

Common fixed point $(\alpha_*$ - ψ - β_i)-contractive set-valued mappings on orthogonal Branciari S_b -metric space

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Abstract

In [24], Khan et al. established some fixed point theorems in complete and compact metric spaces by using altering distance functions. In [16] Gordji et al. described the notion of orthogonal set and orthogonal metric spaces. In [18] Gungor et al. established fixed point theorems on orthogonal metric spaces via altering distance functions. In [25] Lotfy et al, introduced the notion of α_* - ψ -common rational type mappings on generalized metric spaces with application to fractional integral equations. In [28] K. Royy et al. described the notion of Branciari S_b -metric space and related fixed point theorems with an application. In this paper, we introduce the notion of the common fixed point $(\alpha_*$ - ψ - β_i)-contractive set-valued mappings on orthogonal Branciari S_b -metric space with the application of the existence of a unique solution to an initial value problem.

Keywords: $(\alpha_*$ - ψ - β_i)-contractive, Branciari S_b -metric space, Common fixed point, Solution to an initial value problem

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1 Introduction

We know, that the fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[34]). Harandi et al. [5] introduced the best proximity pairs for upper semi continuous set-valued maps in hyper convex metric spaces. Samet et al [30] introduced the notion of α - ψ -contractive type mappings. Hassanzadeh Asl et al. [19, 20] introduced the notion of common fixed point theorems for α_* - ψ -contractive multifunction. Farajzadeh et al. [13] introduced the on fixed point theorems for (ξ, α, η) -expansive mappings in complete metric spaces. Gungor et al, established fixed point theorems on orthogonal metric spaces via altering distance functions. Lotfy et al. [25] introduced the notion of α_* - ψ -common rational type mappings on generalized metric spaces with application to fractional integral equations. The aim of this paper is to introduce the notion common fixed point $(\alpha_*$ - ψ - β_i)-contractive set-valued mappings on orthogonal Branciari S_b -metric space with application the existence of a unique solution to an initial value problem.

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2 Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis. Let 2^X denote the family of all nonempty subsets of X .

Definition 2.1. ([24]) A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

- (ψ_1) $\psi(0) = 0$ and $\psi(t) > 0$ for all $t \in (0, +\infty)$;
 - (ψ_2) ψ is continuous and no-decreasing;
 - (ψ_3) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$;
 - (ψ_4) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;
- for all $t_1, t_2 \in (0, +\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$. We denote Ψ as the set of altering distance function ψ .

Definition 2.2. Let $X \neq \emptyset$ and $\perp \subseteq 2^X \times 2^X$ be a binary relation. If \perp satisfies the following condition

$$\exists A, B \subseteq X; (\exists y_0 \in B; \forall x \in A, x \perp y_0) \vee (\exists x_0 \in A; \forall y \in B, x_0 \perp y)$$

it is called (X, \perp) an orthogonal set.

Definition 2.3. [16] Let (X, \perp) be an orthogonal set. Any two subset $A, B \subseteq X$ are said to be orthogonally relation if $A \perp B \vee B \perp A$.

In the following, we give some examples of orthogonal sets.

Example 2.4. Let $X = \mathbb{Z}$, $A = \{x \in \mathbb{Z} / |x| \leq 2\}$ and $B = \{x \in \mathbb{Z} / x = 2k, k \in \mathbb{Z}\}$ define $A \perp B$ if there are $m \in A$, $k \in \mathbb{Z}$ and for all $n \in B$ such that $n = km$. It is easy to see that $A \perp B$. Hence (\mathbb{Z}, \perp) is an orthogonal set.

Example 2.5. Let $X = \mathbb{R}^2$, $A = \{(x, y) / y = ax, a \in \mathbb{R}\}$ and $B = \{(x, y) / x^2 + y^2 = r^2, r \in \mathbb{R}\}$ define $A \perp B$ if there are $(x_0, y_0) \in A$, for all $(x, y) \in B$ such that $y'_0 \times y' = -1$ or there are $(x_0, y_0) \in B$, for all $(x, y) \in A$ such that $y' \times y'_0 = -1$. It is easy to see that $A \perp B \wedge B \perp A$. Hence (\mathbb{R}^2, \perp) is an O -set.

The extended line is the ordered space $[-\infty; +\infty]$, considering of all points of the number line \mathbb{R} and two points, denoted by $-\infty, +\infty$ with the usual order relation for points of \mathbb{R} .

Definition 2.6. ([9, 16]) A map $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on the orthogonal set X, \perp . If the followig condition are satisfied, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y :

- (GMS1) $d(x, y) = 0$ if and if $x = y$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
- (GMS2) $d(x, y) = d(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
- (GMS3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for any points x, y, u and $v \in X$ such that $x \perp u, u \perp v, v \perp y$ and $x \perp y$ considering that if $d(x, u) = \infty$ or $d(u, v) = \infty$ or $d(v, y) = \infty$ then $d(x, u) + d(u, v) + d(v, y) = \infty$.

In this case the orthogonal set X is called generalized orthogonal metric space and is denoted by (X, d, \perp) .

In the abave definition, if d satisfies only GMS1 and GMS2, then it is called a semi-metric (see, e.g. [33]).

In (2012), Sedghi et al.[31] introduced a new type of metric structure consisting of three variables known as S-metric. Subsequently in the year (2016), N. Souayah and N. Mlaiki [32] investigated the notion of S_b -metric spaces which generalized the concept of S-metric spaces.

Definition 2.7. ([29, 31]) A map $S : X^3 \rightarrow [0, \infty)$ is called an S -metric on the orthogonal set (X, \perp) . If the following conditions are satisfied, for all $x, y, z, t \in X$ such that they are ortogonally to each other:

- (i) $S(x, y, z) = 0$ if and if $x = y = z$;
- (ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

In this case the orthogonal set (X, \perp) is called orthogonal S -metric space and is denoted by (X, S, \perp) .

Example 2.8. ([31]) (1) Let \mathbb{R} be the real line and $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X . Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X .

(2) Let \mathbb{R} be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on \mathbb{R} . This S-metric on \mathbb{R} is called the usual S-metric on \mathbb{R} .

Definition 2.9. ([27, 32]) A map $S_b : X^3 \rightarrow [0, \infty)$ is called an S_b -metric on the orthogonal set (X, \perp) . If the following conditions are satisfied, for all $x, y, z, t \in X$ and such that they are orthogonally to each other and let $s \geq 1$ be a given real number:

(i) $S_b(x, y, z) = 0$ if and if $x = y = z$;

(ii) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$.

In this case the orthogonal set (X, \perp) is called orthogonal S_b -metric space and is denoted by (X, S_b, \perp) .

Example 2.10. ([32]) Let X be a nonempty set and $\text{card}(X) \geq 5$. suppose $X = X_1 \cup X_2$ a partition of X such that $\text{card}(X_1) \geq 4$. Let $s \geq 1$, then

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

for all $x, y, z, t \in X$. Then S_b is an S_b -metric on X with coefficient s .

Definition 2.11. ([28]) A map $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ is called an Branciari S_b -metric on the orthogonal set (X, \perp) . If the following conditions are satisfied, for all $x, y, z \in X$ and for $a, b \in X \setminus \{x, y, z\}$ with $a \neq b$ and such that they are orthogonally to each other and let $k \geq 1$ be a given real number:

(i) $\lambda(x, y, z) = 0$ if and if $x = y = z$;

(ii)

$$\lambda(x, y, z) \leq k[\lambda(x, x, a) + \lambda(y, y, a) + \lambda(z, z, b) + \lambda(a, a, b)]. \quad (2.1)$$

In this case the orthogonal set (X, \perp) is called orthogonal Branciari S_b -metric space and is denoted by (X, λ, \perp) .

Definition 2.12. ([28]) An orthogonal Branciari S_b -metric on a nonempty set X is said to be symmetric if $\lambda(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$.

Proposition 2.13. ([28]) (i) Let (X, S, λ) be an orthogonal S -metric spaces (see definition (2.7)). The X is also an orthogonal Branciari S_b -metric space for $k = 2$.

(ii) Let (X, S_b, λ) be an orthogonal S_b -metric space with coefficient $s \geq 1$ (see definition (2.9)). The X is also an orthogonal Branciari S_b -metric space for $k = 2s^2$.

Proposition 2.14. ([28]) Shows that any orthogonal S -metric space or S_b -metric space is also an orthogonal Branciari S_b -metric space but there are several orthogonal Branciari S_b -metric spaces which are neither orthogonal S -metric spaces nor orthogonal S_b -metric spaces.

Example 2.15. ([28]) Let $X = \mathbb{N}$ and $\lambda : X^3 \rightarrow \mathbb{R}_0^+$ be defined by

$$\lambda(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

for all $x, y, z, t \in X$. Also we take $\lambda(x, x, y) = \lambda(y, y, x)$ for all $x, y \in X$. Then λ is a symmetric S_b -metric space on X for $k = \frac{5}{3}$ but it is neither an S -metric nor an S_b -metric for any $k \geq 1$.

Definition 2.16. ([28]) Let (X, λ, \perp) be an orthogonal Branciari S_b -metric space. Then

A sequence x_n in an orthogonal Branciari S_b -metric space (X, λ, \perp) is called orthogonal Branciari sequence if

$$(\forall n, k \in \mathbb{N}; x_n \perp x_{n+k}) \vee (\forall n, k \in \mathbb{N}; x_{n+k} \perp x_n)$$

(i) An orthogonal Branciari sequence $\{x_n\}$ in (X, λ, \perp) is said to be orthogonal Branciari convergent to some $z \in X$ if $\lambda(x_n, x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

(ii) An orthogonal Branciari sequence $\{x_n\}$ in (X, λ, \perp) is said to be orthogonal Branciari Cauchy if $\lambda(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

(iii) (X, λ, \perp) is said to be orthogonal Branciari complete if every orthogonal Branciari Cauchy sequence in (X, λ, \perp) is orthogonal Branciari convergent to some element in X .

Definition 2.17. We say that (X, λ, \perp) has the property α -regular orthogonal Branciari S_b -metric space if, either

(i) $\{x_n\}$ is a monotone orthogonal Branciari sequences in X such that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists an orthogonal Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k}, x) \geq 1$ for all k . Or

(ii) $\{x_n\}$ is a monotone orthogonal Branciari sequences in X such that $\alpha(x_{n+1}, x_{n+1}, x_n) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists an orthogonal Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x, x, x_{n_k}) \geq 1$ for all k .

Proposition 2.18. [23, 16] Suppose that $\{x_n\}$ is an orthogonal Branciari Cauchy sequence in a (X, λ, \perp) be a orthogonal Branciari S_b -metric space with $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, u) = 0$ where $u \in X$. Then

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, z) = \lambda(u, u, z)$$

for all $z \in X$. In particular, the orthogonal Branciari sequence $\{x_n\}$ dose not Branciari converge to z if $z \neq u$.

Definition 2.19. Let (X, λ, \perp) be an orthogonal Branciari S_b -metric space. A set-valued mapping $T : X \rightarrow 2^X$ is called orthogonal Branciari order closed if for monotone orthogonal Branciari sequences $x_n \in X$ and $y_n \in Tx_n$, with $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x) \rightarrow 0$ and $\lim_{n \rightarrow \infty} \lambda(y_n, y_n, y) \rightarrow 0$, implies $y \in Tx$.

Definition 2.20. Let (X, λ, \perp) be an orthogonal Branciari S_b -metric space and $T, S : X \rightarrow 2^X$ with given set-valued mappings, $\alpha : X \times X \times X \rightarrow [0, +\infty)$, $\alpha_* : 2^X \times 2^X \times 2^X \rightarrow [0, +\infty)$, $\alpha_*(A, A, B) = \inf\{\alpha(a, a, b) : a \in A, b \in B\}$, $\psi \in \Psi$, $\Lambda(s, s, Ts) = \inf\{\lambda(s, s, z)/z \in Ts\}$, H_λ is the Hausdorff metric

$$H_\lambda(Tx, Tx, Ty) = \max\{\sup_{a \in Tx} \Lambda(a, a, Ty), \sup_{b \in Ty} \Lambda(Tx, Tx, b)\}.$$

$\beta_i : \mathbb{R}^+ - \{0\} \rightarrow [0, 1)$ be four decreasing functions such that $\sum_{i=1}^4 \beta_i(t) \leq 1$ for every $t > 0$.

One says that T, S are α_* - ψ - β_i -orthogonal common contractive set-valued mappings whenever

$$\begin{aligned} \alpha_*(Ax, Ax, By)\psi(H_\lambda(Ax, Ax, By)) &\leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) \\ + \beta_2(\Lambda(x, x, Ax))\psi(\Lambda(x, x, Ax)) &+ \beta_3(\lambda(y, y, By))\psi(\Lambda(y, y, By)) \\ + \beta_4(H_\lambda(Ax, Ax, By)) \min\{\psi(\Lambda(x, x, By)), \psi(\Lambda(y, y, Ax))\}. \end{aligned} \quad (2.2)$$

One says that A, B are an α_* - common admissible if

$$\alpha(x, x, y) \geq 1 \Rightarrow \alpha_*(Ax, Ax, By) \geq 1 \quad (2.3)$$

$A, B = T$ or S , $Ax \perp By \vee By \perp Ax$ for all $x, y \in X$ where $x \perp y$ and $x \neq y$.

One says that a mapping $A, B : X \rightarrow 2^X$ is called common orthogonal preserving (\perp -preserving) if $A(x) \perp B(y) \vee A(y) \perp B(x)$ if $x \perp y$.

Example 2.21. ([28]) Let $X = [0, 1)$ and let the metric on X be the Euclidian metric. Define $x \perp y$ if $xy \leq \{\frac{x}{6}, \frac{y}{6}\}$. X is not complete but it is orthogonal complete. Let $x \perp y$ and $xy \leq \frac{x}{6}$. If x_k is an arbitrary Cauchy orthogonal sequence in X , then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0 \vee x_{k_n} \leq \frac{1}{6}$ for all $n \in \mathbb{N}$. it follows that $\{x_{k_n}\}$ converges to a $x \in [0, 1)$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. Let $T, S : X \rightarrow 2^X$ be set-valued mapping defined by

$$Tx = \begin{cases} [0, \frac{x}{3}] & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} < x < 1 \end{cases} \quad \text{and} \quad Sx = \begin{cases} [0, \frac{x}{2}] & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

Also, $x \perp y$ and $xy \leq \frac{x}{6}$, so $x = 0$ or $y \leq \frac{1}{6}$. We have the following cases:

- case(1) $x = 0$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = \{0\}$ and $Sy = [0, \frac{y}{2}]$;
- case(2) $x = 0$ and $\frac{1}{6} < y \leq \frac{1}{2}$, then $Tx = \{0\}$ and $Sy = [0, \frac{y}{2}]$;
- case(3) $x = 0$ and $\frac{1}{2} < y$, then $Tx = \{0\}$ and $Sy = \{0\}$;
- case(4) $0 \leq x \leq \frac{1}{6}$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
- case(5) $\frac{1}{6} < x \leq \frac{1}{3}$ and $0 \leq y \leq \frac{1}{6}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
- case(6) $\frac{1}{6} < x \leq \frac{1}{3}$ and $\frac{1}{6} < y \leq \frac{1}{2}$, then $Tx = [0, \frac{x}{3}]$ and $Sy = [0, \frac{y}{2}]$;
- case(7) $\frac{1}{3} < x$ and $\frac{1}{2} < y$, then $Tx = \{0\}$ and $Sy = \{0\}$.

These cases implies that $TxSy \leq \frac{Tx}{6}$. Hence T and S are common \perp -preserving. Also, one can see that $\|Tx - Sy\| \leq \frac{1}{2}\|x - y\|$. Hence T, S are common \perp -contraction.

Definition 2.22. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that $\Lambda(B, B, y) = \lambda(z, z, y)$.

Definition 2.23. A set-valued mapping $T : X \rightarrow 2^X$ is said to have an approximate values in X if Tx is an approximation for each $x \in X$.

Definition 2.24. Let (X, \perp, λ) be an orthogonal Branciari S_b -metric space. If $T : X \rightarrow 2^X$ is a set-valued mapping, then $x \in X$ is called fixed point for T if and only if $x \in F(x)$. The set $Fix(T) := \{x \in X/x \in Tx\}$ is called the fixed point set of T .

3 Main result

We should emphasize that throughout this paper we suppose that all set-valued mappings on an orthogonal symmetric S_b -metric space (X, λ, \perp) have closed values.

Lemma 3.1. Let (X, λ, \perp) be an orthogonal symmetric Branciari S_b -metric space. Suppose that $T, S : X \rightarrow 2^X$ are $\alpha_*\text{-}\psi\text{-}\beta_i$ -orthogonal common contractive set-valued mappings satisfies the following conditions:

- (i) T, S are α_* -orthogonal common admissible;
- (ii) there exists $x_0 \in X$ such that,

$$\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0$$

Then $Fix(T) = Fix(S)$.

Proof . We first show that any fixed point of T is also a fixed point of S and conversely. Since $Fix(T) \neq Fix(S)$, we may assume there exists $x^* \in X$ such that $x^* \in Fix(T)$, but $x^* \notin Fix(S)$, since $\Lambda(x^*, x^*, Sx^*) > 0$. Let $x_0 \in X$ such that $\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0$. Define the orthogonal Branciari sequence $\{x_n\}$ in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a common fixed point for T, S . So, we can assume that $x_{2n} \notin Tx_{2n}$ and $x_{2n+1} \notin Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. Define

$$\alpha(x, x, y) = \begin{cases} 1 & x \perp y \vee y \perp x \\ 0 & otherwise \end{cases}$$

Since T, S are α_* -orthogonal common admissible and

$$\{x_0\} \perp Tx_0 \Rightarrow \alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1,$$

we have

$$\begin{aligned} \alpha(x_0, x_0, x_1) &\geq \alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \alpha_*(Tx_0, Tx_0, Sx_1) \geq 1; \\ \alpha(x_1, x_1, x_2) &\geq \alpha_*(Tx_0, Tx_0, Sx_1) \geq 1 \Rightarrow \alpha_*(Sx_1, Sx_1, Tx_2) \geq 1; \\ \alpha(x_2, x_2, x_3) &\geq \alpha_*(Sx_1, Sx_1, Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_2, Tx_2, Sx_3) \geq 1. \end{aligned}$$

Inductively, we have

$$\alpha(x_{2n}, x_{2n}, x_{2n+1}) \geq 1 \Rightarrow \alpha_*(Tx_{2n}, Tx_{2n}, Sx_{2n+1}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+1}, x_{2n+2}) \geq 1 \Rightarrow \alpha_*(Sx_{2n+1}, Sx_{2n+1}, Tx_{2n+2}) \geq 1$$

for all $n \in \mathbb{N}_0$. Let

$$\{x_0\} \perp STx_0 \Rightarrow \alpha_*(\{x_0\}, \{x_0\}, STx_0) \geq 1$$

Similarly, we have

$$\alpha(x_{2n}, x_{2n}, x_{2n+2}) \geq 1 \Rightarrow \alpha_*(Tx_{2n}, Tx_{2n}, STx_{2n}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+1}, x_{2n+3}) \geq 1 \Rightarrow \alpha_*(Sx_{2n+1}, Sx_{2n+1}, TSx_{2n+1}) \geq 1$$

for all $n \in \mathbb{N}_0$. We obtain

$$\begin{aligned} \psi(\Lambda(x^*, x^*, Sx^*)) &\leq \psi(H_\lambda(Tx^*, Tx^*, Sx^*)) \leq \alpha_*(Tx^*, Tx^*, Sx^*)\psi(H_\lambda(Tx^*, Tx^*, Sx^*)) \\ &\leq \beta_1(\lambda(x^*, x^*, x^*))\psi(\lambda(x^*, x^*, x^*)) + \beta_2(\Lambda(x^*, x^*, Tx^*))\psi(\Lambda(x^*, x^*, Tx^*)) + \\ &\beta_3(\Lambda(Sx^*, Sx^*, x^*))\psi(\Lambda(Sx^*, Sx^*, x^*)) + \beta_4(H_\lambda(Tx^*, Tx^*, Sx^*)) \min\{\psi(\Lambda(x^*, x^*, Sx^*)), \\ &\psi(\Lambda(x^*, x^*, Tx^*))\} = \beta_3(\Lambda(Sx^*, Sx^*, x^*))\psi(\Lambda(Sx^*, Sx^*, x^*)) < \psi(\Lambda(Sx^*, Sx^*, x^*)) \\ &\stackrel{\text{Symmetric}}{=} \psi(\Lambda(x^*, x^*, Sx^*)) \end{aligned}$$

This is contradiction establishes that $Fix(T) \subseteq Fix(S)$. A similar argument establishes the reverse containment, and therefore $Fix(T) = Fix(S)$. \square

Theorem 3.2. Let (X, λ, \perp) be a complete orthogonal symmetric Branciari S_b -metric space (not necessarily complete metric space). Suppose that $T, S : X \rightarrow 2^X$ are α_* - ψ - β_i -orthogonal common contractive set-valued mappings satisfies the following conditions:

- (i) T, S are α_* -orthogonal common admissible;
- (ii) there exists $x_0 \in X$ such that,

$$\{x_0\} \perp Tx_0 \vee \{x_0\} \perp STx_0$$

- (iii) X has the property α -regular orthogonal Branciari S_b -metric space,
- (iv) T, S are \perp -preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated orthogonal Branciari sequences $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S .

Proof . By lemma (3.1), we have $Fix(T) = Fix(S)$ and we have

$$\begin{aligned} \alpha(x_n, x_n, x_{n+1}) &\geq 1 \vee \alpha(x_n, x_n, x_{n+2}) \geq 1; \\ \{x_0\} \perp Tx_0 \perp STx_0 \cdots \vee \{x_0\} \perp STx_0 \perp TSTx_0 \cdots ; \\ x_0 \perp x_1 \perp x_2 \cdots \vee x_0 \perp x_2 \perp x_3 \cdots ; \end{aligned}$$

Thus $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}_0$. Without loss of generality, we may assume that $T, S : X \rightarrow 2^X$ are α_* - ψ - β_i -orthogonal common contractive set-valued mappings. Consider equation (2.2), with $x = x_{2n+1}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned} \psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) &\leq \alpha_*(Tx_{2n}, Tx_{2n}, Sx_{2n+1})\psi(H_\lambda(Tx_{2n}, Tx_{2n}, Sx_{2n+1})) \\ &\leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &+ \beta_2(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n})) \\ &+ \beta_3(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1}))\psi(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1})) \\ &\beta_4(H_\lambda(Tx_{2n}, Tx_{2n}, Sx_{2n+1})) \min\{\psi(\Lambda(x_{2n}, x_{2n}, Sx_{2n+1})), \psi(\Lambda(x_{2n+1}, x_{2n+1}, Tx_{2n}))\} \\ &\leq \beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &+ \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &+ \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \\ &\beta_4(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \min\{\psi(\lambda(x_{2n}, x_{2n}, x_{2n+2})), \psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+1}))\}. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} (1 - \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \\ \leq (\beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1})))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \end{aligned} \tag{3.2}$$

and

$$\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \leq \frac{(\beta_1(\lambda(x_{2n}, x_{2n}, x_{2n+1})) + \beta_2(\lambda(x_{2n}, x_{2n}, x_{2n+1})))}{(1 - \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})))} \psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \quad (3.3)$$

Thus

$$\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \leq \psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})). \quad (3.4)$$

Similarly,

$$\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \leq \psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n})), \quad (3.5)$$

for all $n \in \mathbb{N}_0$.

We have

$$\psi(\lambda(x_{n+1}, x_{n+1}, x_{n+2})) \leq \psi(\lambda(x_n, x_n, x_{n+1})) \leq \dots \leq \psi^n(\lambda(x_0, x_0, x_1)), \quad (3.6)$$

for all $n \in \mathbb{N}$. From the property of ψ , we conclude that

$$\lambda(x_n, x_n, x_{n+1}) < \lambda(x_{n-1}, x_{n-1}, x_n), \quad (3.7)$$

for all $n \in \mathbb{N}$, it is clear that

$$\lim_{n \rightarrow \infty} \lambda(x_{n+1}, x_{n+1}, x_{n+2}) = 0. \quad (3.8)$$

Consider equation (2.2), with $x = x_{2n}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned} \psi(\lambda(x_{2n}, x_{2n}, x_{2n+2})) &\leq \alpha_*(Sx_{2n-1}, Sx_{2n-1}, Sx_{2n+1})\psi(H_\lambda(Sx_{2n-1}, Sx_{2n-1}, Sx_{2n+1})) \\ &\leq \beta_1(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1}))\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) \\ &\quad + \beta_2(\Lambda(x_{2n-1}, x_{2n-1}, Sx_{2n-1}))\psi(\Lambda(x_{2n-1}, x_{2n-1}, Sx_{2n-1})) \\ &\quad + \beta_3(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1}))\psi(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n+1})) \\ &\quad + \beta_4(H_\lambda(Sx_{2n-1}, Sx_{2n-1}, Sx_{2n+1})) \min\{\psi(\Lambda(x_{2n-1}, x_{2n-1}, Sx_{2n+1})), \psi(\Lambda(x_{2n+1}, x_{2n+1}, Sx_{2n-1}))\} \\ &\leq \beta_1(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1}))\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) \\ &\quad + \beta_2(\lambda(x_{2n-1}, x_{2n-1}, x_{2n}))\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n})) \\ &\quad + \beta_3(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2}))\psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n+2})) \\ &\quad + \beta_4(\lambda(x_{2n}, x_{2n}, x_{2n+2})) \min\{\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+2})), \psi(\lambda(x_{2n+1}, x_{2n+1}, x_{2n}))\}. \end{aligned} \quad (3.9)$$

Similarly, Consider equation (2.2), with $x = x_{2n-1}$ and $y = x_{2n+1}$. Clearly, we have

$$\begin{aligned} \psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) &\leq \alpha_*(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})\psi(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})) \\ &\leq \beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n})) \\ &\quad + \beta_2(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2}))\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2})) \\ &\quad + \beta_3(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n})) \\ &\quad + \beta_4(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})) \min\{\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n})), \psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n-2}))\} \\ &\leq \beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n})) \\ &\quad + \beta_2(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1})) \\ &\quad + \beta_3(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1})) \\ &\quad + \beta_4(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) \min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1})), \psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n}))\}. \end{aligned}$$

Define $a_{2n} = \lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})$ and $b_{2n} = \lambda(x_{2n}, x_{2n}, x_{2n+1})$. Then

$$\begin{aligned} \psi(a_{2n}) &\leq \beta_1(a_{2n-1})\psi(a_{2n-1}) + \beta_2(b_{2n-1})\psi(b_{2n-1}) + \beta_3(b_{2n})\psi(b_{2n}) + \\ &\quad \beta_4(a_{2n}) \min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1})), \psi(b_{2n-1})\}. \end{aligned} \quad (3.10)$$

From the (3.8) $\lim_{n \rightarrow \infty} b_{2n} = \lim_{n \rightarrow \infty} \lambda(x_{2n}, x_{2n}, x_{2n+1}) = 0$. We get

$$\psi(a_{2n}) \leq \beta_1(a_{2n-1})\psi(a_{2n-1}) \leq \psi(a_{2n-1}) \quad (3.11)$$

and hence,

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} \lambda(x_{2n-1}, x_{2n-1}, x_{2n+1}) = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \lambda(x_{n-1}, x_{n-1}, x_{n+1}) = 0.$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $\lambda(x_p, x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1, m = 2k$

and $n = 2l$ for $k, l \in \mathbb{N}$. Substitute again $x = x_{2l} = x_{2k}$ and $y = x_{2l+1} = x_{2k+1}$ in (2.2), (3.7) which yields

$$\begin{aligned}
\psi(\lambda(x_{2l}, x_{2l}, x_{2l+1})) &= \psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \leq \alpha_*(H_\lambda(Sx_{2k-1}, Sx_{2k-1}, Tx_{2k}))\psi(H(Sx_{2k-1}, Sx_{2k-1}, Tx_{2k})) \\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_2(\Lambda(x_{2k-1}, x_{2k-1}, Sx_{2k-1}))\psi(\Lambda(x_{2k-1}, x_{2k-1}, Sx_{2k-1})) \\
&\quad + \beta_3(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k})) \\
&\quad + \beta_4(H_\lambda(Tx_{2k}, Tx_{2k}, Sx_{2k-1})) \min\{\psi(\Lambda(x_{2k}, x_{2k}, Sx_{2k-1})), \psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k}))\} \\
&\leq \beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \\
&\quad + \beta_4(\lambda(x_{2k+1}, x_{2k+1}, x_{2k})) \min\{\psi(\lambda(x_{2k}, x_{2k}, x_{2k})), \psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k+1}))\} \\
&= (\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) \\
&\quad + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) \\
&\leq (\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))) \\
&\quad + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1})) < \psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))
\end{aligned} \tag{3.12}$$

which is impossible. Now, we shall prove that $\{x_n\}$ is an orthogonal Branciari Cauchy sequence, that is,

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x_{n+k}) = 0 \text{ and } x_n \perp x_{n+k}$$

for all $k \in \mathbb{N}$. We have already proved the cases for $k = 1$ and $k = 2$ in (3.7) and (3.10), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case I: Suppose that $S_n = \lambda(x_n, x_n, x_{n+1})$, $\psi(S_n) = \alpha_n S_n$ and $\alpha_n \in (0, \frac{1}{\sqrt{k}})$. Then

$$\begin{aligned}
S_n &= \lambda(x_n, x_n, x_{n+1}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_n)) = \alpha_{n-1} \lambda(x_{n-1}, x_{n-1}, x_n) \\
&\leq \alpha_{n-1} \psi(\lambda(x_{n-2}, x_{n-2}, x_{n-1})) \leq \dots \leq \alpha_{n-1} \alpha_{n-2} \dots \alpha_1 \alpha_0 \lambda(x_0, x_0, x_1) = \alpha^n S_0
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
S_n^* &= \lambda(x_n, x_n, x_{n+2}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \alpha_{n-1} \lambda(x_{n-1}, x_{n-1}, x_{n+1}) \\
&\leq \alpha_{n-1} \psi(\lambda(x_{n-2}, x_{n-2}, x_n)) \leq \dots \leq \alpha_{n-1} \alpha_{n-2} \dots \alpha_1 \alpha_0 \lambda(x_0, x_0, x_1) = \alpha^n S_0^*
\end{aligned} \tag{3.14}$$

for all $n \geq 1$ and $\alpha = \max_{0 \leq i \leq n-1} \{\alpha_i\}$. Now, we shall prove that $\{x_n\}$ is a orthogonal Branciari Cauchy sequence, that is,

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x_{n+l}) = 0,$$

for all $l \in \mathbb{N}$. We have already proved the cases for $l = 1$ and $l = 2$ in (3.7) and (3.10), respectively. Now for $l = 2m + 1$, where $m \geq 1$. Using the inequality (2.1), we have

$$\begin{aligned}
\lambda(x_n, x_n, x_{n+l}) &\leq k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+l}, x_{n+l}, x_{n+2}) \\
&\quad + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] \\
&= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+l}, x_{n+l}, x_{n+2}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\stackrel{Symmetric}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k\lambda(x_{n+2}, x_{n+2}, x_{n+1}) \\
&\leq 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k(k[\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + \lambda(x_{n+2}, x_{n+2}, x_{n+3}) + \lambda(x_{n+l}, x_{n+l}, x_{n+4}) + \lambda(x_{n+3}, x_{n+3}, x_{n+4})]) \\
&\stackrel{Symmetric}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + 2k^2\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + k^2\lambda(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\lambda(x_{n+4}, x_{n+4}, x_{n+2m+1}) \\
&\leq \dots \\
&\vdots \\
&\leq 2k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] + 2k^2[\lambda(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + \lambda(x_{n+3}, x_{n+3}, x_{n+4})] \\
&\quad + \dots + 2k^m[\lambda(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m-1}) + \lambda(x_{n+2m-1}, x_{n+2m-1}, x_{n+2m})] \\
&\quad + k^m\lambda(x_{n+2m}, x_{n+2m}, x_{n+2m+1}) \\
&\leq 2\{k(\alpha_0^n + \alpha_0^{n+1}) + k^2(\alpha_0^{n+2} + \alpha_0^{n+3}) + \dots + k^m(\alpha_0^{n+2m-2} + \alpha_0^{n+2m-1})\} \\
&\quad + k^m\alpha_0^{n+2m} S_0 = 2k(1 + \alpha_0)\alpha_0^n [1 + k\alpha_0^2 + \dots + k^m\alpha_0^{2m}] S_0 = \frac{2k(1+\alpha_0)}{1+k\alpha_0^2} \alpha_0^n S_0
\end{aligned} \tag{3.15}$$

for all $n \geq 1$. Also for $l = 2m$ we get

$$\lambda(x_n, x_n, x_{n+2m}) \leq \dots \leq \frac{2k(1+\alpha_0)}{1+k\alpha_0^2} \alpha_0^n S_0 + \alpha_0^n (k\alpha_0^2)^{m-1} S_0^* \tag{3.16}$$

for all $n \geq 1$. Thus we proved that $\{x_n\}$ is a orthogonal Branciari Cauchy sequence in the complete metric space (X, λ, \perp) , there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x^*) = 0$ by (X, λ, \perp) has the property α -regular Branciari S_b -metric space. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(\{x_{2n_k+1}\}, \{x_{2n_k+1}\}, \{x^*\}) \geq \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*) \geq 1 \text{ for all } k. \quad (3.17)$$

Thus

$$\begin{aligned} \psi(\Lambda(x^*, x^*, Tx^*)) &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \psi(\Lambda(x_{2n_k+1}, x_{2n_k+1}, Tx^*)) \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*)\psi(H_\lambda(Tx_{2n_k}, Tx_{2n_k}, Tx^*)) \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\ &\quad + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx_{2n_k})) \\ &\quad + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*)), \psi(\Lambda(x^*, x^*, Tx_{2n_k}))\} \\ &\leq \psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \\ &\quad + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1})) \\ &\quad + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*)), \psi(\lambda(x^*, x^*, x_{2n_k+1}))\} \\ &\leq \psi(0) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(0) + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(0) \\ &\quad + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \\ &\quad + \beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*)) \min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*)), \psi(0)\} \\ &\leq \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*)) \leq \psi(\Lambda(x^*, x^*, Tx^*)) \end{aligned} \quad (3.18)$$

for all k , which is impossible. Hence, $\Lambda(x^*, x^*, Tx^*) = \Lambda(Tx^*, Tx^*, x^*) = 0$ and so $x^* \in Tx^*$.

By Lemma (3.1) we have x^* common fixed point of T, S . \square

Corollary 3.3. [24] Let (X, λ, \perp) be an orthogonal symmetric Branciari complete metric space (not necessarily complete metric space), $f, g : X \rightarrow X$ be a self map, $\psi \in \Psi$ be a sub-additive function and $\alpha, \beta, \gamma : \mathbb{R}^+ - \{0\} \rightarrow [0, 1]$ be three decreasing functions such that $(\alpha + 2\beta + \gamma)(t) < 1$ for all $t > 0$. Suppose that f is \perp -preserving self mapping satisfying the inequality

$$\begin{aligned} \psi(\lambda(fx, fx, gy)) &\leq \alpha(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta(\lambda(x, x, y))[\psi(\lambda(x, x, fx)) \\ &\quad + \psi(\lambda(y, y, gy))] + \gamma(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, gy)), \psi(\Lambda(y, y, fx))\}, \end{aligned} \quad (3.19)$$

for all $x, y \in X$ where $x \perp y$ and $x \neq y$. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n x_0\}$ converges to this point. Also, if f is \perp -continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f .

Example 3.4. Let $X = \mathbb{Z}$, $A = \{x \in \mathbb{Z} \mid |x| \leq 2\}$ and $B = \{x \in \mathbb{Z} \mid x = 2k, k \in \mathbb{N}\}$ define $A \perp B$ if there are $m \in A$, $k \in \mathbb{Z}$ and for all $n \in B$ such that $n = km$. It is easy to see that $A \perp B$. Hence (\mathbb{Z}, \perp) is an O -set.

Let $Y \subseteq X$ be a finite set defined as $Y = \{1, 2, 4, 8\}$. Define $\lambda : Y \times Y \times Y \rightarrow [0, \infty)$ as:

$$\lambda(1, 1, 1) = \lambda(2, 2, 2) = \lambda(4, 4, 4) = \lambda(8, 8, 8) = 0,$$

$$\lambda(1, 1, 2) = \lambda(2, 2, 1) = 3,$$

$$\lambda(2, 2, 8) = \lambda(8, 8, 2) = \lambda(1, 1, 8) = \lambda(8, 8, 1) = 1 \text{ and}$$

$$\lambda(1, 1, 4) = \lambda(4, 4, 1) = \lambda(2, 2, 4) = \lambda(4, 4, 2) = \lambda(8, 8, 4) = \lambda(4, 4, 8) = \frac{1}{2}.$$

The function λ is not a metric on Y . Indeed, note

$$3 = \lambda(1, 1, 2) \geq \lambda(1, 1, 8) + \lambda(8, 8, 2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, λ is a symmetric Branciari S_b -metric on Y and moreover (Y, λ) is a complete symmetric Branciari S_b -metric space. Define $T, S : Y \rightarrow 2^Y$ as: $T1 = T2 = T8 = \{2, 4\}$, $T4 = \{1, 8\}$ and $S1 = S2 = S4 = \{2, 8\}$, $S8 = \{1, 2\}$, $\alpha : Y \times Y \times Y \rightarrow [0, +\infty)$, $\alpha_* = \inf \alpha$ as

$$\alpha(x, x, y) = \begin{cases} 1 & x \perp y \vee y \perp x \\ 0 & \text{otherwise} \end{cases}$$

$\psi(t) = \frac{2}{3}t$. Clearly, T, S satisfies the conditions of Theorem (3.2) and has a common fixed point $x = 2$.

4 Some consequences

In this section we give some consequences of the main results presented above. Specifically, we apply our results to generalized metric spaces endowed with a partial order.

4.1 Fixed point theorems for weakly increasing on X has the property α -regular orthogonal symmetric Branciari complete metric space

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([14]). Let X be a topological space and \preceq be a partial order on X .

Definition 4.1. ([14]). Let A, B be two nonempty subsets of X , the relations between A and B are definers follows:

- (r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
- (r_2) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
- (r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 4.2. ([11], [12]). Let (X, \preceq) be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing.

Example 4.3. Let $X = \mathbb{R}^+$ endowed with usual ordering. Let $f, g : X \rightarrow X$ defined by

$$fx = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x < \infty \end{cases} \quad \text{and} \quad gx = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x < \infty \end{cases}$$

then it is obvious that $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$. Thus f and g are weakly increasing mappings. Note that both f and g are not nondecreasing.

Definition 4.4. ([3]) Let (X, \preceq) be a partially ordered set. Two mapping $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to \prec_1 if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to \prec_2 if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Now we give some examples.

Example 4.5. ([3]) Let $X = [1, \infty)$ and \leq be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to \prec_2 but not \prec_1 . Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to \prec_2 but $F2 = [1, 4] \succ_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to \prec_1 .

Example 4.6. ([3]) Let $X = [1, \infty)$ and \leq be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = [0, 1]$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to \prec_1 but not \prec_2 . Indeed, since

$$Fx = [0, 1] \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 [0, 1] = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to \prec_1 but $G1 = 1 \succ_2 0, 1 = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to \prec_2 .

Theorem 4.7. Let $(X, \preceq, \perp, \lambda)$ be a partially ordered orthogonal symmetric Branciari complete metric space (not necessarily complete metric space). Suppose that $T, S : X \rightarrow 2^X$ are α_* - ψ - β_i -orthogonal common contractive set-valued mappings for all $x, y \in X$ with $x \prec_1 y$ or $x \perp y$ satisfies the following conditions:

- (i) T and S be a weakly increasing pair on X w.r.t \prec_1 ;
- (ii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 STx_0$ or $\{x_0\} \perp Tx_0$ and $\{x_0\} \perp STx_0$;
- (iii) X has the property α -regular orthogonal symmetric Branciari complete metric space,
- (iv) T, S are \perp -preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated O -sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S .

Proof . Define the orthogonal sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for T, S . Using that the pair of set-valued mappings T and S is weakly increasing and by define $\alpha : X \times X \times X \rightarrow [o, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1, & x \preceq y \vee x \perp y \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily shown that the orthogonal sequence x_n is nondecreasing w.r.t, \preceq i.e; and $\alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0$, such that $\alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \preceq x_1 \vee x_0 \perp x_1$. Now since T and S are weakly increasing with respect to \prec_1 , we have $x_1 \in Tx_0 \prec_1 Sx_1$. Thus there exist some $x_2 \in Sx_1$ such that $x_1 \preceq x_2 \vee x_1 \perp x_2$. Again since T and S are weakly increasing with respect to \prec_1 , we have $x_2 \in Sx_1 \prec_1 Tx_2$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \preceq x_3 \vee x_2 \perp x_3$. Continue this process, we will get a nondecreasing orthogonal sequence $\{x_n\}_{n=1}^\infty$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, 3, \dots$ We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \dots \preceq x_{2n} \preceq x_{2n+1} \preceq x_{2n+2} \preceq \dots$$

\vee

$$x_0 \perp x_1 \perp x_2 \perp \dots \perp x_{2n} \perp x_{2n+1} \perp x_{2n+2} \perp \dots$$

In particular x_n, x_{n+k} are comparable for all $k \in \mathbb{N}$. $\alpha(x_n, x_n, x_{n+k}) \geq 1$ for all $n \in \mathbb{N}_0$ and by (4) we have $\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x_{n+k}) = 0$. Following the proof of Theorem (3.2). Thus we proved that $\{x_n\}$ is a orthogonal Cauchy sequence in the orthogonal symmetric Branciari complete metric space (X, \perp, λ) , there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x^*) = 0$$

and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then x^* is a common fixed point of T, S . \square

Theorem 4.8. Let $(X, \preceq, \perp, \lambda)$ be a partially ordered orthogonal symmetric Branciari complete metric space (not necessarily complete metric space). Suppose that $T, S : X \rightarrow 2^X$ are α_* - ψ - β_i -orthogonal common contractive set-valued mappings for all $x, y \in X$ with $x \prec_2 y$ or $x \perp y$ satisfies the following conditions:

- (i) T and S be a weakly increasing pair on X w.r.t \prec_2 ;
- (ii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $STx_0 \prec_2 \{x_0\}$ or $Tx_0 \perp \{x_0\}$ and $STx_0 \perp \{x_0\}$;
- (iii) X has the property α -regular orthogonal symmetric Branciari complete metric space,
- (iv) T, S are \perp -preserving set-valued mappings.

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated orthogonal sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S .

Proof . Define the orthogonal sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for T, S . Using that the pair of set-valued mappings T and S is weakly increasing and by define

$$\alpha(x, x, y) = \begin{cases} 1, & x \succeq y \vee x \perp y \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily shown that the sequence x_n is non increasing w.r.t, \preceq i.e; and

$$\alpha_*(x_0, x_0, \{Tx_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \succeq x_1;$$

Now since T and S are weakly increasing with respect to \prec_2 , we have $Sx_1 \prec_2 Tx_0$. Thus there exist some $x_2 \in Sx_1$ such that $x_1 \succeq x_2$. Again since T and S are weakly increasing with respect to \prec_2 , we have $Tx_2 \preceq_2 Sx_1$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \succeq x_3$. Continue this process, we will get a non increasing sequence $\{x_n\}_{n=1}^\infty$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, 3, \dots$ We get

$$x_0 \succeq x_1 \succeq x_2 \succeq \dots \succeq x_{2n} \succeq x_{2n+1} \succeq x_{2n+2} \succeq \dots$$

\vee

$$x_0 \perp x_1 \perp x_2 \perp \dots \perp x_{2n} \perp x_{2n+1} \perp x_{2n+2} \perp \dots$$

In particular x_{n+k}, x_n are comparable for all $k \in \mathbb{N}$, $\alpha(x_{n+k}, x_{n+k}, x_n) \geq 1$ for all $n \in \mathbb{N}_0$ and by (4) we have $\lim_{n \rightarrow \infty} \lambda(x_{n+k}, x_{n+k}, x_n) = 0$. Following the proof of Theorem (3.2), thus we proved that $\{x_n\}$ is a orthogonal Cauchy sequence in the orthogonal complete metric space (X, \perp, d) , there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} \lambda(x_n, x_n, x^*) = 0$$

and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then x^* is a common fixed point of T, S . \square

4.2 Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to the our common fixed point theorem.

Let us consider Cauchy problem for the first order differential equations system

$$\begin{cases} x' = f(t, x(t), y(t)), & t \in R, \quad x(0) = x_0 \\ y' = g(t, y(t), x(t)), & t \in R, \quad y(0) = y_0 \end{cases} \quad (4.1)$$

Theorem 4.9. Given a point $(t_0, x_0, y_0) \in R \times R^n \times R^n$ and consider the differential equations system (4.1). Let P be a Picard mapping defined by

$$\begin{cases} (Px)(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau), y(\tau))d\tau \\ (Py)(t) = y_0 + \int_{t_0}^t g(\tau, y(\tau), x(\tau))d\tau \end{cases} \quad (4.2)$$

Note that $(Px)(t_0) = x_0$ and $(Py)(t_0) = y_0$ for any x, y . The mappings $x, y : I \rightarrow R^n$ are a solution to the differential equations system (4.1) with the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$ if and only if $x = Px$ and $y = Py$.

where the functions $f, g : I \times R \times R \rightarrow R$ is defined in the domain $D = \{(t, x, y); |t - t_0| \leq a, |x - x_0| \leq b, |y - y_0| \leq c\}$, $x_0, y_0 \in R$. and satisfied the condition

$$|f(t, x_1, y_1) - g(t, x_2, y_2)| \leq \frac{K}{2|t - t_0|} (|x_1 - x_2| + |y_1 - y_2|), \quad 0 < K < 1. \quad (4.3)$$

Let $M = \max_{(t, x(t), y(t)) \in D} \{|f(t, x(t), y(t))|, |g(t, x(t), y(t))|\}$. There exists $d = \min\{a, \frac{b}{M}, \frac{c}{M}\}$ such that

$$D_0 = \{(t, x, y)/|t - t_0| \leq d, |x - x_0| \leq M|t - t_0|, |y - y_0| \leq M|t - t_0|\}, \quad (4.4)$$

lies in D . We are trying to find a solution $\varphi(t, x, y)$ and $\varphi(t, y, x)$ for the differential equations system (4.1) with initial condition $\varphi(t_0, x, y) = x_0$ and $\varphi(t_0, y, x) = y_0$ expressed in the form $\varphi(t, x, y) = x_0 + h(t, x, y)$ and $\varphi(t, y, x) = y_0 + h(t, y, x)$. Then the mapping φ defined on the $\{(t, x, y); |t - t_0| \leq d, |x - x_0| \leq b, |y - y_0| \leq c\}$ is the general solution of (4.1). Let

$$X = \{h(t, x, y)/(t, x, y) \in D_0\}.$$

Note that $h(t_0, x, y) = 0$ for any $h \in X$. In space X , we define a relation \perp by

$$h_1 \perp h_2 \iff \|h_1\| \|h_2\| \leq d(\|h_1\| \vee \|h_2\|), \quad (4.5)$$

where $\|h_1\| \vee \|h_2\| = \|h_1\| \vee \|h_2\|$ which is an orthogonality relation on X . Let $\lambda : X \times X \times X \rightarrow [0, \infty]$ be given by

$$\lambda(x, y, z) = \|x - z\| + \|y - z\|$$

then

$$\lambda(h_1, h_1, h_2) = \|h_1 - h_2\| + \|h_1 - h_2\| = 2 \sup_{(t, x, y) \in D_0} |h_1(t, x, y) - h_2(t, x, y)|. \quad (4.6)$$

Hence the orthogonal symmetric Branciari metric space (X, \perp, λ) is complete. A mappings $A, B : (X, \perp, \lambda) \rightarrow (X, \perp, \lambda)$ can be defined by

$$\begin{cases} (Ah)(t, x, y) = \int_{t_0}^t f(\tau, x_0 + h(\tau, x, y), y_0 + h(\tau, y, x))d\tau \\ (Bh)(t, y, x) = \int_{t_0}^t g(\tau, y_0 + h(\tau, y, x), x_0 + h(\tau, x, y))d\tau \end{cases} \quad (4.7)$$

We now discuss some properties of mappings A and B .

i) A and B are \perp -preserving mappings;

ii) $\lambda(Ah_1, Ah_1, Bh_2) \leq \delta \lambda(h_1, h_1, h_2)$ for any h_1 and h_2 in X such that $h_1 \perp h_2$ and $0 \leq \delta < 1$;

iii) A or B is \perp -continuous mapping;

Proof . i) We recall that A and B are \perp -preserving mappings if for $h_1, h_2 \in X, h_1 \perp h_2$, we have $Ah_1 \perp Bh_2$.

$$\begin{aligned} |(Ah_1)(t, x, y)| &= \left| \int_{t_0}^t f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x))d\tau \right| \\ &\leq \int_{t_0}^t |f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x))|d\tau \\ &\leq \int_{t_0}^t M d\tau = M|t - t_0| \leq M \frac{d}{M} = d \end{aligned} \quad (4.8)$$

So,

$$\|Ah_1\| \|Bh_2\| \leq d \|Bh_2\|. \quad (4.9)$$

Meaning that $\|Ah_1\| \perp \|Bh_2\|$.

ii) Let h_1, h_2 in X and $h_1 \perp h_2$ we have

$$\begin{aligned} & |(Ah_1)(t, x, y) - (Bh_2)(t, y, x)| = \\ & \left| \int_{t_0}^t f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x)) d\tau - \right. \\ & \left. \int_{t_0}^t g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x)) d\tau \right| = \\ & \left| \int_{t_0}^t (f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x)) - \right. \\ & \left. g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x))) d\tau \right| \leq \\ & \int_{t_0}^t |f(\tau, x_0 + h_1(\tau, x, y), y_0 + h_1(\tau, y, x)) - \\ & g(\tau, x_0 + h_2(\tau, x, y), y_0 + h_2(\tau, y, x))| d\tau \leq \\ & \int_{t_0}^t \left(\frac{K}{2|t-t_0|} |x_0 + h_1(\tau, x, y) - x_0 - h_2(\tau, x, y)| + \right. \\ & \left. \frac{K}{2|t-t_0|} |y_0 + h_1(\tau, y, x) - y_0 - h_2(\tau, y, x)| \right) d\tau = \\ & \int_{t_0}^t \frac{K}{2|t-t_0|} (2|h_1(\tau, x, y) - h_2(\tau, x, y)|) d\tau = K \|h_1 - h_2\| \end{aligned} \quad (4.10)$$

$$\|Ah_1 - Bh_2\| \leq K \|h_1 - h_2\| \quad (4.11)$$

iii) Suppose $\{h_n\}$ is an orthogonal sequence in X such that $\{h_n\}$ converging to $h \in X$. Because A or B is \perp -preserving, $\{Ah_n\}$ or $\{Bh_n\}$ is an orthogonal sequence in X . For any $n \in \mathbf{N}$, by ii we have

$$\|Ah_n(t, x, y) - Ah(t, x, y)\| \leq K \|h_n - h\|. \quad (4.12)$$

As n goes to infinity, it follows that A is \perp -continuous mapping. The mapping A or B defined above is \perp -preserving and \perp -continuous on generalized orthogonal metric space (X, λ, \perp) . Mapping A and B satisfies of Theorem (3.2). Thus, existence and uniqueness of its fixed point $h_0 \in X$ has been guaranteed by Theorem (3.2). We are looking for solutions expressed in the form $\varphi(t, x, y) = x_0 + h(t, x, y)$ and $\varphi(t, y, x) = y_0 + h(t, y, x)$. If h is a common fixed point of A and B then $\psi(t, x, y) = x_0 + Ah(t, x, y)$ and $\varphi(t, y, x) = y_0 + Bh(t, y, x)$ is a common fixed point of our Picard $P(\varphi)$. Hence

$$\begin{aligned} P(\varphi(t, x, y)) &= x_0 + (Ah)(t, x, y) = \\ & x_0 + \int_{t_0}^t f(\tau, x_0 + h(\tau, x, y), y_0 + h(\tau, y, x)) d\tau = . \\ & x_0 + \int_{t_0}^t f(\tau, \psi(t, x, y), \varphi(t, y, x)) d\tau = \psi(t, x, y) \end{aligned} \quad (4.13)$$

Similarly $P(\varphi(t, y, x)) = \varphi(t, y, x)$. By Theorem (3.2), $\varphi(t, x, y)$ and $\varphi(t, y, x)$ are a solutions of the differential equations system (4.1) if and only if $P(\varphi(t, y, x)) = \varphi(t, y, x)$ and $P(\varphi(t, x, y)) = \varphi(t, x, y)$.

□

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