

On cubic convex functions and applications in information theory

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Abstract

In this paper, we introduce the cubic convex function and investigate Jensen type inequality, Fejér-Hermite-Hadamard type inequality and Mercer type inequality for cubic convex functions. Also, we give some applications in means and information theory by applying those inequalities.

Keywords: cubic convex function, entropy, Fejér-Hermite-Hadamard inequality, Jensen's inequality, Mercer inequality

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1 Introduction

Let I be an interval in \mathbb{R} . A function $\varphi : I \rightarrow \mathbb{R}$ is named convex if

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) \quad (1.1)$$

for all $x, y \in I$ and $0 \leq t \leq 1$.

Jensen's inequality [4], Hermite-Hadamard inequality [3] and Mercer inequality [5] for convex functions are some of the best known used inequalities in various fields of optimization and mathematics.

In [2], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality.

Be reason of the extensive applications of the Jensen's, Hermite-Hadamard and Mercer inequalities, some authors generalized their works via mappings of different kinds. For example, we can refer to [1, 6, 13] and the references therein.

The concept of convexity is important in information theory [9] and Shannon entropy [7]-[12].

The aim of this article is to introduce cubic convex function and establish some new Jensen, Fejér-Hermite-Hadamard and Mercer type inequality for cubic convex functions. Next, applying those inequalities in analysis and information theory to obtain a new bound for Shannon entropy.

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2 Jensen inequality for cubic convex functions

Definition 2.1. Let I be an interval in \mathbb{R} . A function $\varphi : I \rightarrow \mathbb{R}$ is named cubic convex if the following inequality

$$\varphi\left(\sqrt[3]{tx^3 + (1-t)y^3}\right) \leq t\varphi(x) + (1-t)\varphi(y) \quad (2.1)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Example 2.2. Let a be a positive real number and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ define by $\varphi(x) = ax^3$. Then φ is cubic convex but is not convex.

Example 2.3. Let a be a positive real number and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ define by $\varphi(x) = ax$. Then φ is convex but is not cubic convex because for $x = -1, y = 2$ and $t = \frac{1}{2}$ we have $\sqrt[3]{\frac{7}{2}} > \frac{1}{2}$.

Example 2.4. Let $\varphi : (0, \infty) \rightarrow \mathbb{R}$ define by $\varphi(x) = -\log(x)$. Then φ is convex and cubic convex function.

Remark 2.5. Let φ be an increasing function on $[a, b]$. Then

1. if $a < b \leq 0$ and φ is convex on $[a, b]$, then φ is cubic convex on $[a, b]$,
2. if $0 \leq a < b$ and φ is cubic convex on $[a, b]$, then φ is convex on $[a, b]$.

Lemma 2.6. Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a cubic convex function. If $a \leq u < v < w \leq b$. Then

1. $\frac{\varphi(v) - \varphi(u)}{2} \leq \varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{u^3 + w^3}{2}}\right)$.
2. $\frac{\varphi(w) - \varphi(v)}{2} \geq \varphi\left(\sqrt[3]{\frac{u^3 + w^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{u^3 + v^3}{2}}\right)$.

Proof . Let $a \leq u < v < w \leq b$.

1. Since $u^3 < v^3 < \frac{v^3 + w^3}{2} < w^3$, there exist $t, s \in [0, 1]$, $t + s = 1$ such that

$$v = \sqrt[3]{tu^3 + s\frac{v^3 + w^3}{2}}.$$

So

$$\begin{aligned} & \frac{\varphi(u) - \varphi(v)}{2} + \varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) \\ &= \frac{1}{2} \left[\varphi(u) - \varphi\left(\sqrt[3]{tu^3 + s\frac{v^3 + w^3}{2}}\right) \right] + \varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) \\ &\geq \frac{1}{2} \left[\varphi(u) - \left(t\varphi(u) + s\varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) \right) \right] + \varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) \\ &= \frac{s}{2}\varphi(u) + \left(1 - \frac{s}{2}\right) \varphi\left(\sqrt[3]{\frac{v^3 + w^3}{2}}\right) \\ &\geq \varphi\left(\sqrt[3]{\frac{s}{2}u^3 + \left(1 - \frac{s}{2}\right)\frac{v^3 + w^3}{2}}\right) \\ &= \varphi\left(\sqrt[3]{\frac{s}{2}u^3 + \frac{v^3 + w^3}{2} - \frac{1}{2}(v^3 - tu^3)}\right) = \varphi\left(\sqrt[3]{\frac{u^3 + w^3}{2}}\right). \end{aligned}$$

2. Since $u^3 < \frac{u^3 + v^3}{2} < v^3 < w^3$, there exist $t, s \in [0, 1]$, $t + s = 1$ such that $v^3 = t\frac{u^3 + v^3}{2} + sw^3$. Thus $v = \sqrt[3]{t\frac{u^3 + v^3}{2} + sw^3}$ and the proof is similar to the proof of part (1).

This completes the proof. \square

Theorem 2.7. Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a cubic convex function and let

$$\Delta(p, q) := \varphi(p) + \varphi(q) - 2\varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right)$$

for all $p, q \in [a, b]$. Then

$$\max_{p, q} \Delta(p, q) = \Delta(a, b).$$

Proof . Setting $u = a, v = p$ and $w = b$ in the part (i) of Lemma 2.6 and $u = p, v = q$ and $w = b$ in the part (ii) of Lemma 2.6, we get

$$\frac{\varphi(p) - \varphi(a)}{2} \leq \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{a^3 + b^3}{2}}\right), \quad (2.2)$$

$$\frac{\varphi(b) - \varphi(q)}{2} \geq \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right), \quad (2.3)$$

respectively.

From (2.3) we arrive at

$$\frac{\varphi(q) - \varphi(b)}{2} \leq \varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right). \quad (2.4)$$

Adding (2.2) from (2.4), we obtain the desired result. \square

In the following theorem we get Jensen type inequality for cubic convex functions.

Theorem 2.8. Let $\varphi : I \rightarrow \mathbb{R}$ be a cubic convex function, let $x_i \in I$, and let $t_i \geq 0$ for all $i = 1, \dots, n$, with $\sum_{i=1}^n t_i = 1$. Then

$$\varphi\left(\sqrt[3]{\sum_{i=1}^n t_i x_i^3}\right) \leq \sum_{i=1}^n t_i \varphi(x_i). \quad (2.5)$$

Proof . The proof can be done by using method induction. For $n = 2$ (2.5) is holds.

To prove the step of induction for $n \geq 2$, suppose that (2.5) satisfies for all convex combinations having less than or equal to n elements. Without loss of generality, we assume that $t_n + t_{n+1} \neq 0$. Therefore,

$$\begin{aligned} \varphi\left(\sqrt[3]{\sum_{i=1}^{n+1} t_i x_i^3}\right) &= \varphi\left(\sqrt[3]{\sum_{i=1}^{n-1} t_i x_i^3 + (t_n + t_{n+1}) \times \frac{t_n x_n^3 + t_{n+1} x_{n+1}^3}{t_n + t_{n+1}}}\right) \\ &\leq \sum_{i=1}^{n-1} t_i \varphi(x_i) + (t_n + t_{n+1}) \varphi\left(\sqrt[3]{\frac{t_n x_n^3 + t_{n+1} x_{n+1}^3}{t_n + t_{n+1}}}\right) \\ &\leq \sum_{i=1}^{n-1} t_i \varphi(x_i) + (t_n + t_{n+1}) \left(\frac{t_n}{t_n + t_{n+1}} \varphi(x_n) + \frac{t_{n+1}}{t_n + t_{n+1}} \varphi(x_{n+1})\right) \\ &= \sum_{i=1}^{n+1} t_i \varphi(x_i). \end{aligned}$$

This completes the proof. \square

Theorem 2.9. Let $\varphi : I \rightarrow \mathbb{R}$ be a cubic convex function, let $x_i \in I$, and let $t_i \geq 0$ for all $i = 1, \dots, n$, with $\sum_{i=1}^n t_i = 1$. Then

$$\begin{aligned} \mathcal{J}\mathcal{C}_\varphi(\mathbf{t}, \mathbf{x}) &:= \sum_{i=1}^n t_i \varphi(x_i) - \varphi \left(\sqrt[3]{\sum_{i=1}^n t_i x_i^3} \right) \\ &\geq \max_{1 \leq r < s \leq n} \left\{ t_r \varphi(x_r) + t_s \varphi(x_s) - (t_r + t_s) \varphi \left(\sqrt[3]{\frac{t_r x_r^3 + t_s x_s^3}{t_r + t_s}} \right) \right\} \end{aligned}$$

where $\mathbf{x} = \{x_i\}$, $\mathbf{t} = \{t_i\}$.

Proof . Suppose that $r, s \in \{1, \dots, n\}$ are arbitrary. So

$$\begin{aligned} \varphi \left(\sqrt[3]{\sum_{i=1}^n t_i x_i^3} \right) &= \varphi \left(\sqrt[3]{\sum_{i \neq r, s} t_i x_i^3 + (t_r + t_s) \frac{t_r x_r^3 + t_s x_s^3}{t_r + t_s}} \right) \\ &\leq \sum_{i \neq r, s} t_i \varphi(x_i) + (t_r + t_s) \varphi \left(\sqrt[3]{\frac{t_r x_r^3 + t_s x_s^3}{t_r + t_s}} \right). \end{aligned}$$

Therefore,

$$\mathcal{J}\mathcal{C}_\varphi(\mathbf{t}, \mathbf{x}) \geq t_r \varphi(x_r) + t_s \varphi(x_s) - (t_r + t_s) \varphi \left(\sqrt[3]{\frac{t_r x_r^3 + t_s x_s^3}{t_r + t_s}} \right).$$

This completes the proof. \square

Theorem 2.10. Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a cubic convex function, let $\mathbf{x} := \{x_i\} \in I$, and let $\mathbf{t} := \{t_i\}$, $t_i \geq 0$ for all $i = 1, \dots, n$, with $\sum_{i=1}^n t_i = 1$ and $\lambda, \mu \geq 0$, $\lambda + \mu = 1$. Then

$$\mathcal{J}\mathcal{C}_\varphi(\mathbf{t}, \mathbf{x}) \leq \max_{\lambda} \left[\lambda \varphi(a) + \mu \varphi(b) - \varphi \left(\sqrt[3]{\lambda a^3 + \mu b^3} \right) \right] := \Lambda_\varphi(a, b). \quad (2.6)$$

Proof . Let $x_i \in [a, b]$. Then there exists a $\gamma_i \in [0, 1]$ such that $x_i^3 = \gamma_i a^3 + (1 - \gamma_i) b^3$ for all $i = 1, 2, \dots, n$. So

$$\begin{aligned} &\sum_{i=1}^n t_i \varphi(x_i) - \varphi \left(\sqrt[3]{\sum_{i=1}^n t_i x_i^3} \right) \\ &= \sum_{i=1}^n t_i \varphi \left(\sqrt[3]{\gamma_i a^3 + (1 - \gamma_i) b^3} \right) - \varphi \left(\sqrt[3]{\sum_{i=1}^n t_i (\gamma_i a^3 + (1 - \gamma_i) b^3)} \right) \\ &\leq \sum_{i=1}^n t_i (\gamma_i \varphi(a) + (1 - \gamma_i) \varphi(b)) - \varphi \left(\sqrt[3]{a^3 \sum_{i=1}^n t_i \gamma_i + b^3 \sum_{i=1}^n t_i (1 - \gamma_i)} \right) \\ &= \lambda \varphi(a) + \mu \varphi(b) - \varphi \left(\sqrt[3]{\lambda a^3 + \mu b^3} \right), \end{aligned}$$

where $\lambda = \sum_{i=1}^n t_i \gamma_i$ and $\mu = 1 - \lambda$. Hence

$$\mathcal{J}\mathcal{C}_\varphi(\mathbf{t}, \mathbf{x}) \leq \max_{\lambda} \left[\lambda \varphi(a) + \mu \varphi(b) - \varphi \left(\sqrt[3]{\lambda a^3 + \mu b^3} \right) \right], \quad (2.7)$$

which ends the proof. \square

In the following theorem we obtain Fejér-Hermite-Hadamard type inequality for cubic convex functions.

Theorem 2.11. Let φ be a real-valued and cubic convex function on $[a, b]$ and $\omega : [a^3, b^3] \rightarrow \mathbb{R}$ be a nonnegative, integrable and symmetric with respect to $\frac{a^3+b^3}{2}$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \int_{a^3}^{b^3} \omega(x) dx \leq \int_{a^3}^{b^3} \varphi(\sqrt[3]{x}) \omega(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \int_{a^3}^{b^3} \omega(x) dx.$$

Proof . Assume that $0 \leq t \leq 1$, we have

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \leq \frac{\varphi\left(\sqrt[3]{ta^3+(1-t)b^3}\right) + \varphi\left(\sqrt[3]{(1-t)a^3+tb^3}\right)}{2} \leq \frac{\varphi(a) + \varphi(b)}{2}.$$

Multiply the above inequalities by $\omega(ta^3 + (1-t)b^3)$ and integrating with respect to t over $[0, 1]$, we arrive at

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \int_{a^3}^{b^3} \omega(x) dx \leq \int_{a^3}^{b^3} \varphi(\sqrt[3]{x}) \omega(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \int_{a^3}^{b^3} \omega(x) dx,$$

and the proof is complete. \square

If we set $\omega(x) = 1$ in Theorem 2.11, then it implies that the Hermite–Hadamard type inequality for cubic convex functions.

Corollary 2.12. Let φ be a real-valued and cubic convex function on $[a, b]$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \leq \frac{1}{b^3-a^3} \int_{a^3}^{b^3} \varphi(\sqrt[3]{x}) dx \leq \frac{\varphi(a) + \varphi(b)}{2}. \tag{2.8}$$

Now, we show that Hermite-Hadamard inequality has a self-improving property.

Corollary 2.13. Let φ be a real-valued and cubic convex function on $[a, b]$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \leq \frac{1}{b^3-a^3} \int_{a^3}^{b^3} \varphi(\sqrt[3]{x}) dx \leq \frac{1}{2} \left[\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) + \frac{\varphi(a) + \varphi(b)}{2} \right]. \tag{2.9}$$

Proof . Using twice the right part of (2.8) yield

$$\frac{2}{b^3-a^3} \int_{a^3}^{\frac{a^3+b^3}{2}} \varphi(\sqrt[3]{x}) dx \leq \frac{1}{2} \left[\varphi(a) + \varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \right], \tag{2.10}$$

and

$$\frac{2}{b^3-a^3} \int_{\frac{a^3+b^3}{2}}^{b^3} \varphi(\sqrt[3]{x}) dx \leq \frac{1}{2} \left[\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) + \varphi(b) \right]. \tag{2.11}$$

From (2.10) and (2.11) clearly leads to (2.9). \square

Theorem 2.14. Let φ be a cubic convex function on $[a, b]$. Then

$$\varphi\left(\sqrt[3]{a^3+b^3-\sum_{i=1}^n t_i x_i^3}\right) \leq \varphi(a) + \varphi(b) - \sum_{i=1}^n t_i \varphi(x_i), \tag{2.12}$$

where $x_i \in [a, b]$ and $t_i \geq 0$ for all $i = 1, \dots, n$ with $\sum_{i=1}^n t_i = 1$.

Proof . Assume that $x \in [a, b]$. Then there exists $t \in [0, 1]$ such that $x^3 = ta^3 + (1-t)b^3$. Thus

$$\begin{aligned} \varphi(x) + \varphi\left(\sqrt[3]{a^3 + b^3 - x^3}\right) &= \varphi\left(\sqrt[3]{ta^3 + (1-t)b^3}\right) + \varphi\left(\sqrt[3]{(1-t)a^3 + tb^3}\right) \\ &\leq \varphi(a) + \varphi(b). \end{aligned}$$

Hence

$$\varphi(x) + \varphi\left(\sqrt[3]{a^3 + b^3 - x^3}\right) \leq \varphi(a) + \varphi(b),$$

for all $x \in [a, b]$. So

$$\begin{aligned} \varphi\left(\sqrt[3]{a^3 + b^3 - \sum_{i=1}^n t_i x_i^3}\right) &= \varphi\left(\sqrt[3]{\sum_{i=1}^n t_i (a^3 + b^3 - x_i^3)}\right) \\ &\leq \sum_{i=1}^n t_i \varphi\left(\sqrt[3]{a^3 + b^3 - x_i^3}\right) \leq \varphi(a) + \varphi(b) - \sum_{i=1}^n t_i \varphi(x_i). \end{aligned}$$

This completes the proof of the inequality (2.12). \square

3 Applications

Let $\mathbf{x} = \{x_i\} \subseteq [a, b]$, $\mathbf{t} = \{t_i\}$ and $\sum_{i=1}^n t_i = 1$. Define

$$A(\mathbf{t}, \mathbf{x}^3) = \sum_{i=1}^n t_i x_i^3, \quad \tilde{A}(\mathbf{t}, \mathbf{x}^3) := a + b - A(\mathbf{t}, \mathbf{x}^3),$$

$$G(\mathbf{t}, \mathbf{x}) = \prod_{i=1}^n x_i^{t_i} \quad \text{and} \quad \tilde{G}(\mathbf{t}, \mathbf{x}) = \frac{ab}{G(\mathbf{t}, \mathbf{x})}.$$

Definition 3.1. For probability distribution \mathbf{t} the Shannon entropy of \mathbf{t} is defined by

$$H(\mathbf{t}) := - \sum_{i=1}^n t_i \log(t_i).$$

Proposition 3.2. Let $0 < a \leq x_i \leq b$. Then

$$a \sqrt[3]{A(\mathbf{t}, \mathbf{x}^3)} \leq b G(\mathbf{t}, \mathbf{x}).$$

Proof . The inequality follows from Theorem 2.10 with $\varphi(x) := -\log(x)$ and

$$\log\left(\frac{\sqrt[3]{\lambda a^3 + \mu b^3}}{a^\lambda b^\mu}\right) \leq \log\left(\frac{b}{a}\right)$$

where $0 \leq \lambda \leq 1$ and $\mu = 1 - \lambda$. \square

Proposition 3.3. Let $\eta := \max\{t_i : 1 \leq i \leq n\}$ and $\theta := \min\{t_i : 1 \leq i \leq n\}$. Then

$$0 \leq \log n - H(\mathbf{t}) \leq \log\left(\frac{\eta}{\theta}\right).$$

Proof . The inequality follows from Theorem 2.10 with $\varphi(x) := -\log(x)$ and $x_i := \frac{1}{\sqrt[3]{t_i}}$ for all $i = 1, \dots, n$. \square

Proposition 3.4. Let $0 < a \leq x_i \leq b$. Then

$$\tilde{G}(\mathbf{t}, \mathbf{x}) \leq \sqrt[3]{\tilde{A}(\mathbf{t}, \mathbf{x}^3)}.$$

Proof . The inequality follows from Theorem 2.14 with $\varphi(x) := -\log(x)$. \square

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