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Common coupled fixed point theorem on fuzzy bipolar metric spaces

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Abstract

We prove a common coupled fixed point theorem on fuzzy bipolar metric spaces. An application of our key results is given to solve a system of integral equations. Our results generalize and expand the literature's well-known results.

Keywords: Fuzzy bipolar metric space, common coupled fixed point, w-compatible
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1 Introduction

Zadeh [16] introduced the notion of fuzzy sets. Using this concept of fuzziness, Kramosil and Michalek [9] introduced the fuzzy metric spaces. Subsequently George and Veeramani [5] further modified the idea of fuzzy metric spaces. Grabeic [6] and Azam et al. [3, 2] extend the well known Banach fixed point theorem to fuzzy metric spaces in the sense of Karamosil and Michalek [9]and also refer [1]. After that, Gregori and Sapena [7] extended the fuzzy Banach contraction principle to fuzzy metric space in the sense George and Veeramaniâ [5]. Recently, Mutlu and Gurdal [12] presented bipolar metric spaces by generalizing metric spaces and proved some fixed point results and also refer [17, 11, 10]. Bartwal, Dimri and Prasad, [4] extended it to fuzzy bipolar metric space and obtained several fixed point theorems. Afterward Mutlu, Ozkan and Gurdal [13] studied coupled fixed point on bipolar metric space and also refer [15, 8, 14].

The aim of this paper is to prove common coupled fixed point theorem on fuzzy bipolar metric space with an application to solve a system of integral equations.

2 Preliminaries

Now we present some basic definitions:

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Definition 2.1. [4] Let Ω and Υ be two non-void sets. We say that quadruple $(\Omega, \Upsilon, \Gamma, *)$ fuzzy bipolar metric space if $*$ is continuous ϱ-norm and Γ is a fuzzy set on $\Omega \times \Upsilon \times (0, \infty)$, fulfill the following conditions for all $\varrho, \omega, \mathfrak{r} > 0$:

1. $\Gamma(\vartheta, \eta, \varrho) > 0$ for all $(\vartheta, \eta) \in \Omega \times \Upsilon$;
2. $\Gamma(\vartheta, \eta, \varrho) = 1$ iff $\vartheta = \eta$ for $\vartheta \in \Omega$ and $\eta \in \Upsilon$;
3. $\Gamma(\vartheta, \eta, \varrho) = \Gamma(\eta, \vartheta, \varrho)$ for all $\vartheta, \eta \in \Omega \cap \Upsilon$;
4. $\Gamma(\vartheta_1, \eta_2, \varrho + \omega + \mathfrak{r}) \geq \Gamma(\vartheta_1, \eta_1, \varrho) * \Gamma(\vartheta_2, \eta_1, \omega) * \Gamma(\vartheta_2, \eta_2, \mathfrak{r})$ for all $\vartheta_1, \vartheta_2 \in \Omega$ and $\eta_1, \eta_2 \in \Upsilon$;
5. $\Gamma(\vartheta, \eta, .) : [0, \infty) \longrightarrow [0, 1]$ is left continuous;
6. $\Gamma(\vartheta, \eta, .)$ is non-decreasing for all $\vartheta \in \Omega$ and $\eta \in \Upsilon$.

Definition 2.2. [4] Let $(\Omega, \Upsilon, \Gamma, *)$ be a fuzzy bipolar metric space. A point $\eta \in \Omega \cup \Upsilon$ is called a left point if $\eta \in \Omega$, a right point if $\eta \in \Upsilon$ and a central point if it is both left and right point. Similarly a sequence $\{\vartheta_\alpha\}$ on the set Ω is called a left sequence and a sequence $\{\eta_\alpha\}$ on Υ is called a right sequence. In a fuzzy bipolar metric space, a left or a right sequence is called simply a sequence. A sequence $\{\eta_\alpha\}$ is said to be convergent to a point η, iff $\{\eta_\alpha\}$ is a left sequence, η is a right point and $\lim_{\alpha \to \infty} \Gamma(\eta_\alpha, \eta, \varrho) = 1$. A bisequence $(\{\vartheta_\alpha\}, \{\eta_\alpha\})$ on $(\Omega, \Upsilon, \Gamma, *)$ is a sequence on the set $\Omega \times \Upsilon$. If the sequence $\{\vartheta_\alpha\}$ and $\{\eta_\alpha\}$ are convergent, then the bisequence $(\{\vartheta_\alpha\}, \{\eta_\alpha\})$ is said to be convergent, and if $\{\vartheta_\alpha\}$ and $\{\eta_\alpha\}$ converge to a common point, then $(\{\vartheta_\alpha\}, \{\eta_\alpha\})$ is called biconvergent. A bisequence $(\{\vartheta_\alpha\}, \{\eta_\alpha\})$ is a Cauchy bisequence, if $\lim_{\alpha, \beta \to \infty} \Gamma(\vartheta_\alpha, \eta_\beta, \varrho) = 1$. In a fuzzy bipolar metric space, every convergent Cauchy bisequence is biconvergent. A fuzzy bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.3. Let $(\Omega, \Upsilon, \Gamma, *)$ be a fuzzy bipolar metric space, $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ be two functions. An element $(\vartheta, \eta) \in \Omega^2 \cup \Upsilon^2$ is called a coupled coincidence point of Φ and $\mathfrak{g}$ if $\Phi(\vartheta, \eta) = \mathfrak{g}\vartheta$, $\Phi(\eta, \vartheta) = \mathfrak{g}\eta$.

Definition 2.4. Let $(\Omega, \Upsilon, \Gamma, *)$ be a fuzzy bipolar metric space, $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ be two functions. An element $(\vartheta, \eta) \in \Omega^2 \cup \Upsilon^2$ is called a common coupled fixed point of Φ and $\mathfrak{g}$ if $\Phi(\vartheta, \eta) = \mathfrak{g}\vartheta = \vartheta$, $\Phi(\eta, \vartheta) = \mathfrak{g}\eta = \eta$.

Definition 2.5. Let $(\Omega, \Upsilon, \Gamma, *)$ be a fuzzy bipolar metric space, $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ be two functions. An element $(\vartheta, \eta) \in \Omega^2 \cup \Upsilon^2$ is called a common coupled fixed point of Φ and $\mathfrak{g}$ if $\Phi(\vartheta, \eta) = \mathfrak{g}\vartheta = \vartheta$, $\Phi(\eta, \vartheta) = \mathfrak{g}\eta = \eta$.

Definition 2.6. Let $(\Omega, \Upsilon, \Gamma, *)$ be a fuzzy bipolar metric space, $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ be two functions are called w-compatible if $\mathfrak{g}(\Phi(\vartheta, \eta)) = \Phi(\mathfrak{g}\vartheta, \mathfrak{g}\eta)$ and $\mathfrak{g}(\Phi(\eta, \vartheta)) = \Phi(\mathfrak{g}\eta, \mathfrak{g}\vartheta)$, whenever $\Phi(\vartheta, \eta) = \mathfrak{g}\vartheta$ and $\Phi(\eta, \vartheta) = \mathfrak{g}\eta$.

Example 2.7. Let $\Omega = [0, 1]$, $\Upsilon = \{0\} \cup \mathbb{N} - \{1\}$. Define $\Gamma(\vartheta, \eta, \varrho) = e^{-\frac{(\vartheta - \eta)}{\varrho}}$ for all $\varrho > 0$ and $\vartheta \in \Omega$ and $\eta \in \Upsilon$. Clearly, $(\Omega, \Upsilon, \Gamma, *)$ is a complete fuzzy bipolar metric space, where $*$ is a continuous ϱ-norm defined as $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Define $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ defined by

$$\Phi(\vartheta, \eta) = \begin{cases} \frac{\vartheta + \eta}{4}, & \text{if } \vartheta, \eta \in \Omega^2, \\ 0, & \text{if } \vartheta, \eta \in \Upsilon^2, \end{cases}$$

for all $\vartheta, \eta \in \Omega^2 \cup \Upsilon^2$ and

$$\mathfrak{g}(\vartheta) = \begin{cases} \vartheta, & \text{if } \vartheta, \eta \in \Omega, \\ 0, & \text{if } \vartheta, \eta \in \Upsilon, \end{cases}$$

for all $\vartheta, \eta \in \Omega \cup \Upsilon$.

Motivated by Mutlu, Ozkan and Gurdal [13], we prove common coupled fixed point theorem on fuzzy bipolar metric space with an application.

3 Main Results

Theorem 3.1. Let $\mathfrak{a} * \mathfrak{b} \geq \mathfrak{a}\mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in [0,1]$ and $(\Omega, \Upsilon, \Gamma, *)$ be a complete fuzzy bipolar metric space. Let $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ be two functions such that

$$\Gamma(\Phi(\vartheta, \eta), \Phi(\mathfrak{u}, \mathfrak{v}), \mathfrak{k}\varrho) \geq \Gamma(\mathfrak{g}\vartheta, \mathfrak{g}\mathfrak{u}, \varrho)^{\frac{1}{2}} * \Gamma(\mathfrak{g}\eta, \mathfrak{g}\mathfrak{v}, \varrho)^{\frac{1}{2}} \tag{3.1}$$

for all $\vartheta, \eta \in \Omega$ and $\mathfrak{u}, \mathfrak{v} \in \Upsilon$, where $0 < \mathfrak{k} < 1$, $\Phi(\Omega^2 \cup \Upsilon^2) \subseteq \mathfrak{g}(\Omega \cup \Upsilon)$, $\mathfrak{g}(\Omega \cup \Upsilon)$ is complete and the pair $(\Phi, \mathfrak{g})$ is compatible. Then the mappings Φ and $\mathfrak{g}$ have unique common coupled fixed point.

Proof . Let $\vartheta_0, \eta_0 \in \Omega$ and $\mathfrak{u}_0, \mathfrak{v}_0 \in \Upsilon$. Since $\Phi(\Omega^2 \times \Upsilon^2) \subseteq \mathfrak{g}(\Omega \cup \Upsilon)$, we can construct bisequence $(\{\vartheta_\alpha\}, \{\mathfrak{u}_\alpha\})$, $(\{\eta_\alpha\}, \{\mathfrak{v}_\alpha\})$ such that

$$\begin{aligned} \mathfrak{g}\vartheta_{\alpha+1} &= \Phi(\vartheta_\alpha, \eta_\alpha) \text{ and } \mathfrak{g}\eta_{\alpha+1} = \Phi(\eta_\alpha, \vartheta_\alpha), \\ \mathfrak{g}\mathfrak{u}_{\alpha+1} &= \Phi(\mathfrak{u}_\alpha, \mathfrak{v}_\alpha) \text{ and } \mathfrak{g}\mathfrak{v}_{\alpha+1} = \Phi(\mathfrak{v}_\alpha, \mathfrak{u}_\alpha), \end{aligned} \tag{3.2}$$

for all $\alpha \geq 0$. Now, we denote

$$\delta_{\alpha-1}(\varrho) = \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_\alpha, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_\alpha, \varrho)\Big)^{\frac{1}{2}}.$$

From (3.1) and (3.2), we have

$$\begin{aligned} \Gamma(\mathfrak{g}\vartheta_\alpha, \mathfrak{g}\mathfrak{u}_{\alpha+1}, \mathfrak{k}\varrho) &= \Gamma(\Phi(\vartheta_{\alpha-1}, \eta_{\alpha-1}), \Phi(\mathfrak{u}_\alpha, \mathfrak{v}_\alpha), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_\alpha, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_\alpha, \varrho)\Big)^{\frac{1}{2}} \\ &= \delta_{\alpha-1}(\varrho). \end{aligned} \tag{3.3}$$

Similarly, from (3.1) and (3.2),

$$\begin{aligned} \Gamma(\mathfrak{g}\eta_\alpha, \mathfrak{g}\mathfrak{v}_{\alpha+1}, \mathfrak{k}\varrho) &= \Gamma(\Phi(\eta_{\alpha-1}, \vartheta_{\alpha-1}), \Phi(\mathfrak{v}_\alpha, \mathfrak{u}_\alpha), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_\alpha, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_\alpha, \varrho)\Big)^{\frac{1}{2}} \\ &= \delta_{\alpha-1}(\varrho). \end{aligned} \tag{3.4}$$

Adding by ϱ - norm $*$ (3.3) and (3.4), we obtain

$$\delta_\alpha(\mathfrak{k}\varrho) \geq \delta_{\alpha-1}(\varrho) * \delta_{\alpha-1}(\varrho) \geq \delta_{\alpha-1}(\varrho). \tag{3.5}$$

Thus, we have

$$\delta_\alpha(\varrho) \geq \delta_{\alpha-1}\left(\frac{\varrho}{\mathfrak{k}}\right) \geq \ldots \geq \delta_0\left(\frac{\varrho}{\mathfrak{k}^\alpha}\right). \tag{3.6}$$

Since $\lim\limits_{\alpha\to\infty} \delta_0\left(\frac{\varrho}{\mathfrak{k}^\alpha}\right) = 1$, for all $\varrho > 0$, we have

$$\lim_{\alpha\to\infty} \delta_\alpha(\varrho) = 1, \text{ for all } \varrho > 0.$$

On the other hand, we denote

$$\gamma_{\alpha-1}(\varrho) = \Big(\Gamma(\mathfrak{g}\vartheta_\alpha, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_\alpha, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}}.$$

From (3.1) and (3.2), we have

$$\begin{aligned} \Gamma(\mathfrak{g}\vartheta_{\alpha+1}, \mathfrak{g}\mathfrak{u}_\alpha, \mathfrak{k}\varrho) &= \Gamma(\Phi(\vartheta_\alpha, \eta_\alpha), \Phi(\mathfrak{u}_{\alpha-1}, \mathfrak{v}_{\alpha-1}), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\vartheta_\alpha, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_\alpha, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} \\ &= \gamma_{\alpha-1}(\varrho). \end{aligned} \tag{3.7}$$

Similarly, from (3.1) and (3.2),

$$\begin{aligned}\Gamma(\mathfrak{g}\eta_{\alpha+1}, \mathfrak{g}\mathfrak{v}_{\alpha}, \mathfrak{k}\varrho) &= \Gamma(\Phi(\eta_{\alpha}, \vartheta_{\alpha}), \Phi(\mathfrak{v}_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\eta_{\alpha}, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha}, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} \\ &= \gamma_{\alpha-1}(\varrho). \end{aligned} \tag{3.8}$$

Adding by ϱ - norm $*$ (3.7) and (3.8), we obtain

$$\gamma_{\alpha}(\mathfrak{k}\varrho) \geq \gamma_{\alpha-1}(\varrho) * \gamma_{\alpha-1}(\varrho) \geq \gamma_{\alpha-1}(\varrho). \tag{3.9}$$

Thus, we have

$$\gamma_{\alpha}(\varrho) \geq \gamma_{\alpha-1}\left(\frac{\varrho}{\mathfrak{k}}\right) \geq \ldots \geq \gamma_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha}}\right). \tag{3.10}$$

Since $\lim_{\alpha\to\infty} \gamma_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha}}\right) = 1$, for all $\varrho > 0$, we have

$$\lim_{\alpha\to\infty} \gamma_{\alpha}(\varrho) = 1, \text{ for all } \varrho > 0.$$

Moreover,

$$\mu_{\alpha-1}(\varrho) = \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}}.$$

From (3.1) and (3.2), we have

$$\begin{aligned}\Gamma(\mathfrak{g}\vartheta_{\alpha}, \mathfrak{g}\mathfrak{u}_{\alpha}, \mathfrak{k}\varrho) &= \Gamma(\Phi(\vartheta_{\alpha-1}, \eta_{\alpha-1}), \Phi(\mathfrak{u}_{\alpha-1}, \mathfrak{v}_{\alpha-1}), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} \\ &= \mu_{\alpha-1}(\varrho). \end{aligned} \tag{3.11}$$

Similarly, from (3.1) and (3.2),

$$\begin{aligned}\Gamma(\mathfrak{g}\eta_{\alpha}, \mathfrak{g}\mathfrak{v}_{\alpha}, \mathfrak{k}\varrho) &= \Gamma(\Phi(\eta_{\alpha-1}, \vartheta_{\alpha-1}), \Phi(\mathfrak{v}_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \mathfrak{k}\varrho) \\ &\geq \Big(\Gamma(\mathfrak{g}\eta_{\alpha-1}, \mathfrak{g}\mathfrak{v}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} * \Big(\Gamma(\mathfrak{g}\vartheta_{\alpha-1}, \mathfrak{g}\mathfrak{u}_{\alpha-1}, \varrho)\Big)^{\frac{1}{2}} \\ &= \mu_{\alpha-1}(\varrho). \end{aligned} \tag{3.12}$$

Adding by ϱ - norm $*$ (3.11) and (3.12), we obtain

$$\mu_{\alpha}(\mathfrak{k}\varrho) \geq \mu_{\alpha-1}(\varrho) * \mu_{\alpha-1}(\varrho) \geq \mu_{\alpha-1}(\varrho).$$

Thus, we have

$$\mu_{\alpha}(\varrho) \geq \mu_{\alpha-1}\left(\frac{\varrho}{\mathfrak{k}}\right) \geq \ldots \geq \mu_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha}}\right). \tag{3.13}$$

Since $\lim_{\alpha\to\infty} \mu_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha}}\right) = 1$ for all $\varrho > 0$, we have

$$\lim_{\alpha\to\infty} \mu_{\alpha}(\varrho) = 1, \text{ for all } \varrho > 0.$$

Using the property 4, we get

$$\begin{aligned}\Gamma(\mathfrak{g}\vartheta_{\alpha}, \mathfrak{g}\mathfrak{u}_{\beta}, \varrho) &\geq \Gamma(\mathfrak{g}\vartheta_{\alpha}, \mathfrak{g}\mathfrak{u}_{\alpha+1}, \frac{\varrho}{3}) * \Gamma(\mathfrak{g}\vartheta_{\alpha+1}, \mathfrak{g}\mathfrak{u}_{\alpha+1}, \frac{\varrho}{3}) * \cdots * \Gamma(\mathfrak{g}\vartheta_{\beta-1}, \mathfrak{g}\mathfrak{u}_{\beta}, \frac{\varrho}{3^{\beta-1}}) \\ \Gamma(\mathfrak{g}\vartheta_{\alpha+1}, \mathfrak{g}\mathfrak{u}_{\alpha+1}, \frac{\varrho}{3}) &\geq \Gamma(\mathfrak{g}\eta_{\alpha}, \mathfrak{g}\mathfrak{v}_{\alpha+1}, \frac{\varrho}{3}) * \Gamma(\mathfrak{g}\eta_{\alpha+1}, \mathfrak{g}\mathfrak{v}_{\alpha+1}, \frac{\varrho}{3}) * \cdots * \Gamma(\mathfrak{g}\eta_{\beta-1}, \mathfrak{g}\mathfrak{v}_{\beta}, \frac{\varrho}{3^{\beta-1}}) \end{aligned} \tag{3.14}$$

and

$$\Gamma(\mathfrak{g}\vartheta_\beta,\mathfrak{gu}_\alpha,\varrho)\geq\Gamma(\mathfrak{g}\vartheta_\beta,\mathfrak{gu}_{\beta-1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\beta-1},\mathfrak{gu}_{\beta-1},\frac{\varrho}{3})*\cdots*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{gu}_\alpha,\frac{\varrho}{3^\alpha})$$
$$\Gamma(\mathfrak{g}\eta_\beta,\mathfrak{gv}_\alpha,\varrho)\geq\Gamma(\mathfrak{g}\eta_\beta,\mathfrak{gv}_{\beta-1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\eta_{\beta-1},\mathfrak{gv}_{\beta-1},\frac{\varrho}{3})*\cdots*\Gamma(\mathfrak{g}\eta_{\alpha+1},\mathfrak{gv}_\alpha,\frac{\varrho}{3^\alpha}),\tag{3.15}$$

for each $\alpha,\beta\in\mathbb{N}$, $\alpha<\beta$. Then, from (3.6), (3.9), (3.13), (3.14) and (3.15), we have

$$\begin{aligned}\Gamma(\mathfrak{g}\vartheta_\alpha,\mathfrak{gu}_\beta,\varrho)*\Gamma(\mathfrak{g}\eta_\alpha,\mathfrak{gv}_\beta,\varrho)&\geq(\Gamma(\mathfrak{g}\vartheta_\alpha,\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_\alpha,\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3}))*(\Gamma(\mathfrak{g}\eta_{\alpha+1},\mathfrak{gv}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\eta_{\alpha+1},\mathfrak{gv}_{\alpha+1},\frac{\varrho}{3}))\\&\quad*\cdots*(\Gamma(\mathfrak{g}\vartheta_{\beta-1},\mathfrak{gu}_\beta,\frac{\varrho}{3^{\beta-1}})*\Gamma(\mathfrak{g}\eta_{\beta-1},\mathfrak{gv}_\beta,\frac{\varrho}{3^{\beta-1}}))\\&\geq\delta_\alpha^2\left(\frac{\varrho}{3}\right)*\mu_{\alpha+1}^2\left(\frac{\varrho}{3}\right)*\delta_{\alpha+1}^2\left(\frac{\varrho}{3}\right)*\cdots*\mu_{\beta-1}^2\left(\frac{\varrho}{3^{\beta-1}}\right)*\delta_{\beta-1}^2\left(\frac{\varrho}{3^{\beta-1}}\right)\\&\geq\delta_0^2\left(\frac{\varrho}{3\mathfrak{k}^\alpha}\right)*\mu_0^2\left(\frac{\varrho}{3\mathfrak{k}^{\alpha+1}}\right)*\cdots*\delta_0^2\left(\frac{\varrho}{3^{\beta-1}\mathfrak{k}^{\beta-1}}\right)\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}\Gamma(\mathfrak{g}\vartheta_\beta,\mathfrak{gu}_\alpha,\varrho)*\Gamma(\mathfrak{g}\eta_\beta,\mathfrak{gv}_\alpha,\varrho)&\geq(\Gamma(\mathfrak{g}\vartheta_\beta,\mathfrak{gu}_{\beta-1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\eta_\beta,\mathfrak{gv}_{\beta-1},\frac{\varrho}{3}))*(\Gamma(\mathfrak{g}\vartheta_{\beta-1},\mathfrak{gu}_{\beta-1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\eta_{\beta-1},\mathfrak{gv}_{\beta-1},\frac{\varrho}{3}))\\&\quad*\cdots*(\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{gu}_\alpha,\frac{\varrho}{3^n})*\Gamma(\mathfrak{g}\eta_{\alpha+1},\mathfrak{gv}_\alpha,\frac{\varrho}{3^n}))\\&\geq\gamma_{\beta-1}^2\left(\frac{\varrho}{3}\right)*\mu_{\beta-1}^2\left(\frac{\varrho}{3}\right)*\gamma_{\beta-2}^2\left(\frac{\varrho}{3}\right)*\cdots*\mu_{\alpha+1}^2\left(\frac{\varrho}{3^{\alpha+1}}\right)*\gamma_\alpha^2\left(\frac{\varrho}{3^\alpha}\right)\\&\geq\gamma_0^2\left(\frac{\varrho}{3\mathfrak{k}^{\beta-1}}\right)*\mu_0^2\left(\frac{\varrho}{3\mathfrak{k}^{\beta-1}}\right)*\cdots*\gamma_0^2\left(\frac{\varrho}{3^\alpha\mathfrak{k}^\alpha}\right).\end{aligned}\tag{3.17}$$

As $\alpha,\beta\to\infty$, we have

$$\lim_{\alpha,\beta\to\infty}(\Gamma(\mathfrak{g}\vartheta_\alpha,\mathfrak{gu}_\beta,\varrho)*\Gamma(\mathfrak{g}\eta_\alpha,\mathfrak{gv}_\beta,\varrho))=1$$

and

$$\lim_{\alpha,\beta\to\infty}(\Gamma(\mathfrak{g}\vartheta_\beta,\mathfrak{gu}_\alpha,\varrho)*\Gamma(\mathfrak{g}\eta_\beta,\mathfrak{gv}_\alpha,\varrho))=1.$$

Therefore $(\{\mathfrak{g}\vartheta_\alpha\},\{\mathfrak{gu}_\alpha\})$ and $(\{\mathfrak{g}\eta_\alpha\},\{\mathfrak{gv}_\alpha\})$ are Cauchy bisequences. Since $\mathfrak{g}(\Omega\cup\Upsilon)$ is a complete subspace of $(\Omega,\Upsilon,\Gamma,*)$, so $\{\mathfrak{g}\vartheta_\alpha\},\{\mathfrak{gu}_\alpha\},\{\mathfrak{g}\eta_\alpha\},\{\mathfrak{gv}_\alpha\}\subseteq g(\Omega\cup\Upsilon)$ are converges in the complete bipolar metric space $(g(\Omega),g(\Upsilon),\Gamma,*)$. Therefore, there exist $\vartheta,\eta\in\mathfrak{g}(\Omega)$ and $\mathfrak{u},\mathfrak{v}\in\mathfrak{g}(\Upsilon)$ such that

$$\lim_{\alpha\to\infty}\mathfrak{g}\vartheta_\alpha=\mathfrak{u},\ \lim_{\alpha\to\infty}\mathfrak{g}\eta_\alpha=\mathfrak{v}$$

and

$$\lim_{\alpha\to\infty}\mathfrak{gu}_\alpha=\vartheta,\ \lim_{\alpha\to\infty}\mathfrak{gv}_\alpha=\eta.$$

Since $\mathfrak{g}:\Omega\cup\Upsilon\to\Omega\cup\Upsilon$ and $\vartheta,\eta\in\mathfrak{g}(\Omega)$, $\mathfrak{u},\mathfrak{v}\in\mathfrak{g}(\Upsilon)$, there exist $\mathfrak{l},\beta\in\Omega,\mathfrak{r},\omega\in\Upsilon$ such that $\mathfrak{gl}=\vartheta$, $\mathfrak{g}\beta=\eta$ and $\mathfrak{gr}=\mathfrak{u}$, $\mathfrak{g}\omega=\mathfrak{v}$. Using the property 4, we get

$$\begin{aligned}\Gamma(\Phi(\mathfrak{l},\beta),\mathfrak{u},\varrho)&\geq\Gamma(\Phi(\mathfrak{l},\beta),\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{u},\frac{\varrho}{3})\\&=\Gamma(\Phi(\mathfrak{l},\beta),\Phi(\mathfrak{u}_\alpha,\mathfrak{v}_\alpha),\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{u},\frac{\varrho}{3})\\&\geq\left(\Gamma(\mathfrak{gl},\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3\mathfrak{k}})\right)^{\frac{1}{2}}*\left(\Gamma(\mathfrak{g}\beta,\mathfrak{gv}_\alpha,\frac{\varrho}{3\mathfrak{k}})\right)^{\frac{1}{2}}*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{gu}_{\alpha+1},\frac{\varrho}{3})*\Gamma(\mathfrak{g}\vartheta_{\alpha+1},\mathfrak{u},\frac{\varrho}{3}).\end{aligned}$$

As $\alpha\to\infty$, we have

$$\lim_{\alpha\to\infty}\Gamma(\Phi(\mathfrak{l},\beta),\mathfrak{u},\varrho)=1.$$

Therefore $\Phi(\mathfrak{l},\beta)=\mathfrak{u}=\mathfrak{g}\mathfrak{r}$. Similarly, we can prove that $\Phi(\beta,\mathfrak{l})=\mathfrak{v}=\mathfrak{g}\omega$, $\Phi(\mathfrak{r},\omega)=\vartheta=\mathfrak{g}\mathfrak{l}$ and $\Phi(\omega,\mathfrak{r})=\eta=\mathfrak{g}\beta$. Since $(\Phi,\mathfrak{g})$ are w-compatible mappings, we have $\Phi(\vartheta,\eta)=\mathfrak{g}\vartheta$, $\Phi(\eta,\vartheta)=\mathfrak{g}\eta$ and $\Phi(\mathfrak{u},\mathfrak{v})=\mathfrak{g}\mathfrak{u}$, $\Phi(\mathfrak{v},\mathfrak{u})=\mathfrak{g}\mathfrak{v}$. Now we show that $\mathfrak{g}\vartheta=\vartheta$, $\mathfrak{g}\eta=\eta$ and $\mathfrak{g}\mathfrak{u}=\mathfrak{u}$, $\mathfrak{g}\mathfrak{v}=\mathfrak{v}$. Now, we denote
$\lambda_\alpha(\varrho)=\left(\Gamma(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{u}_\alpha,\varrho)\right)^{\frac{1}{2}}*\left(\Gamma(\mathfrak{g}\mathfrak{v},\mathfrak{g}\mathfrak{v}_\alpha,\varrho)\right)^{\frac{1}{2}}$. Then

$$\Gamma(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{u}_\alpha,\mathfrak{k}\varrho)=\Gamma(\Phi(\mathfrak{u},\mathfrak{v}),\Phi(\mathfrak{u}_{\alpha-1},\mathfrak{v}_{\alpha-1}),\mathfrak{k}\varrho)\geq\left(\Gamma(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{u}_{\alpha-1},\varrho)\right)^{\frac{1}{2}}*\left(\Gamma(\mathfrak{g}\mathfrak{v},\mathfrak{g}\mathfrak{v}_{\alpha-1},\varrho)\right)^{\frac{1}{2}}=\lambda_{\alpha-1}(\varrho)$$

$$\Gamma(\mathfrak{g}\mathfrak{v},\mathfrak{g}\mathfrak{v}_\alpha,\mathfrak{k}\varrho)=\Gamma(\Phi(\mathfrak{v},\mathfrak{u}),\Phi(\mathfrak{v}_{\alpha-1},\mathfrak{u}_{\alpha-1}),\mathfrak{k}\varrho)\geq\left(\Gamma(\mathfrak{g}\mathfrak{v},\mathfrak{g}\mathfrak{v}_{\alpha-1},\varrho)\right)^{\frac{1}{2}}*\left(\Gamma(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{u}_{\alpha-1},\varrho)\right)^{\frac{1}{2}}=\lambda_{\alpha-1}(\varrho).$$

Therefore

$$\lambda_\alpha(\varrho)\geq\lambda_{\alpha-1}\left(\frac{\varrho}{\mathfrak{k}}\right)\geq\cdots\geq\lambda_0\left(\frac{\varrho}{\mathfrak{k}^\alpha}\right),$$

$$\Gamma(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{v}_\alpha,\mathfrak{k}\varrho)\geq\lambda_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha-1}}\right)$$

and

$$\Gamma(\mathfrak{g}\mathfrak{v},\mathfrak{g}\mathfrak{u}_\alpha,\mathfrak{k}\varrho)\geq\lambda_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha-1}}\right).$$

Since $\lim_{\alpha\to\infty}\lambda_0\left(\frac{\varrho}{\mathfrak{k}^{\alpha-1}}\right)=1$, we get

$$\lim_{\alpha\to\infty}\mathfrak{g}\mathfrak{u}_\alpha=\mathfrak{g}\mathfrak{u}$$

and

$$\lim_{\alpha\to\infty}\mathfrak{g}\mathfrak{v}_\alpha=\mathfrak{g}\mathfrak{v}.$$

This shows that $\mathfrak{g}\mathfrak{u}=\mathfrak{u}$ and $\mathfrak{g}\mathfrak{v}=\mathfrak{v}$. Similarly, we can show that $\mathfrak{g}\vartheta=\vartheta$ and $\mathfrak{g}\eta=\eta$. Therefore,

$$\begin{aligned}\Phi(\mathfrak{u},\mathfrak{v})&=\mathfrak{g}\mathfrak{u}=\mathfrak{u}=\mathfrak{g}\mathfrak{r}=\Phi(\mathfrak{l},\beta)\\ \Phi(\mathfrak{v},\mathfrak{u})&=\mathfrak{g}\mathfrak{v}=\mathfrak{v}=\mathfrak{g}\omega=\Phi(\beta,\mathfrak{l})\\ \Phi(\vartheta,\eta)&=\mathfrak{g}\vartheta=\vartheta=\mathfrak{g}\mathfrak{l}=\Phi(\mathfrak{r},\omega)\\ \Phi(\eta,\vartheta)&=\mathfrak{g}\eta=\eta=\mathfrak{g}\beta=\Phi(\omega,\mathfrak{r}).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}\Gamma(\mathfrak{g}\mathfrak{l},\mathfrak{g}\mathfrak{r},\varrho)&=\Gamma(\vartheta,\mathfrak{u},\varrho)=\Gamma(\lim_{\alpha\to\infty}\mathfrak{g}\mathfrak{u}_\alpha,\lim_{\alpha\to\infty}\mathfrak{g}\vartheta_\alpha,\varrho)\\&=\lim_{\alpha\to\infty}\Gamma(\mathfrak{g}\mathfrak{u}_\alpha,\mathfrak{g}\vartheta_\alpha,\varrho)\\&=1\end{aligned}$$

and

$$\begin{aligned}\Gamma(\mathfrak{g}\beta,\mathfrak{g}\omega,\varrho)&=\Gamma(\eta,\mathfrak{v},\varrho)=\Gamma(\lim_{\alpha\to\infty}\mathfrak{g}\mathfrak{v}_\alpha,\lim_{\alpha\to\infty}\mathfrak{g}\eta_\alpha,\varrho)\\&=\lim_{\alpha\to\infty}\Gamma(\mathfrak{g}\mathfrak{v}_\alpha,\mathfrak{g}\eta_\alpha,\varrho)\\&=1.\end{aligned}$$

Therefore $\vartheta = \mathfrak{u}$ and $\eta = \mathfrak{v}$. Hence, $(\vartheta, \eta) \in \Omega^2 \cap \Upsilon^2$ is common coupled fixed point of Φ and $\mathfrak{g}$. Let $(\vartheta^*, \eta^*) \in \Omega^2 \cup \Upsilon^2$ be another common coupled fixed point of Φ and $\mathfrak{g}$. If $(\vartheta^*, \eta^*) \in \Omega^2$, then

$$\begin{aligned}\Gamma(\vartheta, \vartheta^*, \mathfrak{k}\varrho) = \Gamma(\Phi(\vartheta, \eta), \Phi(\vartheta^*, \eta^*), \mathfrak{k}\varrho) &\geq (\Gamma(\mathfrak{g}\vartheta, \mathfrak{g}\vartheta^*, \varrho))^{\frac{1}{2}} * (\Gamma(\mathfrak{g}\eta, \mathfrak{g}\eta^*, \varrho))^{\frac{1}{2}} \\ &= (\Gamma(\vartheta, \vartheta^*, \varrho))^{\frac{1}{2}} * (\Gamma(\eta, \eta^*, \varrho))^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\Gamma(\eta, \eta^*, \mathfrak{k}\varrho) = \Gamma(\Phi(\eta, \vartheta), \Phi(\eta^*, \vartheta^*), \mathfrak{k}\varrho) &\geq (\Gamma(\mathfrak{g}\eta, \mathfrak{g}\eta^*, \varrho))^{\frac{1}{2}} * (\Gamma(\mathfrak{g}\vartheta, \mathfrak{g}\vartheta^*, \varrho))^{\frac{1}{2}} \\ &= (\Gamma(\eta, \eta^*, \varrho))^{\frac{1}{2}} * (\Gamma(\vartheta, \vartheta^*, \varrho))^{\frac{1}{2}}.\end{aligned}$$

Adding by ϱ - norm $*$, we obtain

$$\Gamma(\vartheta, \vartheta^*, \mathfrak{k}\varrho) * \Gamma(\eta, \eta^*, \mathfrak{k}\varrho) \geq \Gamma(\vartheta, \vartheta^*, \varrho) * \Gamma(\eta, \eta^*, \varrho).$$

Therefore

$$\begin{aligned}\Gamma(\vartheta, \vartheta^*, \varrho) * \Gamma(\eta, \eta^*, \varrho) &\geq \Gamma(\vartheta, \vartheta^*, \frac{\varrho}{\mathfrak{k}}) * \Gamma(\eta, \eta^*, \frac{\varrho}{\mathfrak{k}}) \\ &\vdots \\ &\geq \Gamma(\vartheta, \vartheta^*, \frac{\varrho}{\mathfrak{k}^\alpha}) * \Gamma(\eta, \eta^*, \frac{\varrho}{\mathfrak{k}^\alpha}).\end{aligned}$$

As $\alpha \to \infty$, we have

$$\begin{aligned}\Gamma(\vartheta, \vartheta^*, \varrho) &= 1 \\ \text{and } \Gamma(\eta, \eta^*, \varrho) &= 1.\end{aligned}$$

Therefore $\vartheta = \vartheta^*$ and $\eta = \eta^*$. Similarly, if $(\vartheta^*, \eta^*) \in \Upsilon^2$, then $\vartheta = \vartheta^*$ and $\eta = \eta^*$. Hence $(\vartheta, \eta) \in \Omega^2 \cap \Upsilon^2$ is a unique common coupled fixed point of Φ and $\mathfrak{g}$. □

Example 3.2. Let $\Omega = [0, 1]$, $\Upsilon = \{0\} \cup \mathbb{N} - \{1\}$. Define $\Gamma(\vartheta, \eta, \varrho) = e^{-\frac{(\vartheta - \eta)}{\varrho}}$ for all $\varrho > 0$ and $\vartheta \in \Omega$ and $\eta \in \Upsilon$. Clearly, $(\Omega, \Upsilon, \Gamma, *)$ is a complete fuzzy bipolar metric space, where $*$ is a continuous ϱ-norm defined as $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Define $\Phi : \Omega^2 \cup \Upsilon^2 \to \Omega \cup \Upsilon$ and $\mathfrak{g} : \Omega \cup \Upsilon \to \Omega \cup \Upsilon$ defined by

$$\Phi(\vartheta, \eta) = \begin{cases} \frac{\vartheta + \eta}{2}, & \text{if } \vartheta, \eta \in \Omega^2, \\ 0, & \text{if } \vartheta, \eta \in \Upsilon^2, \end{cases}$$

for all $\vartheta, \eta \in \Omega^2 \cup \Upsilon^2$ and

$$\mathfrak{g}(\vartheta) = \begin{cases} \vartheta, & \text{if } \vartheta, \eta \in \Omega, \\ 0, & \text{if } \vartheta, \eta \in \Upsilon, \end{cases}$$

for all $\vartheta, \eta \in \Omega \cup \Upsilon$. Then

$$\begin{aligned}\Gamma(\Phi(\vartheta, \eta), \Phi(\mathfrak{u}, \mathfrak{v}), \mathfrak{k}\varrho)^2 &= \left(e^{-(\frac{\vartheta - \mathfrak{u} + \eta - \mathfrak{v}}{2\mathfrak{k}\varrho})}\right)^2 \\ &\geq e^{-(\frac{\vartheta - \mathfrak{u} + \eta - \mathfrak{v}}{\varrho})} \\ &= e^{-(\frac{\mathfrak{g}\vartheta - \mathfrak{g}\mathfrak{u} + \mathfrak{g}\eta - \mathfrak{g}\mathfrak{v}}{\varrho})} \\ &= \Gamma(\mathfrak{g}\vartheta, \mathfrak{g}\mathfrak{u}, \varrho) * \Gamma(\mathfrak{g}\eta, \mathfrak{g}\mathfrak{v}, \varrho).\end{aligned}$$

Clearly, all the hypotheses of Theorem 3.1 are satisfied. Hence Φ and $\mathfrak{g}$ have a unique common coupled fixed point, i.e., $(0, 0)$.

4 Application

In this section, we study the existence and unique common solution to a system of integral equations as an application of Theorem 3.1.

Theorem 4.1. Let us consider the system of integral equations:

$$\vartheta(\mathfrak{p}) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{E}_1\cup\mathcal{E}_2} \mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega))d\omega,\ \mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2,$$

$$\eta(\mathfrak{p}) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{E}_1\cup\mathcal{E}_2} \mathcal{G}(\mathfrak{p},\omega,\eta(\omega),\vartheta(\omega))d\omega,\ \mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2,$$

$$\mathfrak{g}(\vartheta(\mathfrak{p})) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{E}_1\cup\mathcal{E}_2} \mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega))d\omega,\ \mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2$$

and

$$\mathfrak{g}(\eta(\mathfrak{p})) = \mathfrak{b}(\mathfrak{p}) + \int_{\mathcal{E}_1\cup\mathcal{E}_2} \mathcal{G}(\mathfrak{p},\omega,\eta(\omega),\vartheta(\omega))d\omega,\ \mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2,$$

where $\mathcal{E}_1\cup\mathcal{E}_2$ is a Lebesgue measurable set. Suppose

1. $\mathcal{G}:(\mathcal{E}_1^2\cup\mathcal{E}_2^2)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ and $b\in L^\infty(\mathcal{E}_1)\cup L^\infty(\mathcal{E}_2)$,
2. there is a continuous function $\theta:\mathcal{E}_1^2\cup\mathcal{E}_2^2\to[0,\infty)$ and $\mathfrak{g}:L^\infty(\mathcal{E}_1)\cup L^\infty(\mathcal{E}_2)\to L^\infty(\mathcal{E}_1)\cup L^\infty(\mathcal{E}_2)$, $\mathfrak{k}\in(0,1)$ such that
$$|\mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega))-\mathcal{G}(\mathfrak{p},\omega,\mathfrak{u}(\omega),\mathfrak{v}(\omega))|\le\theta(\mathfrak{p},\omega)(|\mathfrak{g}\vartheta(\mathfrak{p})-\mathfrak{g}\mathfrak{u}(\mathfrak{p})|+ |\mathfrak{g}\eta(\mathfrak{p})-\mathfrak{g}\mathfrak{v}(\mathfrak{p})|),$$
for $\mathfrak{p},\omega\in\mathcal{E}_1^2\cup\mathcal{E}_2^2$,
3. $\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\int_{\mathcal{E}_1\cup\mathcal{E}_2}\theta(\mathfrak{p},\omega)d\omega\le1$.

Then the integral equations have a unique common solution in $L^\infty(\mathcal{E}_1)\cup L^\infty(\mathcal{E}_2)$.

Proof . Let $\varOmega=L^\infty(\mathcal{E}_1)$ and $\varUpsilon=L^\infty(\mathcal{E}_2)$ be two normed linear spaces, where $\mathcal{E}_1,\mathcal{E}_2$ are Lebesgue measurable sets and $m(\mathcal{E}_1\cup\mathcal{E}_2)<\infty$. Consider $\varGamma:\varOmega\times\varUpsilon\times(0,\infty)\to[0,1]$ by

$$\varGamma(\vartheta,\eta,\varrho)=e^{-\frac{\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}|\vartheta(\mathfrak{p})-\eta(\mathfrak{p})|}{\varrho}}.$$

for all $\vartheta\in\varOmega,\eta\in\varUpsilon$. Then $(\varOmega,\varUpsilon,\varGamma,\star)$ is a complete fuzzy bipolar metric space. Define the mapping $\varPhi:\varOmega^2\times\varUpsilon^2\to\varOmega\cup\varUpsilon$ by

$$\varPhi(\vartheta(\mathfrak{p}),\eta(\mathfrak{p}))=\mathfrak{b}(\mathfrak{p})+\int_{\mathcal{E}_1\cup\mathcal{E}_2}\mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega)d\omega,\ \mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2.$$

Now, we have

$$\begin{aligned}
&\varGamma(\varPhi(\vartheta(\mathfrak{p}),\eta(\mathfrak{p})),\varPhi(\mathfrak{u}(\mathfrak{p}),\mathfrak{v}(\mathfrak{p})))^2\\
&=\left(e^{-\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{|\varPhi(\vartheta(\mathfrak{p}),\vartheta(\mathfrak{p}))-\varPhi(\mathfrak{u}(\mathfrak{p}),\mathfrak{v}(\mathfrak{p}))|}{\mathfrak{k}\varrho}}\right)^2\\
&=\left(e^{-\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{|\mathfrak{b}(\mathfrak{p})+\int_{\mathcal{E}_1\cup\mathcal{E}_2}\mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega))d\omega-\mathfrak{b}(\mathfrak{p})-\int_{\mathcal{E}_1\cup\mathcal{E}_2}\mathcal{G}(\mathfrak{p},\omega,\mathfrak{u}(\omega),\mathfrak{v}(\omega))d\omega)|}{\mathfrak{k}\varrho}}\right)^2\\
&\ge e^{-\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{\int_{\mathcal{E}_1\cup\mathcal{E}_2}|\mathcal{G}(\mathfrak{p},\omega,\vartheta(\omega),\eta(\omega))d\omega-\mathcal{G}(\mathfrak{p},\omega,\mathfrak{u}(\omega),\mathfrak{v}(\omega))|d\omega}{\varrho}}\\
&\ge e^{-\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{\int_{\mathcal{E}_1\cup\mathcal{E}_2}\theta(\mathfrak{p},\omega)(|\mathfrak{g}\vartheta(\mathfrak{p})-\mathfrak{g}\mathfrak{u}(\mathfrak{p})|+|\mathfrak{g}\eta(\mathfrak{p})-\mathfrak{g}\mathfrak{v}(\mathfrak{p})|)d\omega}{\varrho}}\\
&\ge e^{-\sup_{\mathfrak{p}\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{|\mathfrak{g}\vartheta(\mathfrak{p})-\mathfrak{g}\mathfrak{u}(\mathfrak{p})|+|\mathfrak{g}\eta(\mathfrak{p})-\mathfrak{g}\mathfrak{v}(\mathfrak{p})|}{\varrho}}\\
&=\varGamma(\mathfrak{g}\vartheta,\mathfrak{g}\mathfrak{u},\varrho)*\varGamma(\mathfrak{g}\eta,\mathfrak{g}\mathfrak{v},\varrho).
\end{aligned}$$

Hence all the hypotheses of a Theorem 3.1 are verified and consequently, the integral equation has a unique common solution. □

5 Conclusion

In this paper, we proved common coupled fixed point theorem on fuzzy bipolar metric space. An illustrative example and application on fuzzy bipolar metric space is given.

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