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Some differential subordinations and superordinations results for analytic univalent functions using Theyab-Atshan-Lupas-Abdullah operator

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Abstract

In this paper, we consider some differential subordinations and superordinations results for univalent functions by using the operator $(H_{\sigma,\rho,\tau,\mu,y,n})$ Also, we introduce some sandwich theorems.

Keywords: Univalent Function, Analytic Function, Subordination, Superordination, Sandwich theorem
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1 Introduction

Assume that $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in N$ and $a \in \mathbb{C}$, $H[a, n]$ is the subclass of H with the following form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., \qquad (a \in \mathbb{C}) \tag{1.1}$$

Let M be the subclass of H, consisting of analytic and univalent functions f in U of the form:

$$f(1) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in U) \tag{1.2}$$

If f and g are analytic functions in H, then we say that f is subordinate to g in U and written $f \prec g$, if there exists a Schwarz function w in U, with $w(0) = 0$, and $|w(z)| < 1, (z \in U)$, where $f(z) = g(w(z))$. In this situation, we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$ [11]. In addition, if the function g is univalent in U, then $f \prec g \Longleftrightarrow f(0) = g(0)$ and $f(U) \subset g(z)$ [14, 15].

Definition 1.1. [18] Let $\varphi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic function in U and satisfies the second-order differential subordination:

$$\varphi(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \tag{1.3}$$

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then $p(z)$ is called a solution of the differential subordination (1.3), and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.3), moreover simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.3) is said to be the best dominant of (1.3).

Definition 1.2. [14] Let $\varphi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if $p(z)$ satisfies the second -order differential superordination:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.4}$$

then $p(z)$ is called the solution of the differential superordination (1.4). An analytic function $q(z)$ is called subordinant of the solution of the differential superordination (1.4), or more simply a subordinant, if $q(z) \prec p(z)$ for all the functions p satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all the subordinant $q(z)$ of (1.4) is said to be the best subordinant.

Several authors,like, [1, 2, 3, 9, 18, 19, 20] recently attained the sufficient conditions on the functions h, p and φ for which the following conclusion is true:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$$

then

$$q(z) \prec p(z) \tag{1.5}$$

By using the results of other authors (see [4, 5, 6, 7, 8, 10, 15]) to get sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, several authors derived some differential subordination and superordination results with sandwich theorems (like [2, 4, 5, 6, 10, 11, 12, 13, 16, 17, 19]).

Suppose $f \in M$, the modern operator defined by Theyab et al. [20] is as follows: $H_{\sigma,\rho,\tau,\mu,y,n}f(z) : M \longrightarrow M$, where σ, ρ are integer numbers; $\tau, \mu, y, n \in \mathbb{C}\backslash Z_0^-$, $Z_0^- = \{0, -1, -2, ...\}$ and

$$H_{\sigma,\rho,\tau,\mu,y,n}f(z) = z + \sum_{n=2}^{\infty} \left[\frac{\tau y + \mu + n}{\tau y + \mu + 1}\right]^{\sigma+\rho+1} a_n z^n. \tag{1.6}$$

From (1.6), we note that

$$z\left(H_{\sigma,\rho,\tau,\mu,y,n}f(z)\right)' = (\tau y + \mu + 1)H_{\sigma-1,\rho,\tau,\mu,y,n}f(z) - (\tau y + \mu)H_{\sigma,\rho,\tau,\mu,y,n}f(z). \tag{1.7}$$

The major aim of the paper is to identify the necessary conditions for particular normalized analytic function f to satisfy:

$$q_1(z) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{Z}\right]^{\eta} \prec q_2(z),$$

and

$$q_1(z) \prec \left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \prec q_2(z),$$

where the functions q_1 and q_2 are univalent in U and $q_1(0) = q_2(0) = 1$.

In this paper, using the operator $H_{\sigma,\rho,\tau,\mu,y,n}f(z)$, we derive certain sandwich theorems.

2 Preliminaries

The following lemmas and definitions are necessary to prove our results.

Definition 2.1. [14] Denote by Q the set of all functions f that are analytic and injective on $\bar{U}\backslash E(f)$, where $\bar{U} = U \bigcup \{z \in \partial U\}$, and

$$E(f) = \{\varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty\}$$

and are such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U \backslash E(f)$. Further, let the subclass of Q for which $f(0) = a$ be denoted by $Q(a), Q(0) = Q_0$ and $Q(1) = Q_1$.

Lemma 2.2. [15] Let $q(z)$ be a convex univalent function in U and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{0\}$ with

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $\mathcal{T}(z)$ is analytic function in U and

$$\alpha\mathcal{T}(z) + \beta z\mathcal{T}'(z) \prec \alpha q(z) + \beta z q'(z), \tag{2.1}$$

then $\mathcal{T}(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3. [15] Let q be convex univalent function in U, and let θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set

$$Q(z) = zq'(z)\phi h(q(z)) \text{ and } h(z) = \theta(q(z)) + Q(z).$$

Suppose that

$$1 - Q(z) \text{ is starlike univalent in } U.$$

$$2 - Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0 \text{ for } z \in U.$$

If $\mathcal{T}$ is analytic in U, with $\mathcal{T}(0) = q(0), \mathcal{T}(U) \subseteq \mathcal{D}$ and

$$\theta(\mathcal{T}(z)) + z\mathcal{T}'(z)\phi(\mathcal{T}(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $\mathcal{T} \prec q$ and q is the best dominant.

Lemma 2.4. [14] Let q be a convex univalent in U and let $\beta \in \mathbb{C}$, that $Re(\beta) > 0$. If $\mathcal{T} \in \mathcal{H}[q(0), 1] \bigcap Q$ and $\mathcal{T}(z) + \beta z\mathcal{T}'(z)$ is univalent in U, then

$$q(z) + \beta z q'(z) \prec \mathcal{T}(z) + \beta z\mathcal{T}'(z), \tag{2.3}$$

which implies that $q \prec \mathcal{T}$ and q is the best subordinant.

Lemma 2.5. [14] Let q be a convex univalent function in U and let θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(U)$. Suppose that

$$1 - Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0, \text{ for } z \in U.$$

$2 - Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $\mathcal{T} \in \mathcal{H}[q(0), 1] \bigcap Q$, with $\mathcal{T}(U) \subset \mathcal{D}, \theta(\mathcal{T}(z)) + z\mathcal{T}'(z)\phi(\mathcal{T}(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\mathcal{T}(z)) + z\mathcal{T}'(z)\phi(\mathcal{T}(z)), \tag{2.4}$$

then $q \prec \mathcal{T}$ and q is the best subordinant.

3 Differential Subordination Results

Theorem 3.1. Assume that the function $q(z)$ is convex univalent in unit disk U with $q(0)=1, \zeta \in \mathbb{C}\backslash\{0\}, \eta>0$ such that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,-Re\left(\frac{\eta}{\zeta}\right)\right\}. \tag{3.1}$$

If $f \in M$ satisfies the subordination condition:

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}+\zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)\prec q(z)+\frac{\zeta}{\eta}zq'(z) \tag{3.2}$$

then

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\prec q(z), \tag{3.3}$$

and $q(z)$ is the best dominant.

Proof . Putting

$$\mathcal{T}(z)=\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}, \tag{3.4}$$

then the function $\mathcal{T}(z)$ is analytic in U and $\mathcal{T}(0)=1$. By differentiating (3.4) with respect to z, we have

$$\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}=\eta\left[\frac{z(H_{\sigma-1,\rho,\tau,\mu,y,n}f(z))'}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right]. \tag{3.5}$$

Now, by using the identity (1.7) in (3.5), we get

$$\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}=\eta\left[(\tau y+\mu+1)\left(\frac{z(H_{\sigma-1,\rho,\tau,\mu,y,n}f(z))'}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)\right].$$

Therefore,

$$\frac{z\mathcal{T}'(z)}{\eta}=\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left[(\tau y+\mu+1)\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)\right].$$

The subordination (3.2) because of the assumption becomes

$$\mathcal{T}(z)+\frac{\zeta}{\eta}z\mathcal{T}'(z)\prec q(z)+\frac{\zeta}{\eta}zq'(z).$$

We will use Lemma 2.2 with $\beta=\frac{\zeta}{\eta}$ and $\alpha=1$, to prove our result. Therefore, the subordination (3.2) implies that $\mathcal{T}(z)\prec q(z)$ and $q(z)$ is the best dominant. This completes the proof. □

By putting the convex function $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ $(-1\le B<A\le 1)$ in theorem 3.1, we have the next result.

Corollary 3.2. Let $\zeta \in \mathbb{C}\backslash\{0\}, \eta>0$ and

$$Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,-Re\left(\frac{\eta}{\zeta}\right)\right\}.$$

If $f \in M$ satisfies the subordination condition:

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}+\zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)\prec\frac{1+Az}{1+Bz}+\frac{\zeta}{\eta}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\prec\left(\frac{1+Az}{1+Bz}\right)$$

and $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

Putting $A = 1$ and $B = -1$ in above corollary, we obtain the next result.

Corollary 3.3. Let $\zeta \in \mathbb{C}\backslash\{0\}, \eta > 0$ and

$$Re\left\{1+\frac{2z}{1-z}\right\} > \max\left\{0, -Re\left(\frac{\eta}{\zeta}\right)\right\}.$$

If $f \in M$ satisfies the subordination condition:

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right) \prec \left(\frac{1-z^2+2\frac{\zeta}{\eta}z}{(1-z)^2}\right),$$

then

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \prec \left(\frac{1+z}{1-z}\right),$$

and $q(z) = \left(\frac{1+z}{1+z}\right)$ is the best dominant.

Theorem 3.4. Assume that the function $q(z)$ is convex univalent function in U with $q(0) = 1, q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If q satisfy the next condition:

$$Re\left\{1+\frac{\upsilon}{\gamma}q(z)+\frac{2\tau\rho}{\gamma}-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\} > 0, \tag{3.6}$$

where $\upsilon, \tau, \rho \in \mathbb{C},\ \gamma \in \mathbb{C}\backslash\{0\}$ and $z \in U$. If $f \in M$ satisfies:

$$\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z) \prec t+\upsilon q(z)+\tau\alpha q(z)^2+\gamma z\frac{q'(z)}{q(z)}, \tag{3.7}$$

where,

$$\begin{aligned}\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z) = &\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right)^{\eta}\left(\upsilon+\tau\alpha\left(\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right)^{\eta}+t\right)\right)\\ &+\gamma\eta\left[(\tau y+\mu+1)\left(\frac{H_{\sigma-2,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}-\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right)\right],\end{aligned} \tag{3.8}$$

then

$$\left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \prec q(z) \tag{3.9}$$

and $q(z)$ is the best dominant.

Proof . Putting

$$\mathcal{T}(z) = \left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta}, \tag{3.10}$$

then the function $\mathcal{T}(z)$ is analytic in U and $\mathcal{T}(0) = 1$. By differentiating (3.10) with respect to z, and by using identity (1.7) in the resulting equation, we get

$$\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)} = \eta\left[(\tau y+\mu+1)\left(\frac{H_{\sigma-2,\rho,\tau,\delta,\mu,y,n}f(z)}{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}-\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right)\right].$$

By setting

$$\theta(w) = t+\upsilon w+\tau\alpha w^2 \quad \text{and} \quad \phi(w) = \frac{\gamma}{w},\ w \neq 0.$$

It is simple to see that $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. As well, if we let

$$Q(z) = zq'(z)\phi(q(z)) = \gamma\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=t+\upsilon q(z)+\tau\alpha(q(z))^2+\gamma\frac{zq'(z)}{q(z)}.$$

We see that $Q(z)$ is starlike univalent in U, we get

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}=Re\left\{1+\frac{\upsilon}{\gamma}q(z)+\frac{2\tau\rho}{\gamma}-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0.$$

Through simple calculation, we find that

$$\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)=t+\upsilon\mathcal{T}(z)+\tau\alpha\mathcal{T}^2(z)+\gamma\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}, \tag{3.11}$$

where $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)$ is given by (3.8). From (3.7) and (3.11), we have

$$t+\upsilon\mathcal{T}(z)+\tau\alpha\mathcal{T}(z)^2+\gamma\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\prec t+\upsilon q(z)+\tau\alpha q(z)^2+\gamma\frac{zq'(z)}{q(z)} \tag{3.12}$$

We will use Lemma 2.3, to prove our result. Therefore, the subordination (3.7) implies that $\mathcal{T}(z)\prec q(z)$ and $q(z)$ is the best dominant. This completes the proof. □

By putting the convex function $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ $(-1\le B<A\le 1)$ in theorem 3.4, we get the next result.

Corollary 3.5. Let $\upsilon,\tau,\rho\in\mathbb{C},\gamma\in\mathbb{C}\backslash\{0\},z\in U$ and

$$Re\left\{1+\frac{\upsilon}{\gamma}\left(\frac{1+Az}{1+Bz}\right)+\frac{2\tau\rho}{\gamma}+\frac{(A+B)z}{(1+Bz)(1+Az)}-\left(\frac{2Bz}{1+Bz}\right)\right\}>0,$$

if $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)\prec t+\upsilon\left(\frac{1+Az}{1+Bz}\right)+\tau\alpha\left(\frac{1+Az}{1+Bz}\right)^2+\gamma\frac{z(A-B)}{(1+Bz)(1+Az)}$, where $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)$ is given by (3.8), then

$$\left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^\eta\prec\left(\frac{1+Az}{1+Bz}\right)$$

and $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

putting $A=1$ and $B=-1$ in above corollary, we obtain the next result.

Corollary 3.6. Let $\upsilon,\tau,\rho\in\mathbb{C},\gamma\in\mathbb{C}\backslash\{0\},z\in U$ and

$$Re\left\{1+\frac{\upsilon}{\gamma}\left(\frac{1+z}{1+z}\right)+\frac{2\tau\rho}{\gamma}+\frac{2z^2}{1-z^2}-\left(\frac{1+z}{1-z}\right)\right\}>0,$$

if $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)\prec t+\upsilon\left(\frac{1+z}{1+z}\right)+\tau\alpha\left(\frac{1+z}{1+z}\right)^2+\gamma z\frac{2}{(1+z)(1+z)}$, where $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)$ is given by (3.8), then

$$\left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^\eta\prec\left(\frac{1+z}{1+z}\right)$$

and $q(z)=\left(\frac{1+z}{1+z}\right)$ is the best dominant.

4 Differential Superordination Results

Theorem 4.1. Assume that the function $q(z)$ is a convex univalent in U with $q(0)=1,Re\{\zeta\}>0,\eta>0$ and $f\in M$ such that

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^\eta\in\mathcal{H}[q(0),1]\bigcap Q$$

and

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)$$

be univalent in U. If

$$q(z)+\frac{\zeta}{\eta}zq'(z) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right), \tag{4.1}$$

then

$$q(z) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \tag{4.2}$$

and $q(z)$ is the best subordinant.

Proof . Putting

$$\mathcal{T}(z)=\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}. \tag{4.3}$$

Differentiating (4.3) with respect to z, we get

$$\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}=\eta\left[\frac{z\left(H_{\sigma,\rho,\tau,\mu,y,n}f(z)\right)'}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right]. \tag{4.4}$$

After some computation and using (1.7), form (4.4), we get

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)=\mathcal{T}(z)+\frac{\zeta}{\eta}z\mathcal{T}'(z),$$

we will use Lemma 2.4, to prove our result. Therefore, we get $T \prec q$ and q is the best subordinant. The proof is complete. □

By putting the convex function $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ $(-1 \le B < A \le 1)$ in theorem 4.1, we get the next result.

Corollary 4.2. Let $Re\{\zeta\}>0, \eta>0$, suppose that

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \in \mathcal{H}[q(0),1]\bigcap Q$$

and

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)$$

be univalent in U, if

$$\left(\frac{1-z^2+2\left(\frac{\zeta}{\eta}\right)z}{(1-z^2)}\right) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right),$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta},$$

and $q(z)=\left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4.3. Assume that the function $q(z)$ is a convex univalent in U with $q(0)=1, q'(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If q satisfy the next condition:

$$Re\left\{\frac{q(z)}{\gamma}(2\tau\alpha q(z)+\upsilon)q'(z)\right\}>0, \tag{4.5}$$

where $\upsilon \in \mathbb{C}, \gamma \in \mathbb{C}\backslash\{0\}$ and $z \in U$. Let $f \in M$ satisfies

$$\left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \in \mathcal{H}[q(0),1]\bigcap Q,$$

and $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)$ is univalent in U, where is given by (3.8). If

$$t+\upsilon q(z)+\tau\alpha q^2(z)+\gamma\frac{zq'(z)}{q(z)} \prec \Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{H_{\sigma-1,\rho,\tau,\delta,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \tag{4.7}$$

and $q(z)$ is the best subordinant.

Proof . Putting

$$\mathcal{T}(z)=\left[\frac{H_{\sigma-1,\rho,\tau,\delta,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta}. \tag{4.8}$$

Differentiating (4.8) with respect to z, we get

$$\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}=\eta\left[(\tau y+\mu+1)\left(\frac{H_{\sigma-2,\rho,\tau,\delta,\mu,y,n}f(z)}{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}-\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right)\right].$$

By setting

$$\theta(w)=t+\upsilon w+\tau\alpha w^2 \quad \text{and} \quad \phi(w)=\frac{\gamma}{w}, w\neq 0.$$

It is simple to see that $\theta(w)$ is analytic in $\mathbb{C}$, and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0, w\in\mathbb{C}\backslash\{0\}$. As well, if we let

$$Q(z)=zq'(z)Q(q(z))=\gamma\frac{zq'(z)}{q(z)}.$$

We can observe that $Q(z)$ is starlike univalent in U, we get

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=Re\left\{\frac{q(z)}{\gamma}(2\tau\alpha q(z)+\upsilon)q'(z)\right\}>0.$$

Through simple calculation, we find that

$$\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)=t+\upsilon\mathcal{T}(z)+\tau\alpha\mathcal{T}^2(z)+\gamma z\frac{\mathcal{T}'(z)}{\mathcal{T}(z)}, \tag{4.9}$$

where $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z)$ is given by (3.8). Form (4.6) and (4.9), we get

$$t+\upsilon q(z)+\tau\alpha q(z)^2+\gamma\frac{zq'(z)}{q(z)} \prec t+\upsilon\mathcal{T}(z)+\tau\alpha\mathcal{T}(z)^2+\gamma\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}. \tag{4.10}$$

we will use Lemma 2.5, to prove our result. Therefore, we get $\mathcal{T}\prec q$ and q is the best subordinant. The proof is complete. □

5 Sandwich Results

We arrive at the next sandwich theorem by combining theorems 3.1 and 4.1.

Theorem 5.1. Suppose that the functions $q_1(z)$ and $q_2(z)$ is convex univalent in U with $q_1(0)=q_2(0)=1$ and q_2 satisfies (3.1), and suppose that $\eta>0$ and $Re\{\zeta\}>0$. Let $f\in M$ satisfies:

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \in \mathcal{H}[1,1]\bigcap Q$$

and

$$\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta}\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right)$$

be univalent in U. If

$$q_1(z)+\frac{\zeta}{\eta}zq_1'(z) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} + \zeta(\tau y+\mu+1)\left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \times\left(\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}-1\right) \prec q_2(z)+\frac{\zeta}{\eta}zq_2'(z),$$

then

$$q_1(z) \prec \left[\frac{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}{z}\right]^{\eta} \prec q_2(z),$$

and q_1 and q_2 respectively the best subordinant and the best dominant.

We arrive at the next sandwich theorem by combining theorems 3.4 and 4.3.

Theorem 5.2. Suppose that the functions $q_1(z)$ and $q_2(z)$ are convex univalent in U with $q_1(0) = q_2(0) = 1$, and let $q_1(z)$ satisfies 4.5 and $q_2(z)$ satisfies (3.6). Assume that $f \in M$ satisfies:

$$\left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \in \mathcal{H}[1,1]\bigcap Q.$$

If

$$t+\upsilon q_1(z)+\tau\alpha q_1(z)^2+\gamma\frac{zq_1'(z)}{q_1(z)} \prec \Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z) \prec t+\upsilon q_2(z)+\tau\alpha q_2(z)^2+\gamma\frac{zq_2'(z)}{q_2(z)}$$

such that $\Psi(\eta,\rho,\tau,\mu,y,n,\upsilon;z) \prec t+\upsilon q_2(z)+\tau\alpha q_2(z)^2+\gamma\frac{zq_2'(z)}{q_2(z)}$ is univalent in U, and given by (3.8). Then

$$q_1(z) \prec \left[\frac{H_{\sigma-1,\rho,\tau,\mu,y,n}f(z)}{H_{\sigma,\rho,\tau,\mu,y,n}f(z)}\right]^{\eta} \prec q_2(z)$$

and q_1 and q_2 respectively the best subordinant and the best dominant.

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