows from that $q \to 0$ and then $[k, q] \to k$.

We finally want to use the function $\mathcal{T}_{q,\mu+1}^{-1}(z)$ which has been defined by Arif et. al [8] and define $\mathcal{N}_q^{\mu} f(z)$ as a subclass of functions with negative and fixed finitely many coefficient. We have

$$\mathcal{T}_{q,\mu+1}^{-1} * \mathcal{T}_{q,\mu+1}(z) = z \partial_q f(z) \qquad (\mu > -1),$$

where

$$\mathcal{T}_{q,\mu+1}(z) = z - \sum_{k=2}^{\infty} \frac{[\mu+1, q]_{k-1}}{[k, q]!} z^k.$$

The right-hand side of the above equality is absolutely convergent in $\mathbb{D}$. We now define the Noor integral operator $\mathcal{N}_q^{\mu} f(z)$ by using the definition of q-derivatives and Hadamard product as follows

$$\begin{aligned} \mathcal{N}_q^{\mu} f(z) &= \mathcal{T}_{q,\mu+1}^{-1}(z) * f(z) \\ &= z - \sum_{k=2}^{\infty} \Psi_{k-1} a_k z^k \qquad (z \in \mathbb{D}), \end{aligned} \tag{1.4}$$

where

$$\Psi_{k-1} = \frac{[k, q]!}{[\mu+1, q]_{k-1}} \tag{1.5}$$

also

$$[k,q]! = \begin{cases} 1, & k=0, \\ [1,q][2,q]\cdots[k,q], & k\in\mathbb{N}. \end{cases}$$

It can be easily checked that

$$\mathcal{N}_q^0 f(z) = z\partial_q f(z), \qquad \mathcal{N}_q^1 f(z) = f(z)$$

and

$$\lim_{q\to 1^-} \mathcal{N}_q^\mu f(z) = z - \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k,$$

which is the familiar Noor integral operator, see [17, 18]. For $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta < 1$, the function $f \in \mathcal{N}$ is in the class $\mathcal{N}_q^\mu(\alpha,\beta)$ if it satisfies

$$\operatorname{Re}\left\{ \frac{z\partial_q\big(\mathcal{N}_q^\mu f(x)\big) + \alpha z^2 \partial_q^2\big(\mathcal{N}_q^\mu f(x)\big)}{\alpha z\partial_q\big(\mathcal{N}_q^\mu f(x)\big) + (1-\alpha)\mathcal{N}_q^\mu f(x)} \right\} > \beta, \tag{1.6}$$

where ∂_q and $\mathcal{N}_q^\mu$ are defined in (1.2) and (1.4) respectively. Also $\partial_q^2\big(\mathcal{N}_q^\mu f(z)\big)$ means $\partial_q\big(\partial_q(\mathcal{N}_q^\mu f(z))\big)$. Now, we consider the class $\mathcal{N}_q^\mu(\alpha,\beta,d_m)$ consisting of functions with negative and fixed finitely many coefficient of the following form

$$\begin{aligned} f(z) \quad = \quad & z - \sum_{m=2}^{n} \frac{1-\beta}{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)} d_m z^m \\ & - \sum_{k=n+1}^{\infty} a_k z^k, \end{aligned} \tag{1.7}$$

where satisfies (1.6). We need the following Lemma which has been proved in a general case in [15].

Lemma 1.2. $f(z) \in \mathcal{N}$ is in the class $\mathcal{N}_q^\mu(\alpha,\beta)$ if and only if

$$\sum_{k=n+1}^{\infty} \Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right) a_k < 1-\beta,$$

where Ψ_{k-1} and $[k,q]$ are given by (1.5) and (1.3), respectively.

2 Main results

In this section, we obtain a sharp coefficient bound for functions in the class $\mathcal{N}_q^\mu(\alpha,\beta,d_m)$. We also investigate the convexity of $\mathcal{N}_q^\mu(\alpha,\beta,d_m)$.

Theorem 2.1. The function $f(z)$ of the form (1.7) is in the class $\mathcal{N}_q^\mu(\alpha,\beta,\, d_m)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta} a_k < 1 - \sum_{m=2}^{n} d_m. \tag{2.1}$$

Proof . Consider

$$a_m = \frac{(1-\beta)}{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)} d_m.$$

Since $\mathcal{N}_q^\mu(\alpha,\beta,d_m) \subset \mathcal{N}_q^\mu(\alpha,\beta)$, so $f \in \mathcal{N}_q^\mu(\alpha,\beta,d_m)$ if and only if

$$\sum_{m=2}^{n} \frac{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta} a_m$$

$$+\sum_{k=n+1}^{\infty}\frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta}a_k<1$$

or

$$\sum_{k=n+1}^{\infty}\frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta}a_k<1-\sum_{m=2}^{n}d_m$$

and this gives the desired result. □

Remark 2.2. By (2.1) we conclude that for $k\geqslant n+1$ the following inequality holds

$$a_k\leqslant\frac{(1-\beta)}{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}\left(1-\sum_{m=2}^{n}d_m\right).$$

Remark 2.3. Define $F(z)$ by setting

$$\begin{aligned}F(z) &= z-\sum_{m=2}^{n}\frac{1-\beta}{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}d_m z^m\\ &\quad -\frac{1-\beta}{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}\left(1-\sum_{m=2}^{n}d_m\right)z^k.\end{aligned}$$

The inequality (2.1) is sharp for $F(z)$.

Theorem 2.4. The class $\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$ is a convex set.

Proof . We have to show that if

$$\begin{aligned}f_j(z)=z&-\sum_{m=2}^{n}\frac{(1-\beta)}{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}d_m z^m\\ &-\sum_{k=n+1}^{\infty}a_{k,j}z^k\end{aligned}$$

is in $\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$ for $j=1,2,\ldots,t$, then the function $F(z)=\sum_{j=1}^{t}\lambda_j f_j(z)$ is also in $\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$ where

$$\sum_{j=1}^{t}\lambda_j=1,\ 0\leqslant\sum_{m=2}^{n}d_m\leqslant 1$$

and $0\leqslant d_m\leqslant 1$. By Theorem 2.1 we have

$$\sum_{k=n+1}^{\infty}\frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta}a_{k,j}<1-\sum_{m=2}^{n}d_m$$

for every $j=1,2,\ldots,t$. Since

$$\begin{aligned}F(z)&=\sum_{j=1}^{t}\lambda_j f_j(z)\\ &=z-\sum_{m=2}^{n}\frac{(1-\beta)d_m}{\Psi_{m-1}\left([m,q]\left(1+\alpha[m,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}-\sum_{k=n+1}^{\infty}\left(\sum_{j=1}^{t}\lambda_j a_{k,j}\right)z^k\end{aligned}$$

and

$$\sum_{k=n+1}^{\infty}\frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta}\left(\sum_{j=1}^{t}\lambda_j a_{k,j}\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{t}\left(\sum_{k=n+1}^{\infty}\left[\frac{\Psi_{k-1}\left([k,q]\left(1+\alpha[k,q]-\alpha\beta\right)+\beta(1-\alpha)\right)}{1-\beta}\right]\lambda_j\right)\\
&< \sum_{j=1}^{t}\Big(1-\sum_{m=2}^{n} d_m\Big)\lambda_j\\
&= 1-\sum_{m=2}^{n} d_m,
\end{aligned}
$$

so by Theorem 2.1 we get $F(z) \in \mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$. □

3 Geometric properties of $\mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$

In this section, we introduce the extreme points of $\mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$. The special geometric property and convolution-preserving concept are also investigated.

Theorem 3.1. The extreme points of the class $\mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$ are the functions $f_n(z)$ and $f_k(z)$ $(k \geqslant n+1)$ defined by setting

$$
\begin{aligned}
f_n(z) &= z - \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m,\\
f_k(z) &= z - \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m z^m\\
&\quad - \frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}\big(1-\sum_{m=2}^{n} d_m\big) z^k \qquad (k \geqslant n+1).
\end{aligned}
$$

Proof . We show that $F(z) \in \mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$ if and only if it can be expressed in the following form

$$
F(z) = \sum_{k=n}^{\infty} \lambda_k f_k(z)
$$

where $\lambda_k \geqslant 0$ $(k \geqslant n)$ and $\sum_{k=n}^{\infty} \lambda_k = 1$. Let $F(z) = \sum_{k=n}^{\infty} \lambda_k f_k(z)$. Then

$$
\begin{aligned}
F(z) &= \lambda_n f_n(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)\\
&= \lambda_n z - \lambda_n \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m z^m\\
&+ \sum_{k=n+1}^{\infty} \lambda_k z - \sum_{k=n+1}^{\infty} \lambda_k \left(\sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m z^m\right)\\
&- \sum_{k=n+1}^{\infty} \lambda_k \left(\frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}\big(1-\sum_{m=2}^{n} d_m\big) z^k\right)\\
&= z - \Big(\lambda_n + \sum_{k=n+1}^{\infty} \lambda_k\Big) \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m\\
&- \sum_{k=n+1}^{\infty} \frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}\big(1-\sum_{m=2}^{n} d_m\big)\lambda_k z^k\\
&= z - \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} d_m z^m\\
&- \sum_{k=n+1}^{\infty} \frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}\big(1-\sum_{m=2}^{n} d_m\big)\lambda_k z^k.
\end{aligned}
$$

Since

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))(1-\beta)}{(1-\beta)\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}(1-\sum_{m=2}^{n}d_m)\lambda_k\\
&=\Big(1-\sum_{m=2}^{n}d_m\Big)\sum_{k=n+1}^{\infty}\lambda_k\\
&=\Big(1-\sum_{m=2}^{n}d_m\Big)(1-\lambda_n)\\
&<1-\sum_{m=2}^{n}d_m,
\end{aligned}$$

so by Theorem 2.1 we deduce $F(z)\in\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$.

Conversely, suppose $F(z)\in\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$. By putting

$$\lambda_k=\frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n}d_m\big)}a_k \qquad (k\geqslant n+1)$$

we have $\lambda_k\geqslant 0$ and if we set $\lambda_n=1-\sum_{k=n+1}^{\infty}\lambda_k$, we reach

$$\begin{aligned}
F(z)&=z-\sum_{m=2}^{n}\frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))}d_m z^m\\
&-\sum_{k=n+1}^{\infty}\frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}(1-\sum_{m=2}^{n}d_m)\lambda_k z^k\\
&=f_n(z)-\sum_{k=n+1}^{\infty}\left(z-\sum_{m=2}^{n}\frac{(1-\beta)d_m}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))}z^m-f_k(z)\right)\lambda_k\\
&=f_n(z)-\sum_{k=n+1}^{\infty}\big(f_n(z)-f_k(z)\big)\lambda_k\\
&=\Big(1-\sum_{k=n+1}^{\infty}\lambda_k\Big)f_n(z)+\sum_{k=n+1}^{\infty}\lambda_k f_k(z)\\
&=\sum_{k=n}^{\infty}\lambda_k f_k(z).
\end{aligned}$$

Hence, the proof is complete. □

Theorem 3.2. Let $f(z)\in\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$. If

$$c_m=\frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))}d_m^2 \quad (2\leqslant m\leqslant n), \tag{3.1}$$

then the function G defined by

$$G(z)=z-\sum_{m=2}^{n}\frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))}c_m z^m-\sum_{k=n+1}^{\infty}a_k z^k$$

is also in $\mathcal{N}_q^{\mu}(\alpha,\beta,d_m)$.

Proof . Since $\Psi_{m-1}([m,q]\,(1+\alpha\,[m,q]-\alpha\beta)+\beta(1-\alpha))>1$, we get

$$c_m=\frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))}d_m^2$$

$$< d_m$$
$$\leqslant 1.$$

So, $0 \leqslant \sum_{m=2}^{n} c_m < \sum_{m=2}^{n} d_m \leqslant 1$ and hence

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} a_k \\
&< \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} d_m\big)} a_k \\
&< 1.
\end{aligned}$$

This completes the proof. □

Theorem 3.3. Let $f, g \in \mathcal{N}_q^{\mu}(\alpha, \beta, d_m)$. Then

$$\begin{aligned}
(f*g)(z) = z &- \sum_{m=2}^{n} \frac{(1-\beta)^2}{\big[\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))\big]^2} d_m^2 z^m \\
&- \sum_{k=n+1}^{\infty} a_k b_k z^k,
\end{aligned}$$

is in $\mathcal{N}_q^{\mu}(\alpha, \beta_0, c_m)$, where c_m $(2 \leqslant m < n)$ is defined by (3.1) and

$$\beta_0 \leqslant \frac{M - [k,q] - \alpha([k,q])^2}{M + \alpha([k,q]-1)+1},$$

$$M = \frac{\Psi_{k-1}}{1-\sum_{m=2}^{n} c_m} \left(\frac{[k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)}{1-\beta} \right)^2. \tag{3.2}$$

Proof . By (3.1) we have

$$\begin{aligned}
(f*g)(z) = z &- \sum_{m=2}^{n} \frac{(1-\beta)}{\Psi_{m-1}([m,q](1+\alpha[m,q]-\alpha\beta)+\beta(1-\alpha))} c_m z^m \\
&- \sum_{k=n+1}^{\infty} a_k b_k z^k.
\end{aligned}$$

By applying Theorem 3.2 we get

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} a_k < 1$$

and

$$\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} b_k < 1.$$

It is now sufficient to show that

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} a_k b_k \\
&\leq \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} \sqrt{a_k b_k} \\
&\leqslant 1.
\end{aligned}$$

We use the Cauchy-Schwartz inequality for this purpose and find the largest β_0 such that

$$
\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}\big([k,q](1+\alpha[k,q]-\alpha\beta_0\big)+\beta_0(1-\alpha)}{(1-\beta_0)\big(1-\sum_{m=2}^{n} c_m\big)} a_k b_k \\
&\leqslant \sum_{k=n+1}^{\infty} \frac{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}{(1-\beta)\big(1-\sum_{m=2}^{n} c_m\big)} \sqrt{a_k b_k} \\
&\leqslant 1
\end{aligned}
$$

or equivalently

$$
\sqrt{a_k b_k} \leqslant \frac{(1-\beta_0)\big([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha)\big)}{(1-\beta)\big([k,q](1+\alpha[k,q]-\alpha\beta_0)+\beta_0(1-\alpha)\big)}, \quad (k \geqslant n+1).
$$

This inequality holds when

$$
\frac{(1-\beta)}{\Psi_{k-1}([k,q](1+\alpha[k,q]-\alpha\beta)+\beta(1-\alpha))}\big(1-\sum_{m=2}^{n} c_m\big)
$$

$$
\leqslant \frac{(1-\beta_0)\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}{(1-\beta)\big([k,q](1+\alpha[k,q]-\alpha\beta_0)+\beta_0(1-\alpha)\big)}
$$

or equivalently

$$
\beta_0 \leqslant \frac{M-[k,q]-\alpha([k,q])^2}{M+\alpha([k,q]-1)+1},
$$

where M is given by (3.2). This completes the proof. □

In the forthcoming article, we verify the connection between the class defined in this article and bi-univalent functions. We consider Lucas polynomials [6] and Faber polynomial [14] and we discuss their characteristics through the q-analogue of the Noor integral operator.

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References

[1] B. Ahmad, M.G. Khan, B.A. Fasin, M.K. Aouf, T. Abdeljawad, W.K. Mashwani and M. Arif, *On q-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain*, AIMS Math. **6** (2021), 3037—3052.

[2] A. Akgül and B. Alçin, *Based on a family of bi-univalent functions introduced through the q-analogue of Noor integral operator*, Int. J. Open Prob. Compt. Math. **15** (2022), 20—33.

[3] A. Akgül and F.M. Sakar, *A certain subclass of bi-univalent analytic functions introduced by means of the q-analogue of Noor integral operator and Horadam polynomials*, Turk. J. Math. **43** (2019), 2275—2286.

[4] H. Aldweby and M. Darus, *q-analogue of Ruscheweyh operator and its applications to certain subclass of uniformly starlike functions*, AIP Conf. Proc. **1602** (2014), no. 1, 767–771.

[5] S.G. Ali Shah, S. Khan, S. Hussain and M. Darus, *q-Noor integral operator associated with starlike functions and q-conic domains*, AIMS Math. **7** (2022), 10842–10859.

[6] S. Altinkaya, *Inclusion properties of Lucas polynomials for bi-univalent functions introduced through the q-analogue of Noor integral operator*, Turk. J. Math. **43** (2019), 620—629.

[7] A. Aral, E. Deniz and H. Erbay, *The Picard and Gauss–Weierstrass Singular Integrals in (p, q)-Calculus*, Bull. Malays. Math. Sci. Soc. **43** (2020), 1569-–1583.

[8] M. Arif, M.U. Haq and J.L. Liu, *A subfamily of univalent functions associated with-analogue of Noor integral operator*, J. Funct. Biomater. **2018** (2018), 1—5.

[9] M. Arif, H.M. Srivastava, and S. Umar, *Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions*, RACSAM. **113** (2019), 1211—1221.

[10] P.L. Duren, *Univalent functions, Grundlehren der mathematischen Wissenschaften*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1993.

[11] B. Khan, H.M. Srivastava, M. Tahir, M. Darus, Q.Z. Ahmed and N. Khan, *Applications of a certain q- integral operator to the subclasses of analytic and bi-univalent functions*, AIMS Math. **6** (2020), 1024—1039.

[12] P. Long, J. Liu, M. Gangadharan and W. Wang, *Certain subclass of analytic functions based on q-derivative operator associated with the generalized Pascal snail and its applications*, AIMS Math. **7** (2022), 13423–13441.

[13] S. Mahmood and J. Sokoł, *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math. **71** (2017), 1-–13.

[14] F. Müge Sakar and A. Akgül, *Based on a family of bi-univalent functions introduced through the Faber polynomial expansions and Noor integral operator*, AIMS Math. **7** (2022), 5146–5155.

[15] S. Najafzadeh, *Some results on univalent holomorphic functions based on q-analogue of Noor operator*, Int. J. Appl. Math. **32** (2019), 775–784.

[16] S. Najafzadeh and D.O. Makinde, *Certain subfamily of harmonic functions related to Salagean q-differential operator*, Int. J. Anal. Appl. **18** (2020), 254–261.

[17] K.I. Noor, Ş. Altinkaya and S. Yalçin, *Coefficient inequalities of analytic functions equipped with conic domains involving q-analogue of Noor integral operator*, Tbilisi Math. J. **14** (2021), 1–14.

[18] K.I. Noor and M.A. Noor, *On integral operators*, J. Math. Anal. Appl. **238** (1999), 341–352.

[19] S.H. Sayedain Boroujeni, S. Najafzadeh, *Error function and certain subclasses of analytic univalent functions*, Sahand Commun. Math. Anal. **20** (2023), no. 1, 107–117.

[20] T.M. Seoudy and M.K. Aouf, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Ineq. **10** (2016), 135-–145.

[21] M. Tamer and E. Amnah, *Certain subclasses of spiral-like functions associated with q-analogue of Carlson-Shaffer operator*, AIMS Math. **6** (2021), 2525–2538.

[22] B. Wang and R. Srivastava and L.L. Jin, *Certain properties of multivalent analytic functions defined by q-difference operator involving the Janowski function*, AIMS Math. **6** (2021), 8497–8508.