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An efficient finite difference scheme for fractional partial differential equation arising in electromagnetic waves model

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Abstract

We present an unconditionally stable finite difference scheme (FDS) for the fractional partial differential equation (PDE) arising in the electromagnetic waves, which contains both initial and Dirichlet boundary conditions. The Riemann-Liouville fractional derivatives in time are discretized by a finite difference scheme of order $\mathcal{O}\left(\Delta t^{3-\alpha}\right)$ and $\mathcal{O}\left(\Delta t^{3-\beta}\right)$, $1<\beta<\alpha<2$ and the Laplacian operator is discretized by central difference approximation. The proposed stable FDS schemes transform the fractional PDE into a tridiagonal system. Theoretically, uniqueness, unconditionally stability, error bound, and convergence of FDS are investigated. Moreover, the accuracy of the order of convergence $\mathcal{O}\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x^{2}\right)$ of the scheme is investigated. Finally, numerical results are reported to illustrate our optimal error bound, order of convergence, and efficiency of proposed schemes.

Keywords: Fractional PDE, Finite difference scheme, Reimann-Liouville fractional derivative, Convergence analysis
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1 Introduction

In this paper, we consider the following time fractional partial differential equation arising from electromagnetic wave in dielectric media with Riemann-Liouville time fractional derivative operator [34]:

$$({}_{0}^{RL}D_{t}^{\alpha}u)(t,x)+\lambda_{1}({}_{0}^{RL}D_{t}^{\beta}u)(t,x)-\lambda_{2}\nabla^{2}u(t,x)=f(t,x), x\in\Omega, t\in(0,T], \tag{1.1}$$

subject to initial condition

$$\begin{cases} u(0,x)=g_{1}(x), \\ \left[\frac{\partial u(t,x)}{\partial t}\right]_{t=0}=g_{2}(x), \end{cases} \tag{1.2}$$

and Dirichlet boundary conditions are

$$u(t,x)=0, \qquad x\in\partial\Omega, \quad t\in(0,T], \tag{1.3}$$

where $u(t,x)$ is unknown, $f(t,x)$ is known function for the current density of free charges, space domain $\Omega=[0,L]$, the constant coefficient λ_1 and λ_2 depend on the frequency independent properties of a medium, $1<\beta<\alpha<2$,

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${}_{0}^{RL}D_t^{\alpha}$ & ${}_{0}^{RL}D_t^{\beta}$ both fractional derivatives are defined in the Riemann-Liouville derivative operator sense, and ∇^2 is Laplace operator.

The theory of integrals and fractional derivative, which is a generalization of the classical order calculus, was first introduced by Leibniz in 1965. Over the last few decades, its application to science and engineering has made it extremely popular and important. Recently, the fractional differential operator hypothesis has evolved primarily as an open field in science and engineering. In addition, researchers have found that fractional-order models are more suitable than integer-order models. Fractional differential equations (FDEs) provide powerful and flexible tools for modeling and describing the behavior of real materials [35], Finance [37], Viscoelastic Fluids [28], Signal and image processing [29], Biological systems [20], Control theory [36], Electrochemical processes [23], electromagnetic waves Negheta1998, etc. Moreover, it is difficult to find analytical solution of all FDE. Therefore, the numerical solution of FDE has become a major research topic (for instant see [7]-[10]). Fractional calculus provide a great tool for expressing the memory and genetic characteristics of various materials and processes. Several studies have extended fractions to discover stable numerical and scientific methods for unsolved FPDEs of physical problems. These motivate us to look for an efficient, effective and stable numerical approach to understanding FPDE.

Several numerical approaches to FPDE in time, space and space-time have been proposed by very few researchers. For example, Liu et al. [15] -[17], have found the solution of second order Fokker-Planck equation, a space fractional Fokker-Planck equation, and a modified anomalous partial diffusion equation with a nonlinear source term, and time-space fractional advection-diffusion equation. In [22], Meerschaert has used a finite difference scheme for the numerical solution of fractional advection variance flow equations in Caputo space. Shen & Liu [30] and Liu et al. [31] have proposed the spatial fractional diffusion equation and the numerical solution of RieszFPDE separately. A broad numerical approach also solves FPDE using the finite element method (see [3]), the finite difference method (see [1]), and the spectral method [12] which are limited. Due to the nonlocal nature of the fractional derivation operator, the essential problem with FPDE's numerical solution is to reduce computational effort. Few numerical approaches to calculation cost reduction have been proposed such as alternating direction implicit method (ADI) [22], finite difference scheme [8]-[9], multigrid method [2], and appropriate iterative approach in [14].

Broadly speaking, fractional models can be divided into two main types: spatial fractional differential equations and time fractional differential equations. In other cases, as expert collapse and flow studies show, little work is available to solve time-separated PDEs, but it is still limited. Proposed problem (1.1) has been comprehensively considered by [33]- [26]. Most of schemes could not established uniqueness, stability, and convergence numerical schemes for proposed problem (1.1) (1.3). As far as we know, the proposed problem has only discussed by [26] -[24]. Therefore, this article introduced an unconditional stable numerical scheme of (1.1) − (1.3) based on a finite difference scheme. Our aim to formulate a reliable, efficient, well-versatile and unconditional stable scheme that will be suitable to address any points and queries that may naturally arise with the simulation of proposed fractional PDE (1.1)- (1.3). The main work of this paper can be described by the following points:

- An unconditional stable FDS is presented and analyzed for solving (1.1)-(1.3). The FDS is established by assembling the central difference approximation for Laplace operator along with difference approximations of order $\mathcal{O}(\Delta t^{3-\alpha})$ and $\mathcal{O}(\Delta t^{3-\beta})$ for Riemann- Liouville fractional derivative ${}_{0}^{RL}D_t^{\alpha}$ and ${}_{0}^{RL}D_t^{\beta}$, respectively.
- The uniqueness, unconditionally stability, and error bounds of FDS for proposed problem (1.1)-(1.3) are derived.
- It is proven that the proposed FDS admit the optimal convergence order $\mathcal{O}\left(\Delta t^{3-\alpha} + \Delta t^{3-\beta} + \Delta x^2\right)$ of error estimate; where Δx and Δt represents the equal step sizes of meshes in space and time domain, respectively.
- Some numerical examples are performed to validate the applicability and reliability of the scheme.

The outlines of the rest of the paper are as follows: Section 2 introduced the preliminaries of fractional derivatives in terms of Riemann-Liouvelly derivative operators. In section 3, finite difference scheme for proposed problem (1.1)-(1.3) is established. In sections 4, uniqueness, stability analysis, error bound and order of convergence of scheme are estimated. And finally, two numerical examples have illustrated the efficiency of the proposed FDS approach in section 5.

2 Fractional derivative

In this section, some preliminary results about Caputo's fractional partial differential operator, wavelets, Kronecker multiplications, and function approximations are discussed. The Riemann-Liouville fractional partial differential operator $({}_0^{RL}D_t^{\alpha}u)$ and $({}_0^{RL}D_t^{\beta}u)$ denotes the Riemann-Liouville fractional derivative of order α and β respectively with respect to time t and defined by ([27]):

$$({}_a^{RL}D_t^{\alpha}u)(t,x) = \sum_{k=0}^{m-1} \frac{u^{(k)}(a,x)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{u^{(m)}(s,x)}{(t-s)^{\alpha+1-m}} ds, \quad m-1 \le \alpha < m. \tag{2.1}$$

Constant-order fractional partial derivatives are an extension of constant-order fractional derivatives. Constant-order fractional partial derivatives have been introduced in several sciences and engineering fields. We adopt the following definition of constant-order for fractional analysis of partial derivative at $m = 2$ and $a = 0$:

$$({}_0^{RL}D_t^{\alpha}u)(t,x) = \sum_{k=0}^{1} \frac{u^{(k)}(0,x)(t)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u^{(2)}(s,x)}{(t-s)^{\alpha-1}} ds, \quad 1 < \alpha < 2. \tag{2.2}$$

and similarly

$$({}_0^{RL}D_t^{\beta}u)(t,x) = \sum_{k=0}^{1} \frac{u^{(k)}(0,x)(t)^{k-\beta}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{u^{(2)}(s,x)}{(t-s)^{\beta-1}} ds, \quad 1 < \beta < 2. \tag{2.3}$$

Also, Caputo fractional derivative of order α with respect to time t and defined by

$$({}_a^{C}D_t^{\alpha}u)(t,x) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{u^{(m)}(s,x)}{(t-s)^{\alpha+1-m}} ds, \quad m-1 \le \alpha < m. \tag{2.4}$$

3 Finite difference scheme

In this section, we establish a new finite difference scheme for problem 1.1 to 1.3. Let N and M be positive integers, $\Delta t = T/N$ and $\Delta x = L/M$ denote the uniform sizes of time step and spatial grid, respectively. Let $\Omega_{\Delta t} = \{t_k : t_k = k\Delta t, k = 0,1,...,N\}$ and $\Omega_{\Delta x} = \{x_j : x_j = j\Delta x, j = 0,1,...,M\}$, then the domain $[0,T]\times[0,L]$ is covered by $\Omega_{\Delta t}\times\Omega_{\Delta x}$. At the points (t_k, x_j), the functional values and approximated values of $u(t,x)$ are denoted by $U_j^k = u(t_k, x_j)$ and $\bar{U}_j^k$, respectively. Let $U = U_j : j = 0,1,...,M, U_0 = U_M = 0$ and $V = V_j : j = 0,1,...,M, V_0 = V_M = 0$ represent two grid functions on $\Omega_{\Delta t}\times\Omega_{\Delta x}$.

We introduce the following relations and notations [13]:

$$\delta_t U^n = \frac{U^n - U^{n-1}}{\Delta t}, \quad \delta_x U_{j-\frac{1}{2}} = \frac{U_j - U_{j-1}}{\Delta x}, \quad \delta_x^2 U_j = \frac{\delta_x U_{j+\frac{1}{2}} - \delta_x U_{j-\frac{1}{2}}}{\Delta x},$$

$$U_j^{k-\frac{1}{2}} = \frac{U_j^k + U_j^{k-1}}{2}, \quad |\delta_x U^k| = \sqrt{\Delta x \sum_{i=1}^{M-1} (\delta_x U_j^k)^2},$$

$$\langle U, V\rangle = \sum_{j=1}^{M-1} \Delta x U_j V_j, \quad \|U\|_2^2 = \langle U, U\rangle, \quad \|U^k\|_\infty = \max_{0\le j\le M} |U_j^k|,$$

$$\left\langle \delta_x^2 U, V\right\rangle = -\left\langle \delta_x U, \delta_x V\right\rangle, \quad \left\langle \delta_x^2 U, U\right\rangle = -\left\langle \delta_x U, \delta_x U\right\rangle.$$

Let us recall the Riemann-Liouville fractional time derivative of order $\alpha \in (1,2)$, given by

$$\begin{aligned}({}_0^{RL}D_t^{\alpha}u)(t,x) &= \sum_{k=0}^{1} \frac{u^{(k)}(0,x)(t)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u^{(2)}(s,x)}{(t-s)^{\alpha-1}} ds, \\ &= \sum_{k=0}^{1} \frac{u^{(k)}(0,x)(t)^{k-\alpha}}{\Gamma(k-\alpha+1)} + {}_0^{C}D_t^{\alpha-1}(D_t u(t,x)).\end{aligned} \tag{3.1}$$

Let $v = D_t u$ i.e. $v(t,x) = D_t u(t,x)$, so by using the standard central difference scheme we get

$$V_j^{k+\frac{1}{2}} = \frac{U_j^{k+1} - U_j^k}{\Delta t} + \mathcal{O}(\Delta t^2). \tag{3.2}$$

Then a discrete approximation to the fractional derivative ${}_0^C D_t^{\alpha-1} v(t,x)$ at (t_k, x_j) can be obtained by the following quadrature formula:

$$\begin{aligned}
{}_0^C D_t^{\alpha-1} v(t_k, x_j) &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_k} \frac{v_t(s,x_j)}{(t_k-s)^{\alpha-1}} ds, \\
&= \frac{1}{\Gamma(2-\alpha)} \left[\int_0^{t_{\frac{1}{2}}} \frac{v_t(s,x_j)}{(t_k-s)^{\alpha-1}} ds + \int_{t_{\frac{1}{2}}}^{t_{k-\frac{1}{2}}} \frac{v_t(s,x_j)}{(t_k-s)^{\alpha-1}} ds + \int_{t_{k-\frac{1}{2}}}^{t_k} \frac{v_t(s,x_j)}{(t_k-s)^{\alpha-1}} ds \right], \\
&= \frac{1}{\Gamma(2-\alpha)} \left[-\int_{t_{-\frac{1}{2}}}^{0} \left(\frac{V_j^0 - V_j^{-\frac{1}{2}}}{\Delta t} + O(\Delta t) \right) \frac{ds}{(t_k-s)^{\alpha-1}} \right] \\
&+ \frac{1}{\Gamma(2-\alpha)} \left[\sum_{m=0}^{k-1} \int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \left(\frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t} + v_t(s,x_j) - \frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t} \right) \frac{ds}{(t_k-s)^{\alpha-1}} \right] \\
&+ \frac{1}{\Gamma(2-\alpha)} \left[\int_{t_{k-\frac{1}{2}}}^{t_k} \left(\frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t} + \mathcal{O}(\Delta t) \right) \frac{ds}{(t_k-s)^{\alpha-1}} \right],
\end{aligned} \tag{3.3}$$

Since we have, $U^{-1} = U^0 - \Delta t V^0 = g_1 - \Delta t g_2$, then

$$V_j^{-\frac{1}{2}} = \frac{U_j^0 - U_j^{-1}}{\Delta t} + \mathcal{O}(\Delta t^2) = V_j^0 + O(\Delta t^2). \tag{3.4}$$

Combining the eq. (3.2) with (3.3)-(3.4), we have

$$\begin{aligned}
({}_0^{RL} D_t^{\alpha} u)(t_k, x_j) &= \sum_{k=0}^{1} \frac{u^{(k)}(0,x)(t_k)^{k-\alpha}}{\Gamma(k-\alpha+1)} + {}_0^C D_t^{\alpha-1}(D_t u(t,x)). \\
&= \frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha}) + g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \left[-\int_{-\frac{\Delta t}{2}}^{0} \frac{\Delta t}{(k\Delta t - s)^{\alpha-1}} ds \right] \\
&+ \frac{1}{\Gamma(2-\alpha)} \left[\sum_{m=0}^{k-1} \left(\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} \right) \int_{(m-\frac{1}{2})\Delta t}^{(m+\frac{1}{2})\Delta t} \frac{ds}{(k\Delta t - s)^{\alpha-1}} \right] \\
&+ \frac{1}{\Gamma(2-\alpha)} \left(\frac{U_j^k - 2U_j^{k-1} + U_j^{k-2}}{\Delta t^2} \right) \int_{(m-\frac{1}{2})\Delta t}^{k\Delta t} \frac{ds}{(k\Delta t - s)^{\alpha-1}} + R_1, \\
&= \frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha}) + g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)} + \frac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{m=0}^{k-1} \left(\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\left((k-m+\frac{1}{2})^{2-\alpha} - (k-m-\frac{1}{2})^{2-\alpha}\right)^{-1}} \right) \\
&+ \frac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{m=0}^{k-1} \left(U_j^{m+1} - 2U_j^m + U_j^{m-1} \right) \frac{1}{2^{2-\alpha}} + \mathcal{O}(\Delta t^{3-\alpha}) + R_1,
\end{aligned} \tag{3.5}$$

where,

$$R_1 = \frac{1}{2-\alpha} \sum_{m=0}^{k-1} \int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \left(v_t(s,x_j) - \frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t} \right) \frac{ds}{(t_k-s)^{\alpha-1}}$$

Finally, we have obtained the following approximation:

$$\begin{aligned}
({}_0^{RL} D_t^{\alpha} u)(t_k, x_j) &= \frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha}) + g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)} + \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \left(\frac{\delta_t U_j^k - \delta_t U_j^{k-1}}{2^{2-\alpha}} \right) \\
&+ \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \left[\sum_{m=1}^{k-1} (A_{k-m+1} - A_{k-m}) \delta_t U_j^k + A_1 \delta_t U_j^k - A_k g_{2j} \right] + R_1 + \mathcal{O}(\Delta t^{3-\alpha}),
\end{aligned} \tag{3.6}$$

where $A_k = \left(k+\frac{1}{2}\right)^{2-\alpha} - \left(k-\frac{1}{2}\right)^{2-\alpha}$.

Similarly, a discrete approximation to the fractional derivative ${}_0^{RL}D_t^{\beta-1}u(t,x)$ at (t_k, x_j) can be obtained by the following quadrature formula:

$$\begin{aligned}({}_0^{RL}D_t^{\beta}u)(t_k,x_j) &= \frac{g_1 t_k^{-\beta}}{\Gamma(1-\beta)} + \frac{g_1(1-t_k^{-\beta}) + g_2 t_k^{-\beta}\Delta t}{\Gamma(2-\beta)} + \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left(\frac{\delta_t U_j^k - \delta_t U_j^{k-1}}{2^{2-\beta}}\right) \\ &+ \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}\left(B_{k-m+1} - B_{k-m}\right)\delta_t U_j^k + B_1 \delta_t U_j^k - B_k g_{2j}\right] + R_2 + \mathcal{O}(\Delta t^{3-\alpha}),\end{aligned} \tag{3.7}$$

where $B_k = \left(k+\frac{1}{2}\right)^{2-\beta} - \left(k-\frac{1}{2}\right)^{2-\beta}$ and $R_2 = \frac{1}{2-\beta}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\left(v_t(s,x_j) - \frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t}\right)\frac{ds}{(t_k-s)^{\beta-1}}$. Also, we have finite difference scheme for second space derivative of u as follows []:

$$U_{xx}(t_k,x_j) = \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\Delta x} + \mathcal{O}(\Delta x^2) \tag{3.8}$$

Now for solving proposed problem (1.1)-(1.3), we combined Eqs.(1.1) and (3.6)-(3.8) as follows:

$$\begin{aligned}&\frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha}) + g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)} + \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left(\frac{\delta_t U_j^k - \delta_t U_j^{k-1}}{2^{2-\alpha}}\right) \\ &+ \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{m=1}^{k-1}\left(A_{k-m+1} - A_{k-m}\right)\delta_t U_j^m + A_1 \delta_t U_j^k - A_k g_{2j}\right] \\ &+ \lambda_1\left(\frac{g_1 t_k^{-\beta}}{\Gamma(1-\beta)} + \frac{g_1(1-t_k^{-\beta}) + g_2 t_k^{-\beta}\Delta t}{\Gamma(2-\beta)}\right) + \frac{\lambda_1(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left(\frac{\delta_t U_j^k - \delta_t U_j^{k-1}}{2^{2-\beta}}\right) \\ &+ \frac{\lambda_1(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}\left(B_{k-m+1} - B_{k-m}\right)\delta_t U_j^m + B_1 \delta_t U_j^k - B_k g_{2j}\right] \\ &= \lambda_2\left(\frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\Delta x}\right) + F_j^k + R_1 + R_2 + \mathcal{O}\left(\Delta^{3-\alpha} + \Delta^{3-\beta} + \Delta x^2\right).\end{aligned} \tag{3.9}$$

Neglecting the small term $R_1 + R_2 + \mathcal{O}\left(\Delta^{3-\alpha} + \Delta^{3-\beta} + \Delta x^2\right)$ from Eq.(3.9), we get the following finite difference scheme for Eqs.(1.1)-(1.3):

$$\begin{aligned}&\frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha}) + g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)} + \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left(\frac{\delta_t \bar{U}_j^k - \delta_t \bar{U}_j^{k-1}}{2^{2-\alpha}}\right) \\ &+ \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{m=1}^{k-1}\left(A_{k-m+1} - A_{k-m}\right)\delta_t \bar{U}_j^m + A_1 \delta_t \bar{U}_j^k - A_k g_{2j}\right] \\ &+ \lambda_1\left(\frac{g_1 t_k^{-\beta}}{\Gamma(1-\beta)} + \frac{g_1(1-t_k^{-\beta}) + g_2 t_k^{-\beta}\Delta t}{\Gamma(2-\beta)}\right) + \frac{\lambda_1(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left(\frac{\delta_t \bar{U}_j^k - \delta_t \bar{U}_j^{k-1}}{2^{2-\beta}}\right) \\ &+ \frac{\lambda_1(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}\left(B_{k-m+1} - B_{k-m}\right)\delta_t \bar{U}_j^m + B_1 \delta_t \bar{U}_j^k - B_k g_{2j}\right] \\ &= \lambda_2\left(\frac{\bar{U}_{j+1}^k - 2\bar{U}_j^k + \bar{U}_{j-1}^k}{\Delta x}\right) + F_j^k.\end{aligned} \tag{3.10}$$

The algorithm for solving time-fractional partial differential equation (1.1)-(1.3).

Algorithm 1: FDS for time-fractional partial differential equation (1.1)-(1.3).

Input: The numbers $\alpha, \beta \in (1,2), \beta < \alpha, L, T \in \mathbb{R}, M, N \in \mathbb{N}$; the given functions $g_1(x), g_2(x), f(x,t)$.
Output: Numerical solution of (1.1)-(1.3): $\bar{U}$
for *Numerical solution of (1.1)-(1.3) by FDS* **do**
STEP 1. Discretize the physical domain with the uniform mesh $\Omega_{\Delta t} = \{t_k : t_k = k\Delta t, k = 0,1,...,N\}$ and $\Omega_{\Delta x} = \{x_j : x_j = j\Delta x, j = 0,1,...,M\}$.
STEP 2. Convert time-fractional PDE (1.1) into integro-PDE using Riemann-Liouville derivative operator.
STEP 3. Convert integro-PDE into difference scheme by section 3.
STEP 4. For time level k apply the scheme (3.10) to evaluate $\bar{U}_j^k$.
end

4 Analysis of uniqueness, stability, error bound and convergence of scheme

Lemma 4.1. [13] If $U_0^k = 0$ and $U_M^k = 0$, then we have

$$\|U^k\|_\infty \leq \frac{\sqrt{L}}{2}|\delta_x U^k|.$$

Now for the error R_1 and R_2, we need the following lemmas

Lemma 4.2. [16] Suppose that $u(t) \in C^2[0,t_k]$ and $1 < \alpha < 2$, we have

$$\begin{aligned}|R_1| &= \left|\frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\left(v_t(s,x_j) - \frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t}\right)\frac{ds}{(t_k-s)^{\alpha-1}}\right| \\ &\leq C \max_{-t_1 \leq t \leq t_k}|u_{tt}(t,x_j)|\Delta t^{3-\alpha}.\end{aligned} \tag{4.1}$$

Similarly,

Lemma 4.3. Suppose that $u(t) \in C^2[0,t_k]$ and $1 < \beta < 2$, we have

$$\begin{aligned}|R_2| &= \left|\frac{1}{2-\beta}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\left(v_t(s,x_j) - \frac{V_j^{m+\frac{1}{2}} - V_j^{m-\frac{1}{2}}}{\Delta t}\right)\frac{ds}{(t_k-s)^{\beta-1}}\right| \\ &\leq C \max_{-t_1 \leq t \leq t_k}|u_{tt}(t,x_j)|\Delta t^{3-\beta}.\end{aligned} \tag{4.2}$$

Lemma 4.4. [13] For the definition $A_k = \left(k+\frac{1}{2}\right)^{2-\alpha} - \left(k-\frac{1}{2}\right)^{2-\alpha}$ and $B_k = \left(k+\frac{1}{2}\right)^{2-\beta} - \left(k-\frac{1}{2}\right)^{2-\beta}$, $k = 1,2,...,M-1$, we have $A_k > 0,\ B_k > 0$ and $A_{K+1} \leq A_k,\ B_{K+1} \leq B_k$.

Theorem 4.5 (Uniqueness). The finite difference scheme (3.10) has unique solution.

Proof . The tridiagonal matrix associated to finite difference scheme (3.10) is always diagonally dominant, so it is nonsingular. Hence, finite difference scheme (3.10) has unique solution. □

Theorem 4.6 (Stability of scheme). Let $\{\bar{U}_j^k : 0 \leq j \leq M, 0 \leq k \leq N\}$ be the solution of the finite difference scheme (3.10), then and hence proposed finite difference scheme for problem (1.1)-(1.3) is a stable scheme.

Proof . Denoting $r_1 = \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)}, r_2 = \frac{\Delta t^{1-\beta}}{\Gamma(3-\beta)}, r_\alpha = \frac{g_1 t_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{g_1(1-t_k^{-\alpha})+g_2 t_k^{-\alpha}\Delta t}{\Gamma(2-\alpha)}, r_\beta = \frac{g_1 t_k^{-\beta}}{\Gamma(1-\beta)} + \frac{g_1(1-t_k^{-\beta})+g_2 t_k^{-\beta}\Delta t}{\Gamma(2-\beta)}$ and multiplying both side of (3.10) by $\Delta x \delta_t \bar{U}_j^k$ and making summation on space discretization j from 1 to $M-1$, we have

$$\begin{aligned}
&\left\langle r_\alpha, \delta_t \bar{U}^k \right\rangle + \frac{r_1}{2^{2-\alpha}} \left\langle \delta_t \bar{U}^k - \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle + r_1 \sum_{m=1}^{k-1} \left(A_{k-m+1} - A_{k-m}\right) \left\langle \delta_t \bar{U}^m, \delta_t \bar{U}^k \right\rangle \\
&\quad + r_1 A_1 \|\delta_t \bar{U}^k\|^2 - r_1 A_k \left\langle g_2, \delta_t \bar{U}^k \right\rangle + \lambda_1 \left\langle r_\beta, \delta_t \bar{U}^k \right\rangle + \frac{\lambda_1 r_2}{2^{2-\beta}} \left\langle \delta_t \bar{U}^k - \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle \\
&\quad + \lambda_1 r_2 \sum_{m=1}^{k-1} \left(B_{k-m+1} - AB_{k-m}\right) \left\langle \delta_t \bar{U}^m, \delta_t \bar{U}^k \right\rangle + \lambda_1 r_2 B_1 \|\delta_t \bar{U}^k\|^2 + \lambda_1 r_2 B_k \left\langle g_2, \delta_t \bar{U}^k \right\rangle \\
&= \lambda_2 \left\langle \delta_x^2 \bar{U}^k, \delta_t \bar{U}^k \right\rangle + \left\langle F^k, \delta_t \bar{U}^k \right\rangle .
\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}
\left\langle \delta_t \bar{U}^k - \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle &= \|\delta_t \bar{U}^k\|^2 - \left\langle \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle \\
&\geq \|\delta_t \bar{U}^k\|^2 - \frac{1}{2} \left(\|\delta_t \bar{U}^{k-1}\|^2 + \|\delta_t \bar{U}^k\|^2 \right) \\
&\geq \frac{1}{2} \left(\|\delta_t \bar{U}^k\|^2 - \|\delta_t \bar{U}^{k-1}\|^2 \right) .
\end{aligned} \tag{4.4}$$

So,

$$\begin{aligned}
&\frac{r_1}{2^{2-\alpha}} \left\langle \delta_t \bar{U}^k - \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle + \frac{r_2}{2^{2-\beta}} \left\langle \delta_t \bar{U}^k - \delta_t \bar{U}^{k-1}, \delta_t \bar{U}^k \right\rangle \\
&\qquad \geq \left(\frac{r_1}{2^{2-\alpha}} + \frac{r_2}{2^{2-\beta}} \right) \left(\|\delta_t \bar{U}^k\|^2 - \|\delta_t \bar{U}^{k-1}\|^2 \right) .
\end{aligned} \tag{4.5}$$

Now using Cauchy Schwarz inequality and Lemma 4.4, we obtained

$$\begin{aligned}
& r_1 \sum_{m=1}^{k-1} \left(A_{k-m+1} - A_{k-m}\right) \left\langle \delta_t \bar{U}^m, \delta_t \bar{U}^k \right\rangle + \lambda_1 r_2 \sum_{m=1}^{k-1} \left(B_{k-m+1} - B_{k-m}\right) \left\langle \delta_t \bar{U}^m, \delta_t \bar{U}^k \right\rangle \\
&\leq \frac{r_1}{2} \sum_{m=1}^{k-1} \left(A_{k-m+1} - A_{k-m}\right) \left(\|\delta_t \bar{U}^m\|^2 + \|\delta_t \bar{U}^k\|^2 \right) \\
&\quad + \frac{\lambda_1 r_2}{2} \sum_{m=1}^{k-1} \left(B_{k-m+1} - B_{k-m}\right) \left(\|\delta_t \bar{U}^m\|^2 + \|\delta_t \bar{U}^k\|^2 \right) \\
&= \frac{r_1}{2} \sum_{m=1}^{k-1} \left(A_{k-m+1} - A_{k-m}\right) \|\delta_t \bar{U}^m\|^2 + \frac{r_1}{2} (A_1 - A_k) \|\delta_t \bar{U}^k\|^2 \\
&\quad + \frac{\lambda_1 r_2}{2} \sum_{m=1}^{k-1} \left(B_{k-m+1} - B_{k-m}\right) \|\delta_t \bar{U}^m\|^2 + \frac{\lambda_1 r_2}{2} (B_1 - B_k) \|\delta_t \bar{U}^k\|^2 \\
&= \frac{r_1}{2} \sum_{m=1}^{k-1} A_m \|\delta_t \bar{U}^{k-m}\|^2 - \frac{r_1}{2} \sum_{m=1}^{k} A_m \|\delta_t \bar{U}^{k-m+1}\|^2 \\
&\quad + \frac{r_1}{2} \|\delta_t \bar{U}^k\|^2 + \frac{r_1}{2} (A_1 - A_k) \|\delta_t \bar{U}^k\|^2 + \frac{\lambda_1 r_2}{2} \sum_{m=1}^{k-1} B_m \|\delta_t \bar{U}^{k-m}\|^2 \\
&\quad - \frac{\lambda_1 r_1}{2} \sum_{m=1}^{k} B_m \|\delta_t \bar{U}^{k-m+1}\|^2 \qquad + \frac{\lambda_1 r_2}{2} \|\delta_t \bar{U}^k\|^2 + \frac{r_2}{2} (B_1 - B_k) \|\delta_t \bar{U}^k\|^2,
\end{aligned} \tag{4.6}$$

and the first term on the right side of equation (4.3) can be transform as follows:

$$\begin{aligned}
\left\langle \delta_x^2 U^k, \delta_t U^k \right\rangle &= \frac{1}{\Delta t} \left\langle \delta_x^2 U^k, \delta_t U^k - \delta_t U^{k-1} \right\rangle \\
&= \frac{-1}{\Delta t} \left(\left\langle \delta_x U^k, \delta_t U^k \right\rangle - \left\langle \delta_x U^k, \delta_t U^{k-1} \right\rangle \right) \\
&\leq \frac{-1}{2\Delta t} |\delta_x \bar{U}^k|^2 + \frac{1}{2\Delta t} |\delta_x \bar{U}^{k-1}|^2.
\end{aligned} \tag{4.7}$$

Next, we estimate the last term on the R.H.S. and first, fifth, sixth and seventh terms of L.H.S. of eq.(4.3). Using Cauchy-Schwarz inequality, there exist a constant $\epsilon = \frac{(2-\alpha)2^{\alpha-2}}{\Gamma(3-\alpha)}$ to give us:

$$
\begin{aligned}
&(r_1 A_k + \lambda_1 r_2 B_k)\left\langle g_1, \delta_t \bar{U}^k \right\rangle + \left\langle F^k, \delta_t \bar{U}^k \right\rangle - \left\langle r_\alpha, \delta_t \bar{U}^k \right\rangle - \lambda_1 \left\langle r_\beta, \delta_t \bar{U}^k \right\rangle \\
&\quad \leq (r_1 A_k + \lambda_1 r_2 B_k)\left(\|g_1\|^2 + \|\delta_t \bar{U}^k\|^2\right) + \frac{1}{2\epsilon}\|F^k\|^2 \\
&\quad + \frac{3\epsilon}{2}\|\delta_t \bar{U}^k\|^2 + \frac{1}{2\epsilon}\left(\|r_\alpha\|^2 + \lambda_1 \|r_\beta\|^2\right).
\end{aligned} \tag{4.8}
$$

Now, combining Eq.(4.3) with Eqs.(4.4)-(4.8) and multiplying both side of the Eq.(4.3) by Δt, we get

$$
\begin{aligned}
&\frac{r_1 \Delta t}{2^{3-\alpha}}\|\delta_t \bar{U}^k\|^2 + \frac{r_1 \Delta t}{2}\sum_{m=1}^{k} A_m \|\delta_t \bar{U}^{k-m+1}\|^2 + \frac{\lambda_1 r_2 \Delta t}{2^{3-\beta}}\|\delta_t \bar{U}^k\|^2 \\
&+ \frac{\lambda_1 r_2}{2}\sum_{m=1}^{k} B_m \|\delta_t \bar{U}^{k-m+1}\|^2 + \frac{|\lambda_2|}{2}|\delta_t \bar{U}^k|^2 \\
&\leq \frac{r_1 \Delta t}{2^{3-\alpha}}\|\delta_t \bar{U}^{k-1}\|^2 + \frac{r_1 \Delta_t}{2}\sum_{m=1}^{k-1} A_m \|\delta_t \bar{U}^{k-m}\|^2 + \frac{\lambda_1 r_2 \Delta_t}{2^{3-\beta}}\|\delta_t \bar{U}^{k-1}\|^2 \\
&\quad + \frac{\lambda_1 r_2 \Delta_t}{2}\sum_{m=1}^{k-1} B_m \|\delta_t \bar{U}^{k-m}\|^2 + \frac{\lambda_2}{2}|\delta_x \bar{U}^{k-1}|^2 + \left(\frac{r_1 \Delta_t}{2} A_k + \frac{\lambda_1 r_2 \Delta_t}{2}\right)\|g_1\|^2 \\
&+ \frac{\Delta_t}{2\epsilon}\|F^k\|^2 + \frac{3\epsilon \Delta_t}{2}\|\delta_t \bar{U}^k\|^2 + \frac{\Delta_t}{2\epsilon}\left(\|r_\alpha\|^2 + \lambda_1 \|r_\beta\|^2\right).
\end{aligned} \tag{4.9}
$$

Let

$$
\begin{aligned}
S\left(\bar{U}^k\right) &= \frac{r_1 \Delta t}{2^{3-\alpha}}\|\delta_t \bar{U}^k\|^2 + \frac{r_1 \Delta t}{2}\sum_{m=1}^{k} A_m \|\delta_t \bar{U}^{k-m+1}\|^2 + \frac{\lambda_1 r_2 \Delta t}{2^{3-\beta}}\|\delta_t \bar{U}^k\|^2 \\
&+ \frac{\lambda_1 r_2 \Delta t}{2}\sum_{m=1}^{k} B_m \|\delta_t \bar{U}^{k-m+1}\|^2 + \frac{|\lambda_2|}{2}|\delta_t \bar{U}^k|^2.
\end{aligned} \tag{4.10}
$$

So, the inequality (4.9) can be written as

$$
\begin{aligned}
S\left(\bar{U}^k\right) &\leq S\left(\bar{U}^{k-1}\right) + \left(\frac{r_1 \Delta_t}{2} A_k + \frac{\lambda_1 r_2 \Delta_t}{2}\right)\|g_1\|^2 \\
&\quad + \frac{\Delta_t}{2\epsilon}\|F^k\|^2 + \frac{3\epsilon \Delta t}{2}\|\delta_t \bar{U}^k\|^2 + \frac{\Delta_t}{2\epsilon}\left(\|r_\alpha\|^2 + \lambda_1 \|r_\beta\|^2\right).
\end{aligned} \tag{4.11}
$$

Now summing (4.11) with respect to k from 1 to K $(K \leq N)$, we get

$$
\begin{aligned}
S\left(\bar{U}^k\right) &\leq S\left(\bar{U}^{k-1}\right) + \frac{r_1 \Delta_t}{2}\sum_{m=1}^{K} A_m \|g_1\|^2 + \frac{\lambda_1 r_2 \Delta_t}{2}\sum_{m=1}^{K} B_m \|g_1\|^2 \\
&\quad + \frac{\Delta_t}{2\epsilon}\sum_{m=1}^{K}\|F^m\|^2 + \frac{3\epsilon \Delta t}{2}\sum_{m=1}^{K}\|\delta_t \bar{U}^m\|^2 + \frac{\Delta_t}{2\epsilon}\sum_{m=1}^{K}\left(\|r_\alpha\|^2 + \lambda_1 \|r_\beta\|^2\right).
\end{aligned} \tag{4.12}
$$

So, above implies as follows:

$$
\begin{aligned}
\|\bar{U}^n\|^2 \leq 2\|\bar{U}^0\|^2 + C\Bigg(& S(\bar{U}^0) + \frac{r_1 \Delta t}{2}\sum_{m=1}^{n} A_m \|g_1\|^2 + \frac{\lambda_1 r_2 \Delta t}{2}\sum_{m=1}^{n} B_m \|g_1\|^2 \\
&+ \sum_{m=1}^{n}\|F^m\|^2 + \frac{\Delta_t}{2\epsilon}\sum_{m=1}^{n}\left(\|r_\alpha\|^2 + \lambda_1 \|r_\beta\|^2\right)\Bigg)
\end{aligned}
$$

(4.13)

Hence using Eq.(4.13), we get the following required stability inequality for proposed scheme

$$\begin{aligned}\|\bar{U}^n\|^2+|\delta_x\bar{U}^n|^2 \leq C_1\|\bar{U}^0\|^2+C_2|\delta_x\bar{U}^0|^2+C_3\Delta_t^{2-\alpha}\sum_{m=1}^{n}A_m\|g_1\|^2+C_4\Delta_t^{2-\beta}\sum_{m=1}^{n}B_m\|g_1\|^2\\ +C_5\Delta_t\sum_{m=1}^{n}\|F^m\|^2+C_6\Delta_t\sum_{m=1}^{n}\left(\|r_\alpha\|^2+\lambda_1\|r_\beta\|^2\right),\quad 1\leq n\leq N,\end{aligned}\tag{4.14}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are positive real constants independent of Δt. □

Theorem 4.7 (Optimal error bound and convergence of scheme). Let $U(t,x)$ be sufficiently smooth solution of Eqs.(1.1)-(1.3) and $\bar{U}_j^k : 0\leq j\leq M, 0\leq k\leq N$ be the solution of finite difference scheme (3.10), then

$$\|U^k-\bar{U}^k\|_2+\|U^k-\bar{U}^k\|_\infty\leq C\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x\right),\quad 1\leq k\leq N,$$

where C is a positive real constant independent of Δt and Δx.

Proof . Subtracting Eq.(3.10) from Eq.(3.9), we get

$$\begin{aligned} & r_1\left[\sum_{m=1}^{k-1}\left(A_{k-m+1}-A_{k-m}\right)\delta_t E_j^m+A_1\delta_t E_j^k-A_k g_{2j}\right]\\ &\quad+\lambda_1 r_2\left[\sum_{m=1}^{k-1}\left(B_{k-m+1}-B_{k-m}\right)\delta_t E_j^m+B_1\delta_t E_j^k\right]\\ &\quad+r_1\left(\frac{\delta_t E_j^k-\delta_t E_j^{k-1}}{2^{2-\alpha}}\right)+\lambda_1 r_2\left(\frac{\delta_t\bar{U}_j^k-\delta_t\bar{U}_j^{k-1}}{2^{2-\beta}}\right)\\ &=\lambda_2\left(\frac{\bar{U}_{j+1}^k-2\bar{U}_j^k+E_{j-1}^k}{\Delta x}\right),\end{aligned}\tag{4.15}$$

where $r_1=\frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}, r_2=\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}, R_j^k=\mathcal{O}\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x^2\right)$ and $E_j^k=U_j^k-\bar{U}_j^k$.
Now, multiplying Eq.(4.15) by $\Delta x\delta_t E_j^k$ and taking summation on j for $1\leq j\leq M-1$, we have

$$\begin{aligned} &\frac{r_1}{2-\alpha}\left\langle\delta_t E^k-\delta_t E^{k-1},\delta_t E^k\right\rangle+r_1A_1\|\delta_t E^k\|^2\frac{\lambda_1 r_1}{2-\alpha}\left\langle\delta_t E^k-\delta_t E^{k-1},\delta_t E^k\right\rangle+r_2B_1\|\delta_t E^k\|^2\\ &=r_1\sum_{m=1}^{k-1}\left(A_{k-m}-A_{k-m+1}\right)\left\langle\delta_t E^m,\delta_t E^k\right\rangle+\lambda_1 r_2\sum_{m=1}^{k-1}\left(B_{k-m}-B_{k-m+1}\right)\left\langle\delta_t E^m,\delta_t E^k\right\rangle\\ &+\lambda_2\left\langle\delta_x^2 E^k,\delta_t E^k\right\rangle+\left\langle R^k,\delta_t E^k\right\rangle.\end{aligned}\tag{4.16}$$

So, by using similar calculation to the stability analysis in theorem 4.6, we get

$$\begin{aligned} &\frac{r_1\Delta t}{2^{3-\alpha}}\|\delta_t E^k\|^2+\frac{r_1\Delta t}{2}\sum_{m=1}^{k}A_m\|\delta_t E^{k-m+1}\|^2+\frac{\lambda_1 r_2\Delta t}{2^{3-\beta}}+\frac{\lambda_1 r_2\Delta t}{2}\sum_{m=1}^{k}B_m\|\delta_t E^{k-m+1}\|^2\\ &+\frac{|\lambda_2|}{2}|\delta_x E^K|^2\leq\frac{r_1\Delta t}{2^{3-\alpha}}\|\delta_t E^{k-1}\|^2+\frac{r_1\Delta t}{2}\sum_{m=1}^{k-1}B_m\|\delta_t E^{k-m}\|^2+\frac{\lambda_1 r_2\Delta t}{2^{3-\beta}}\|\delta_t E^{k-1}\|^2\\ &+\frac{\lambda_1 r_2\Delta t}{2}\sum_{m=1}^{k-1}B_m\|\delta_t E^{k-m}\|^2+\frac{|\lambda_2|}{2}|\delta_x E^{k-1}|^2+\frac{\Delta t}{2\epsilon}\|R^k\|^2+\frac{\Delta t\epsilon}{2}\|\delta_t E^k\|^2.\end{aligned}\tag{4.17}$$

Now, using the definition of S as in theorem 4.6 Eq.(4.17) can be rewritten as

$$S\left(E^k\right)\leq S\left(E^{k-1}\right)+\frac{\Delta t}{2\epsilon}\|R^k\|^2+\frac{\epsilon\Delta t}{2}\|E^k\|^2\tag{4.18}$$

So, from the similar calculation as in [16], we get

$$S\left(E^k\right) \leq C\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x^2\right)^2, \tag{4.19}$$

and

$$\|E^n\|^2 \leq \|E^0\|^2 + 2T\Delta t \sum_{k=1}^{n} \|\delta_t E^k\|^2, \quad 1 \leq n \leq N. \tag{4.20}$$

Using Lemma 4.1, Eqs.(4.19)-(4.20), we obtain

$$\|U^k - \bar{U}^k\|_2 + \|U^k - \bar{U}^k\|_\infty \leq C\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x^2\right), \forall\ 1 \leq k \leq N, \tag{4.21}$$

where C is positive real constant independent of Δx and Δt. □

5 Numerical examples

Now, we have discussed some examples to show the accuracy and efficiency of the proposed schemes to validate our scheme which verifies the stability and convergence of the finite difference scheme. Let U be the exact solution and $\bar{U}$ be the numerical solution then

$$\text{Absolute } error = \left|U-\bar{U}\right|,$$

$$\left\|U-\bar{U}\right\|_2 = \sqrt{\sum_{j=1}^{M-1}\left|U(x_j,T)-\bar{U}(x_j,T)\right|^2},$$

and

$$\left\|U-\bar{U}\right\|_\infty = \max_{1\leq j\leq N}\left|U(x_j,T)-\bar{U}(x_j,T)\right|.$$

We have used the following formula for calculating the computational orders (COs) of the proposed finite difference scheme:

$$CO(N) = \frac{\log\left(\frac{E_N}{E_{N+1}}\right)}{\left(log\frac{h_N}{h_{N+1}}\right)},$$

where E_N and E_{N+1} are errors corresponding to the grids with mesh size h_N and h_{N+1} respectively.

Example 5.1.

$$({}_cD_t^\alpha B)(x,t) - ({}_cD_t^\beta B)(x,t) - \nabla^2 B(x,t) = f(x,t), \tag{5.1}$$

subject to initial condition

$$\begin{cases} B(x,0) = g(x), \\ \left[\frac{\partial B(x,t)}{\partial t}\right]_{t=0} = h(x), \end{cases} \tag{5.2}$$

and Dirichlet boundary conditions are

$$B(x,t) = 0, x \in \partial\Omega, t \in (0,T], \tag{5.3}$$

and the exact solution of the above test problem is $B(x,t) = t^3 sin(\pi x)$. The value of source term $f(x,t)$ is varies for different choices of α and β.

We have solve Example 5.1 using proposed finite difference scheme is given in $Eq.$(3.10). Figure 1 shows the behavior of absolute errors of Example 5.1 at $L = 1, T = 1, \Delta t = 1/1000$ and $\Delta x = 1/160$, Figure 2 shows the behavior of absolute errors of Example 5.1 at $L = 1, T = 1, \Delta x = 1/1000, \Delta t = 1/160$, and different values of α and β, where $\alpha = 1.5,\ \beta = 1.1;\ \alpha = 1.7,\ \beta = 1.5$ and $\alpha = 1.9,\ \beta = 1.3$, respectively. Also, L_2 errors, L_∞ errors and temporal order of convergence of Example 5.1 at fixed temporal step size $\Delta t = 1/1000$ and different spatial step size Δx are given in Tables 1 and 2 respectively and L_2 errors, L_∞ errors and order of convergence of Example 5.1 at fixed space size $\Delta x = 1/1000$ and different temporal step size Δx are given in Tables 3 and **??** respectively.

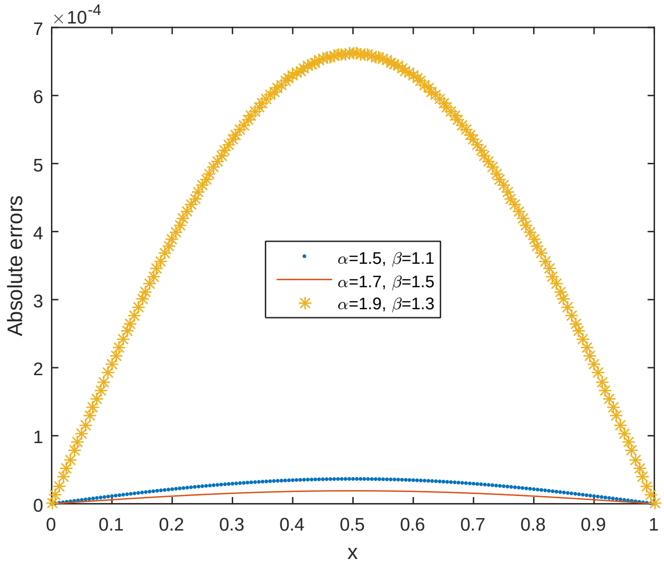

Figure 1: Absolute errors of Example 5.1 with the spatial step size $\Delta x = 1/160$, temporal step size $\Delta t = 1/1000$ and different values of α & β.

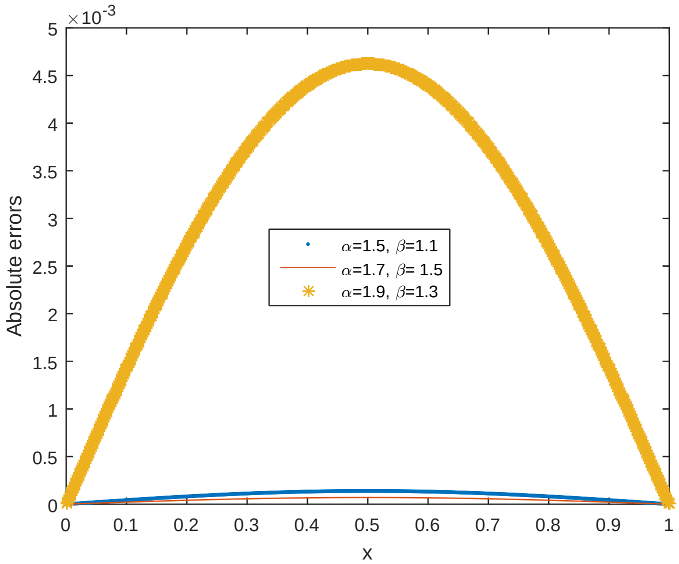

Figure 2: Absolute errors of Example 5.1 with the spatial step size $\Delta x = 1/1000$, temporal step size $\Delta t = 1/1000$ and different values of α & β.

Δx	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs	$\|B - \bar{B}\|_2$	COs
1/10	5.5000E-03	-	5.5000E-03	-	4.0000E-03	-
1/20	1.2000E-03	2.19640	1.4000E-03	1.97400	1.3000E-03	1.62149
1/40	3.1655E-04	1.92253	3.3648E-04	2.05683	6.7702E-04	0.94124
1/80	8.3953E-05	1.91478	7.8056E-05	2.10794	5.0953E-05	0.81003
1/160	2.5813E-05	1.70148	1.3446E-05	2.53733	4.6765E-05	0.72374

Table 1: L_2 errors and order of convergence of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

Δx	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\left\lVert B - \bar{B} \right\rVert_2$	COs	$\left\lVert B - \bar{B} \right\rVert_2$	COs	$\left\lVert B - \bar{B} \right\rVert_2$	COs
1/10	5.5000E-03	-	5.5000E-03	-	4.0000E-03	-
1/20	1.2000E-03	2.19640	1.4000E-03	1.97400	1.3000E-03	1.62149
1/40	3.1655E-04	1.92253	3.3648E-04	2.05683	6.7702E-04	0.94124
1/80	8.3953E-05	1.91478	7.8056E-05	2.10794	5.0953E-05	0.81003
1/160	2.5813E-05	1.70148	1.3446E-05	2.53733	4.6765E-05	0.72374

Table 2: L_2 errors and order of convergence of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

Δx	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\left\lVert B - \bar{B} \right\rVert_\infty$	T	$\left\lVert B - \bar{B} \right\rVert_\infty$	T	$\left\lVert B - \bar{B} \right\rVert_\infty$	T
1/10	7.0000E-03	11.02	7.8000E-03	12.33	5.7000E-03	10.98
1/20	1.8000E-03	16.13	1.9000E-03	15.71	1.9000E-03	14.75
1/40	4.4768E-04	30.21	4.7586E-04	31.81	9.5746E-04	30.97
1/80	1.1873E-04	41.12	1.1039E-04	44.82	7.2058E-04	43.75
1/160	3.6505E-05	70.45	1.9040E-05	72.33	6.6136E-04	70.89

Table 3: L_∞ errors and CPU time in second (T) of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

Δt	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\left\lVert B - \bar{B} \right\rVert_2$	COs	$\left\lVert B - \bar{B} \right\rVert_2$	COs	$\left\lVert B - \bar{B} \right\rVert_2$	COs
1/10	7.3000E-03	-	6.4000E-03	-	4.2900E-02	-
1/20	2.4000E-03	1.60486	1.8000E-03	1.83007	2.4800E-02	0.79064
1/40	7.9704E-04	1.59031	2.7623E-04	2.70405	1.3300E-02	0.89891
1/80	2.7663E-04	1.52669	3.2435E-05	3.09025	6.7000E-03	0.98919
1/160	9.8344E-05	1.49205	4.9671E-05	0.61485	3.3000E-03	1.02170

Table 4: L_2 errors and order of convergence of Example 5.1 for different values of α and β with fixed spatial step size $\Delta x = 1/1000$
table4

Δt	$\alpha = 1.5, \beta = 1.1$		$\alpha = 1.7, \beta = 1.5$		$\alpha = 1.9, \beta = 1.3$	
	$\left\lVert B - \bar{B} \right\rVert_\infty$	COs	$\left\lVert B - \bar{B} \right\rVert_\infty$	COs	$\left\lVert B - \bar{B} \right\rVert_\infty$	COs
1/10	1.0300E-02	-	9.0000E-03		6.0600E-02	
1/20	3.4000E-03	1.59904	2.5000E-03	1.84800	3.5000E-02	0.79196
1/40	1.1000E-03	1.62803	3.9066E-04	2.67794	1.8800E-02	0.89662
1/80	3.9121E-04	1.49149	4.5870E-05	3.09029	9.5000E-03	0.98473
1/160	1.3908E-04	1.49203	7.0246E-05	0.71487	4.6000E-03	1.04629

Table 5: L_∞ errors and order of convergence of Example 5.1 for different values of α and β with fixed spatial step size $\Delta x = 1/1000$

Example 5.2.

$$({}_cD_t^{\alpha}B)(x,t) - ({}_cD_t^{\beta}B)(x,t) - \nabla^2 B(x,t) = f(x,t), \tag{5.4}$$

subject to initial condition

$$\begin{cases} B(x,0) = g(x), \\ \left[\frac{\partial B(x,t)}{\partial t}\right]_{t=0} = h(x), \end{cases} \tag{5.5}$$

and Dirichlet boundary conditions are

$$B(x,t) = 0, x \in \partial\Omega, t \in (0,T], \tag{5.6}$$

and the exact solution of the above test problem is $B(x,t) = t^{\alpha+\beta}x^{1+\alpha+\beta}(1-x)$. The value of source term $f(x,t)$ is varies for different choices of α and β.

We have solve Example 5.2 using proposed finite difference scheme is given in $Eq.$(3.10). Figure 3 shows the behavior of absolute errors of Example 5.2 at $L = 1, T = 1, \Delta t = 1/1000$ and $\Delta x = 1/160$, Figure 4 shows the behavior of absolute errors of Example 5.1 at $L = 1, T = 1, \Delta x = 1/1000, \Delta t = 1/160$, and different values of α and β, where $\alpha = 1.5,\ \beta = 1.1;\ \alpha = 1.7,\ \beta = 1.5$ and $\alpha = 1.9,\ \beta = 1.3$, respectively. Also, L_2 errors, L_∞ errors and temporal order of convergence of Example 5.1 at fixed temporal step size $\Delta t = 1/1000$ and different spatial step size Δx are given in Tables 6 and 7 respectively and L_2 errors, L_∞ errors and order of convergence of Example 5.2 at fixed space size $\Delta x = 1/100$ ... les 8 and 9 respectively.

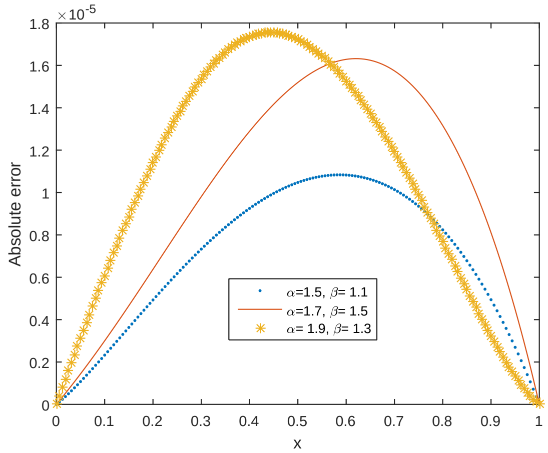

Figure 3: Absolute errors of Example 5.2 with the spatial step size $\Delta x = 1/160$, temporal step size $\Delta t = 1/1000$ and different values of α & β.

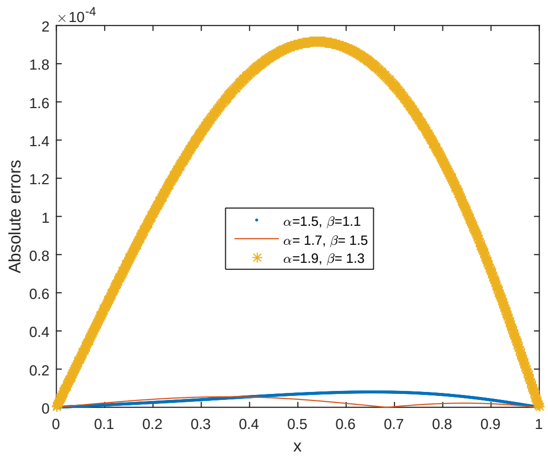

Figure 4: Absolute errors of Example 5.2 with the spatial step size $\Delta x = 1/1000$, temporal step size $\Delta t = 1/160$ and different values of α & β.

Δx	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\lVert B-\bar{B}\right\rVert_2$	COs	$\left\lVert B-\bar{B}\right\rVert_2$	COs	$\left\lVert B-\bar{B}\right\rVert_2$	COs
1/10	2.1000E-03	-	2.8000E-03	-	1.9000E-03	-
1/20	5.1522E-04	2.02713	7.1060E-04	1.97832	4.5900E-04	2.04943
1/40	1.2848E-04	2.00364	1.7794E-04	1.99765	1.0147E-04	2.17744
1/80	3.1841E-05	2.01259	4.4694E-05	1.99324	1.3043E-05	2.95971
1/160	7.6875E-06	2.05030	1.1380E-05	1.97358	3.2023E-06	2.02610

Table 6: L_2 errors and order of convergence of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

Δx	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs
1/10	2.9000E-03	-	4.1000E-03	-	2.8000E-03	-
1/20	7.2700E-04	1.99603	1.0000E-03	2.03562	6.9735E-04	2.00547
1/40	1.8130E-04	2.00358	2.5682E-04	1.96117	1.5710E-04	2.15020
1/80	4.4940E-05	2.01231	6.4431E-05	1.99493	2.2935E-05	2.77606
1/160	1.0835E-05	2.05230	1.6320E-05	1.98111	4.7549E-06	2.27006

Table 7: L_∞ errors and order of convergence of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

Δt	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\lVert B-\bar{B}\right\rVert_2$	COs	$\left\lVert B-\bar{B}\right\rVert_2$	COs	$\left\lVert B-\bar{B}\right\rVert_2$	COs
1/10	4.0632E-04	-	2.7738E-04	-	1.8000E-03	-
1/20	1.3378E-04	1.60275	8.2209E-05	1.75450	1.0000E-03	0.84799
1/40	4.4777E-05	1.59031	1.8872E-05	2.12305	5.5628E-04	0.84612
1/80	1.5448E-05	1.53534	6.1390E-06	1.62017	2.8046E-04	0.98802
1/160	5.3532E-06	1.52895	3.3143E-06	0.88943	1.3638E-04	1.04016

Table 8: L_2 errors and order of convergence of Example 5.1 for different values of α and β with fixed spatial step size $\Delta x = 1/1000$

Δt	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs	$\left\lVert B-\bar{B}\right\rVert_\infty$	COs
1/10	5.8923E-04	-	4.1694E-04		2.5000E-03	
1/20	1.9661E-04	1.58349	1.3378E-04	1.63998	1.5000E-03	0.73697
1/40	6.6467E-05	1.56463	3.5105E-05	1.93011	7.7956E-04	0.94423
1/80	2.3079E-05	1.52606	8.7782E-06	1.99968	3.9346E-04	0.98644
1/160	8.0367E-06	1.52191	5.5136E-06	0.67093	1.9148E-04	1.03902

Table 9: L_∞ errors and order of convergence of Example 5.1 for different values of α and β with fixed spatial step size $\Delta x = 1/1000$

Δx	$\alpha=1.5,\beta=1.1$		$\alpha=1.7,\beta=1.5$		$\alpha=1.9,\beta=1.3$	
	$\left\lVert B-\bar{B}\right\rVert_\infty$	T	$\left\lVert B-\bar{B}\right\rVert_\infty$	T	$\left\lVert B-\bar{B}\right\rVert_\infty$	T
1/10	2.9000E-03	15.22	4.1000E-03	14.89	2.8000E-03	15.11
1/20	7.2700E-04	28.63	1.0000E-03	29.75	6.9735E-04	30.01
1/40	1.8130E-04	51.92	2.5682E-04	52.33	1.5710E-04	51.43
1/80	4.4940E-05	66.21	6.4431E-05	65.11	2.2935E-05	65.03
1/160	1.0835E-05	80.33	1.6320E-05	81.23	4.7549E-06	80.90

Table 10: L_∞ errors and CPU time in second (T) of Example 5.1 for different values of α and β with fixed temporal step size $\Delta t = 1/1000$

6 Conclusion

In this paper, we have presented and analyzed an unconditionally stable and efficient finite difference scheme (FDS) for proposed problem (1.1)-(1.3). Our approach is based on an approximation of Riemann-Liouville fractional derivative operator of order $\mathcal{O}\left(\Delta t^{3-\alpha}\right)$ and $\mathcal{O}\left(\Delta t^{3-\beta}\right)$ in time domain and central difference discretization for Laplacian operator. The uniqueness, unconditionally stability, error bound, and convergence of the scheme are investigated. Also, we have shown that the order of convergence of the proposed FDS is $\mathcal{O}\left(\Delta t^{3-\alpha}+\Delta t^{3-\beta}+\Delta x^{2}\right)$. At last, from the considered examples 5.1 and 5.2, L^2 errors (see Tables 1, 3, 5 and 7), L^∞ errors (see Tables 2, 5, 7 and 9), absolute errors (see Figures 1-4), and comparison with the existing method in [26]-[24] , it can be effectively observed that the FDS for fractional PDE has acquired the outcome as precise as could be expected under the circumstances. Numerical outcomes of examples (see Tables 1-10) confirm the theoretical results and high accuracy of the proposed finite difference scheme. In this paper,we could not provided the results for nonlinear source term and unbounded domain, which is one of our goals and a topic for future study.

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References

[1] B. Beumer, M. Kovcs and M.M. Meerschaert, *Numerical solutions for fractional reaction diffusion equations*, Comput. Math. Appl. **55** (2008), 2212–2226.

[2] L. Chen, R.H. Nochetto, E. Otsarola and A.J. Salgado, *Multilevel methods for nonuniformly elliptic operators and fractional diffusion*, Math. Comput. **85** (2016), 2583–2607.

[3] W. Deng, *Finite element method for the space and time fractional Fokker–Planck equation*, SIAM J. Numer. Anal. **47** (2008), 204–226.

[4] N. Engheta, *Fractional curl operator in electromagnetics*, Microw. Opt. Technol. Lett. **17** (1998), no. 2, 86–91.

[5] J.F. Kelly, R.J. McGough and M.M. Meerschaert, *Analytical time-domain Green's functions for power-law media*, J. Acous. Soc. Amer. **124** (2008), 2861–2872.

[6] S. Kumar, S. Kumar and S. Sumit, *A posteriori error estimation for quasilinear singularly perturbed problems with integral boundary condition*, Numer. Algor. **89** (2022), 791-–809.

[7] S. Kumar, S. Kumar and Sumit, *High-order convergent methods for singularly perturbed quasilinear problems with integral boundary conditions*, Math. Method Appl. Sci, (2020), https://doi.org/10.1002/mma.6854.

[8] K. Kumar, R.K. Pandey, S. Sharma and Y. Xu, *Numerical scheme with convergence for a generalized time-fractional Telegraph-type equation*, Numer. Meth. Partial Differ. Equ. **35** (2019), no. 3, 1164–1183.

[9] K. Kumar, R.K. Pandey and F. Sultana, *Numerical schemes with convergence for generalized fractional integro-differential equations*, J. Comput. Appl. Math. **388** (2021), no. 1, 113318.

[10] S. Kumar, S. Sumit and J.V. Aguiar, *A high order convergent numerical method for singularly perturbed time dependent problems using mesh equidistribution*, Math. Comput. Simul. **199** (2022), 287–306.

[11] S. Kumar, S. Sumit and J.V. Aguiar, *A parameter-uniform grid equidistribution method for singularly perturbed degenerate parabolic convection–diffusion problems*, J. Comput Appl. Math. **404** (2022), 113273.

[12] X. Li and C. Xu, *The existence and uniqueness of the week solution of the space–time fractional diffusion equation and a spectral method approximation*, Commun. Comput. Phys. **8** (2010), 1016–1051.

[13] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC, 2015.

[14] F.R. Lin, S.W. Yang and X.Q. Jin, *Preconditioned iterative methods for fractional diffusion equation*, J. Comput. Phys. **256** (2014), 109–117.

[15] F. Liu, V. Anh and I. Turner, *Numerical solution of the space fractional Fokker–Planck equation*, J. Comput. Appl. Math. **166** (2004), 209–219.

[16] Z. Liu, A. Cheng and X. Li, *A novel finite difference discrete scheme for the time fractional diffusion-wave equation*, Appl. Numer. Math. **134** (2018), 17–30.

[17] Q. Liu, F. Liu, I. Turner and V. Anh, *Approximation of the lévy–Feller advection–dispersion process by random walk and finite difference method*, J. Comput. Phys. **222** (2007), 57–70.

[18] F. Liu, C. Yang, K. Burrage, *Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term*, J. Comput. Appl. Math. **231** (2009), 160–176.

[19] F. Liu, P. Zhuang, V. Anh, I. Turner and K. Burrage, *Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation*, Appl. Math. Comput. **191** (2007), 12–20.

[20] R.L. Magin, *Fractional calculus models of complex dynamics in biological tissues*, Comput. Math. Appl. **59** (2010), no. 5, 1586–1593.

[21] F. Mainardi, *The fundamental solution for the fractional diffusion-wave equation*, Appl. Math. Lett. **9** (1996), 23–28.

[22] M.M. Meerschaert, H.P. Scheffler and C. Tadjeran, *Finite difference methods for two-dimensional fractional dispersion equation*, J. Comput. Phys. **211** (2006), 249–261.

[23] K.B. Oldham, *Fractional differential equations in electrochemistry*, Adv. Eng. Softw. **41** (2010), no. 1, 9–12.

[24] V.K. Patel and D. Bahuguna, *An efficient matrix approach for the numerical solutions of electromagnetic wave model based on fractional partial derivative*, Appl. Numer. Math. **169** (2021), 1–20.

[25] V.K. Patel, S. Singh and V.K. Singh, *Two-dimensional shifted Legendre polynomial collocation method for electromagnetic waves in dielectric media via almost operational matrices*, Math. Meth. Appl. Sci. **40** (2017), 3698–3717.

[26] V.K. Patel, S. Singh and V.K. Singh, *Two-dimensional wavelets collocation method for electromagnetic waves in dielectric media*, J. Comput. Appl. Math. **317** (2017), 307–330.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, vol 198, 1999.

[28] H. Qi and M. Xu, *Stokes' first problem for a viscoelastic fluid with the generalized Oldroyd-B model*, Acta Mech. Sinica **23** (2007), no. 5, 463–469.

[29] M. Shao and C.L. Nikias, *Signal processing with fractional lower order moments: Stable processes and their applications*, Proc. IEEE **81** (1993), no. 7, 986–1010.

[30] S. Shen and F. Liu, *Error analysis of an explicit finite difference approximation for the space fractional diffusion*, ANZIAM J. **46** (2005), 871–887.

[31] S. Shen, F. Liu, V. Anh and I. Turner, *The functional solution and numerical solution of the Riesz fractional advection–dispersion equation*, IMA. J. Appl. Math. **73** (2008), 850–872.

[32] S. Sumit, S. Kumar and M. Kumar, *Optimal fourth-order parameter-uniform convergence of a non-monotone scheme on equidistributed meshes for singularly perturbed reaction–diffusion problems*, Int. J. Comput. Math. **99** (2022), no. 8, 1638–1653.

[33] T.L. Szabo, *Time domain wave equations for lossy media obeying a frequency power law*, J. Acous. Soc. Amer. **96** (1994), 491–500.

[34] V.E. Tarasov, *Fractional integro-differential equations for electromagnetic waves in dielectric media*, The. Math. Phys. 158 (2009) 355-359.

[35] P.J. Torvik and R.L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech. **51** (1984), no. 2, 294–298.

[36] B. Vinagre and V. Feliu, *Modeling and control of dynamic system using fractional calculus: Application to electrochemical processes and flexible structures*, Proc. 41st IEEE Conf. Decision and Control, Las Vegas, NV, 2002, pp. 214–239.

[37] H. Zhang, F. Liu, I. Turner and Q. Yang, *Numerical solution of the time fractional Black–Scholes model governing European options*, Comput. Math. Appl. **71** (2016), no. 9, 1772–1783.