Int. J. Nonlinear Anal. Appl. In Press, 1–7
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29716.4238

Application of optimal stopping to model sales in financial markets: Examination and analysis

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(Communicated by Ehsan Kozegar)

Abstract

High-precision prediction of financial prices is still a deemed long-term challenge that constantly calls for state-of-the-art approaches. Thus, the purpose of the current study was to examine the efficacy of optimal stopping, and use its connection with branching processes to predict several financial markets. For this purpose, the S&P 500 index and dollar, gold, oil and bitcoin markets are predicted at 5-, 10-, 30-, 50-, and 100-day forecast horizons, for each of which the optimal buying and selling point was determined. Closing price data for at least 2200 trading days during the period 2013-2021 was used for the purposes of this study. Moreover, given that any prediction and decision in the financial markets is highly based on probability, and hence risk, two strategies are devised for examination, namely (1) high risk (success rate of at least 50%) and (2) low risk (success rate of at least 70%). The findings indicated that the estimations on all the indices and prices were relevant for the high-risk scenario (that is a success rate of at least 50%), while only those on the S& P 500 index and price of gold were relevant for the low-risk scenario (success rate of at least 70%).

Keywords: Optimal stopping, Branching processes, Simulation modeling, Financial markets forecasting
2020 MSC: 91G15, 62P05

1 Introduction

Predicting the dynamism of financial markets has long been a sought-after academic realm [11]. In financial markets, real-time data processing and analysis are of paramount importance since real-time transactions are highly correlated with profit margins. In practice, decisions that are influenced by the experiences of experts in fundamental and technical analysis play a pivotal role in the financial viability of firms. The economy transfers real economic resources, offers dividends or profits to market participants, creates liquidity, and facilitates trade between investors in the market. Even though active managers who make hasty portfolio adjustments and are only backed by some rough estimates and poor prediction may improve their market scheduling performance to profit in turbulent times, oversimplification on their side may can lead to inability to make a profit in calmer markets [5]. Many investors employ multiple sources of information to predict and value the target index, and to develop strategies for gaining the upper hand. Owing to recent advances in the field of computation, several data mining models have been proposed, and complex financial data are now employed to perform more in-depth analysis. Currently, the financial markets are

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Received: December 2022 *Accepted:* February 2023

able to generate big data in real time, and competition in the capacity to process different types of real-time data that are characterized as data-oriented by the financial market leads to more optimal guidelines for business decision makers [10]. We use the method of optimal stopping for forecast. Wald, Wolfwitz, Arrow, and Blackwell obtained the first results for the theory of optimal stopping in 1940 followed by Snell (1952). In this theory, a constant value (n) is taken into account the problem's horizon within a random n-member sample $(X_1, X_2, \ldots, X_n)$ with known distribution (F). Values of this random sampling are observed in a row. The final goal is to choose the largest possible value. Accordingly, we decide in each step to select the variable we observe or continue the process. One variable should be chosen finally. As it is supposed to select the largest value, the former variables are not larger than the available variables. Moreover, variable X_n is automatically chosen if stopping does not occur before time n [9, 7]. Because optimal stopping theory is complicated, the connection between this theory and branching processes- which were developed by Assaf et al. [2] and Shishehbor et al. [17] is used. An unexpected connection is available between optimal stopping theory for independent variables and branching processes. In a special state, a random variable X, and joint distribution $X_1, X_2, \ldots, X_n$ of optimal stopping variables exist for branching processes Z_n with offspring distribution Y and probable extinction of generation $n(q_n = P(Z_n = 0))$. In this case, q_n equals the value obtained by optimal stopping (V_n) in this sequence. We consider the finite problem's horizon for the optimal stopping theory. The main result is indeed expressed based on the $Y \to X$ of integer values of branching processes of children's distributions towards a distribution on $[0, 1]$. Branching processes' features are used in critical, subcritical, and supercritical states.

2 Literature review

Today, the regression model has been used in many articles [1, 13, 8, 4]. Assaf et al. [2] conducted a study entitled 'An unexpected connection between branching processes and optimal stopping,' and explained a curious connection exists between the theory of optimal stopping for independent random variables and branching processes. In particular, for the branching process Z n with offspring distribution Y, there exists a random variable X such that the probability $P(Z_n = 0)$ of extinction of the nth generation in the branching process equals the value obtained by optimally stopping the sequence $X_1, \ldots, X_n$, where these variables are i.i.d. distributed as X. Generalizations to the inhomogeneous and infinite horizon cases are also considered. This correspondence furnishes a simple 'stopping rule' method for computing various characteristics of branching processes, including rates of convergence of the nth generation's extinction probability to the eventual extinction probability, for the supercritical, critical, and subcritical Galton-Watson process. Examples, bounds, further generalizations, and a connection to classical prophet inequalities are presented. Throughout, the aim is to show how this unexpected connection can be used to translate methods from one area of applied probability to another, rather than to provide the most general results.

Shishehbor et al. [17] carried out a MA dissertation entitled 'unexpected correlation between branching processes and theory of optimal stopping,' and explained branching processes and theory of optimal stopping are two parts of practical probability that there is no correlation between them. This study shows how results at one level can be used to confirm results on another level. The theory of optimal stopping cannot be used for all problems because it is sophisticated while branching processes are usable for all cases. This thesis indicated a correlation between these two cases, and then solved the problems of optimal stopping by branching processes.

Fathan and Delage [6] carried out a study entitled 'Deep Reinforcement Learning for Optimal Stopping with Application in Financial Engineering.' They explained Optimal stopping is the problem of deciding the right time, at which, to take a particular action in a stochastic system, to maximize an expected reward. It has many applications in areas such as finance, healthcare, and statistics. In this paper, we employ deep Reinforcement Learning (RL) to learn optimal stopping policies in two financial engineering applications: namely option pricing, and optimal option exercise. We present for the first time a comprehensive empirical evaluation of the quality of optimal stopping policies identified by three states of art deep RL algorithms: double deep Q-learning (DDQN), categorical distributional RL (C51), and Implicit Quantile Networks (IQN). In the case of option pricing, our findings indicate that in a theoretical Black-Schole environment, IQN successfully identifies nearly optimal prices. On the other hand, it is slightly outperformed by C51 when confronted with real stock data movements in a put option exercise problem that involves assets from the S&P500 index. More importantly, the C51 algorithm can identify an optimal stopping policy that achieves 8% more out-of-sample returns than the best of four natural benchmark policies. We conclude with a discussion of our findings which should pave the way for relevant future research.

Rotondi [15] conducted a study entitled 'Optimal stopping theory and American options' and explained that the optimal stopping problem is a classical one within stochastic calculus theory. Formally, given a gain process, the optimal stopping problem is about finding the stopping time that maximizes the expected gain. In discrete-time, this problem is solved using a dynamic programming technique and setting up a backward recursion. This delivers both the optimal stopping rule and the optimal value process. This optimal value process coincides with the Snell Envelope of the gain process, namely, the smallest super martingale dominating it. Optimal stopping theory closely relates to the American derivatives valuation problem. American derivatives are financial contracts characterized by a payoff process that depends on an underlying stochastic process (usually the price of a traded asset). The holder of an American derivative chooses when to cash in the payoff, trying to do so by optimizing the gain. Therefore, the fair price of this derivate depends on its optimal stopping time. During my talk, which will be accessible also to non-specialists, I will first describe and then show how to solve the optimal stopping problem in a discrete-time setting. Secondly, I will show how to apply these techniques for the valuation of American derivatives in the most standard-setting. Finally, I will present a result on how to price these derivatives in a more sophisticated market model.

Shah [16] conducted a study entitled 'Optimal Stopping Problems: Autonomous Trading Over an Infinite Time Horizon' and explained that Statistical arbitrage (StatArb) has taken off ever since its advent in the 1980s and is being increasingly used by hedge funds and investment banks to produce profitable, quantitative trading strategies. With financial firms pushing for a larger amount of automation, we are driven to investigate trading methods that decide optimally on our behalf. At the core of StatArb strategies lies mean-reverting pairs trading models that work by exploiting market pricing discrepancies. This thesis is devoted to the study of an optimal double-stopping problem characterized by optimal entry and exit times. We consider a model for both a long and short trading position and then combine both strategies into one joint optimal stopping problem. The theory is idealized for the Ornstein-Uhlenbeck process but has been framed in a general way to open the methodology to other pricing processes. The analysis is given on finding the optimal execution levels, and optimal strategy timing, and a case study of the Ornstein-Uhlenbeck process concludes the study.

Wong [18] wrote a book entitled 'Generalised optimal stopping problems and financial markets' and provided mathematicians and applied researchers with a well-developed framework in which option pricing can be formulated, and a natural transition from the theory of optimal stopping problems to the valuation of different kinds of options. With the introduction of generalized optimal stopping theory, a unifying approach to option pricing is presented.

3 Research methodology

The present paper is used to predict variables using the theory of optimal stopping. The theory of optimal stopping aims to find the maximum or minimum points in the price process of the financial variable within a certain number of days, which is called the planning horizon. For instance, assume that you want to sell the stock of a firm for up to 20 days. The price is given every morning. You should decide to accept or reject the suggested price. A rejected suggestion will be forgotten. Assume that independent sequential suggestions are determined. Naturally, you want to sell your stock for the highest price. The theory of optimal stopping has been designed to achieve this goal [2]. Several random variables and number n are given as planning horizon (the periodic maximum level at which they buy or sell should be done) when solving optimal stopping problems. Random variables are observed sequentially, and the objective is to choose the maximum sell value (minimum buy value). When we observe every variable at each moment, we should choose it or consider the next variable. If we do not choose until the nth time, this variable is selected automatically [3]. It should be explained that the former data are examined in the theory of optimal stopping. In the next step, it is found which market cannot be predicated based on this theory if there is divergence [2]. In general state, these random variables are considered independent co-distributed variables. However, the complexity of this theory makes it difficult to solve problems directly. Therefore, we use the correlation between this theory and branching processes [2]. The specific state of correlation between the theory of optimal stopping and branching processes is the state in which, numbers follow a uniform distribution $[0, 1]$ or numbers converted to a uniform distribution $[0, 1]$ using the probability integral transform theorem [17]. The appendix consists of further steps of this method. The following steps are considered in an optimal stopping problem:

Firstly, the distribution of random variables of optimal stopping $X_1, X_2, \ldots,$ and planning horizon n are determined.

The statistical distribution of values should be transformed by using their distribution variable change as a distribution on the interval $[0, 1]$ to use optimal stopping. In general, the rea optimal stopping occurs when the distribution of variables $X_1, X_2, \ldots,$ has a uniform distribution $U[0, 1]$ that is done using specific transforms. The optimal stopping values are measured for real random variables doing reverse variable change. Homogeneity or non-homogeneity of branching processes should be found before solving these problems. First, we should have the distribution of offspring numbers or the probability generating function of offspring numbers. Then we can measure the probability generating function and distribution of individuals in each generation considering the assumptions of branching processes. In the next step, the obtained formulas are used in terms of the topic to find mathematical expectation or expected value, variance, and probability of generation n extinction:

$$\widetilde{q}_n = P(Z_n = 0) = g^{(n)}(0)$$

and the probability of society extinction:

$$\lim_{n\to\infty} \widetilde{q}_n = \widetilde{\pi}_n$$

Solving equation $g(s) = s$ and finding its smallest root. Generally, the following steps are done to solve a branching process problem:

1. Assume distribution or probability generating function are given, following equations are used to measure the number of offspring Y, probability generating function, size of nth generation Z_n:

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_i$$
$$g^{(n+1)}(s) = g^{(n)}(g(s))$$
$$\vdots$$
$$g(g(\dots g(s))) \qquad\qquad n = 1, 2, \dots$$

2. The state of the process is determined in this step. To do so, $E(Y)$ is calculated. According to previous definitions, the process is subcritical, critical, and supercritical if $E(Y) < 1, E(Y) = 1$, and $E(Y > 1)$, respectively.
3. The probability of each generation q_n extinction can be measured based on the equation $\widetilde{q}_n = P(Z_n = 0) = g^{(n)}(0)$ for all generations.
4. To measure the extinction probability of the whole process π, the state of the process is considered. If the process is at subcritical or critical states, the probability of branching process extinction is equaled $\pi = 1$ without any calculation. If the branching process is at a supercritical state, equation $g(s) = s$ is solved, and its roots are measured. Then the smallest root is considered as the extinction probability.

Table 1: Optimal Stopping Prediction Steps.

Step 1	The analyst finds the planning horizon (in the daily time frame within 5, 10, 30, 50, and 100-day horizons in the present paper)
Step 2	Determine the statistical distribution of values using statistical tests, including the goodness of fit, chi-square, Chebyshev's inequality, and Q-Qplot [12]
Step 3	Transforming it to a normal distribution using Box-Cox Transformations (Cox-Box 1964), and converting to the standard normal distribution (minus mean value divided by standard deviation) [12]
Step 4	Using inverse distribution function (using probability integral transform theorem) [12] and transforming it to considered distribution in branching process and determining convergency or divergence of data [14].
Step 5	Predicting the best point for optimal buy or sell at a determined horizon [2].
Step 6	Reversing all transforms and predicting real values [2].

Note: although the theory of optimal stopping aims to measure the maximum value, the minimum value can be also considered by doing a transform. As we assumed optimal stopping values have a 0,1uniform distribution, the minimum value will be obtained using the $X^* = 1 - X$ transform.

4 Results

4.1 Predicting the price of gold in daily timeframe

The daily timeframe prediction horizons are based on intervals of 5, 10, 30, 50, 100 days, and given that the predictions are based on up to 100 days before, the results are compared with real numbers.

Table 2: Test for the normality of gold price in the daily timeframe.

Variable	K-S statistic	Sig. level	Result
Gold price in daily timeframe (initial data)	10.257	0.0001	Not normal
Gold price in daily timeframe (using the $\frac{1}{\sqrt{y}}$ Cox-Box conversion)	1.481	0.129	Normal

4.1.1 Normality

The Kolmogorov-Smirnov test is used in this study to determine the goodness of fit and normality. The statistical non-significance of the K-S statistic indicates that the variable is normal. The results of this test are as follows (Table 2).

After normalization, the following prediction values are obtained (Table 3).

Table 3: Prediction of Gold price in daily timeframe.

Horizon	Prediction	1^{st} iteration	Success (based on candlestick pattern)	2^{nd} iteration	Success (based on candlestick pattern)	3^{rd} iteration	Success (based on candlestick pattern)
5 days	buying	1777	1	1777	1	1782	1
	selling	1750	0	1751	0	1763	0
10 days	buying	1781	0	1789	0	1809	1
	selling	1749	0	1760	1	1788	0
30 days	buying	1820	1	1820	1	1866	1
	selling	1741	0	1740	0	1762	0
50 days	buying	1820	1	1832	1	1869	1
	selling	1730	1	1721	0	1788	1
100 days	buying	1901	1	1911	1	1847	1
	selling	1740	1	1739	1	1737	1

To measure the accuracy of the estimation, the binomial test is performed on two scenarios of low risk (with 70% accuracy) and high risk (with 50% accuracy) (Table 4).

Table 4: Testing the accuracy of predicting gold price in daily timeframe.

Level of risk	Number of successes	p-value	Result
Low risk (70% accuracy)	19	0.270	A minimum of 70 percent accuracy is achieved
high risk (50% accuracy)	19	0.951	A minimum of 50 percent success is achieved

4.2 Predicting the US dollar index in daily timeframe

The daily timeframe prediction horizons are based on intervals of 5, 10, 30, 50, 100 days, and given that the predictions are based on up to 100 days before, the results are compared with real numbers.

4.2.1 Normality

The Kolmogorov-Smirnov test is used in this study to determine the goodness of fit and normality. The statistical non-significance of the K-S statistic indicates that the normality of the variable. The results of this test are as follows (Table 5).

Table 5: Test for the normality of US Dollar index in the daily timeframe.

Variable	K-S statistic	Sig. level	Result
US Dollar index in daily timeframe (initial data)	7.996	0.0001	Not normal
US Dollar index in daily timeframe (using the $\frac{1}{y^2}$ Cox-Box onversion)	1.887	0.120	Normal

Table 6: Prediction of US Dollar index in daily timeframe.

Horizon	Prediction	1^{st} iteration	Success (based on candlestick pattern)	2^{nd} iteration	Success (based on candlestick pattern)	3^{rd} iteration	Success (based on candlestick pattern)
5 days	buying	93.890	0	91.762	0	94.464	0
	selling	90.090	0	89.955	0	92.29	0
10 days	buying	93.500	0	94.875	0	90.845	1
	selling	92.400	1	94.107	1	89.263	0
30 days	buying	93.570	0	95.597	1	92.289	1
	selling	92.200	1	93.559	1	92.059	1
50 days	buying	93.400	1	96.106	1	91.421	0
	selling	91.700	0	94.050	0	95.176	1
100 days	buying	93.410	1	91.887	1	95.436	1
	selling	89.500	0	87.459	1	90.674	1

After normalization, the following prediction values are obtained (Table 6).

To measure the accuracy of the estimation, the binomial test is performed on two cases of low risk (with 70% accuracy) and high risk (with 50% accuracy) (Table 6).

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