Int. J. Nonlinear Anal. Appl. In Press, 1–12
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.23709.2586

New generalization of norm spaces and applications

Mohammad Reza Haddadi[a,*], Mohammad H. Labbaf Ghasemi[b], Noha Eftekhari[b]

[a]Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran

[b]Department of Pure Mathematics, University of Sharekord, Sharekord, Iran

(Communicated by Ali Jabbari)

Abstract

One of the generalizations that were studied from metric space was multiplicative metric space. The main idea was that the usual triangular inequality was replaced by a multiplicative triangle inequality. The important thing is that logarithm of every multiplicative metric is a metric. In this paper, we introduce multiplicative norm space and present three norms in bounded multiplicative operator spaces and we investigate conditions that bounded multiplicative operator spaces be complete norm multiplicative spaces. It is notable that the logarithm of every multiplicative norm is not a norm and so we have new results in multiplicative norm spaces. We give an important extension of the Hahn-Banach theorem to nonlinear operators and their ramifications and indicate some applications.

Keywords: Hahn Banach theorem, multiplicative calculus, multiplicative metric space
2020 MSC: Primary 54E50; Secondary 46A22, 46A30

1 Introduction

In Bashirov et al. initiated a new kind of spaces, called multiplicative metric spaces[4]. By defining multiplicative distance they provided the foundation for multiplicative metric spaces and many authors give theoretical concepts and applicable results about this subject; we refer the reader to [1, 10].

But until now, the authors of the results in a norm space did not extend to the results of a multiplicative norm space. Of course, Grossman and Katz [8] established a new calculus called multiplicative calculus also termed as exponential calculus in 1970. Florack and Assen [6] used the idea of multiplicative calculus in biomedical image analysis. Bashirov et al.[2] demonstrated the efficiency of multiplicative calculus over the Newtonian calculus. They elaborated that multiplicative calculus is more effective than Newtonian calculus for modeling various problems from different fields. Bashirov and Bashirova [3] used the concept of multiplicative calculus for deriving function that shows dynamics of literary text. The aim of this work is to introduce multiplicative space and normed multiplicative space that we consider in the second section. Also, we present three norm in the bounded multiplicative operator space $\mathcal{M}(\mathcal{A},\mathcal{B})$ for a pair $(\mathcal{A},\mathcal{B})$ of normed multiplicative spaces and we investigate, in what conditions, $\mathcal{M}(\mathcal{A},\mathcal{B})$ will be a complete space. In the third section, we will give an important extension of the Hahn Banach theorem to nonlinear operators and its ramifications and indicate some applications.

In the following we start by definition of multiplicative spaces and normed multiplicative spaces.

*Corresponding author
Email addresses: haddadi@abru.ac.ir (Mohammad Reza Haddadi), mhlgh@stu.sku.ac.ir (Mohammad H. Labbaf Ghasemi), eftekhari.n@sci.sku.ac.ir (Noha Eftekhari)

Received: June 2021 *Accepted:* March 2022

Definition 1.1. A multiplicative space over a field $\mathbb{F}$ is a nonempty set $\mathcal{M}$ of elements $x, y, ...$ together two algebraic operations. These operations are called multiplication and multiplicative of scalars. We denote the multiplicative operator by $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, which associates to any pair $x \circ y$ of $\mathcal{M}$, we call multiply of x by y, *multiplicative operator* associates to any ordered pair (x, y) of $\mathcal{M} \times \mathcal{M}$, an element $x \circ y$ in $\mathcal{M}$ in such a way that, called the multiply of x to y, such that the following properties hold. The multiplicative operator is commutative and associative, that is, for all elements $x, y, z \in \mathcal{M}$ we have

(a) $x \circ y = y \circ x$,

(b) $x \circ (y \circ z) = (x \circ y) \circ z$.
Furthermore, there exists an element $1 \in \mathcal{M}$, called the unit element, such that for any nonzero element $x \in \mathcal{M}$ there exists an element x^{-1}, such that for all $x \in \mathcal{M}$ we have

(c) $x \circ 1 = x$,

(d) $x \circ (x^{-1}) = 1$.
multiplicative of scalar associates to any $x \in \mathcal{M}$ and a scalar $\alpha \in \mathbb{F}$, an element x^α in $\mathcal{M}$. For all elements $x, y \in \mathcal{M}$ and scalars $\alpha, \beta \in \mathbb{F}$, we have

(e) $(x^\alpha)^\beta = x^{\alpha\beta}$,

(f) $x^1 = x$, and $x^0 = 1$,
and the distributive laws

(g) $(x \circ y)^\alpha = x^\alpha \circ y^\alpha$,

(h) $x^{\alpha+\beta} = x^\alpha \circ x^\beta$.

A multiplicative combination of $x_1, ..., x_m$ of a multiplicative space $\mathcal{M}$ is an expression of the form

$$x_1^{\alpha_1} \circ x_2^{\alpha_2} \circ \cdots \circ x_m^{\alpha_m},$$

where the multiplicative $\alpha_1, \ldots, \alpha_m$ are any scalars. Let $\mathcal{M}$ be a multiplicative space. The properties of multiplicative operator and multiplicative of scalars on $\mathcal{M} \times \mathcal{M}$ imply the following statements.

(a) If $x, y, z \in \mathcal{M}$ and $x \circ y = x \circ z$ then $y = z$.

(b) If $x, y \in \mathcal{M}$ and $x \circ y = x$ then $y = 1$.

(c) If $x, y \in \mathcal{M}$ and $x \circ y = 1$ then $y = x^{-1}$.

(d) If $x \in \mathcal{M}$ then $(x^{-1})^{-1} = x$.

A multiplicative subspace of a multiplicative space $\mathcal{A}$ is a subset $\mathcal{B}$ of $\mathcal{A}$ that is a multiplicative space under the operations obtained by restricting those of $\mathcal{A}$ to $\mathcal{B}$. In other words, if $1 \in B$ and for all $y_1, y_2 \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{F}$, we have $y_1^\alpha, y_2^\beta \in \mathcal{B}$.

We define $\star$-norm on a multiplicative space that plays a key role in the main results.

Definition 1.2. Let $\mathcal{M}$ be a multiplicative space. A mapping $\natural.\natural : \mathcal{M} \to [1, \infty)$ is said to be $\star$-norm if

(i) $\natural x \natural = 1 \Leftrightarrow \ x = 1, \ x \in \mathcal{M}$,

(ii) $\natural x^\alpha \natural = \natural x \natural^{|\alpha|}, \ x \in \mathcal{M}, \ \alpha \in \mathbb{R}$,

(iii) $\natural x \circ y \natural \leq \natural x \natural \ \natural y \natural, \ x, y \in \mathcal{M}$,

the ordered pair $(\mathcal{M}, \natural.\natural)$ is called a $\star$-normed multiplicative space.

Remark 1.1. The important thing is that logarithm of every multiplicative metric is a metric, but logarithm of every multiplicative norm is not a norm. Therefore results in the multiplicative metric space are similar to metric space, but we have new results in multiplicative norm spaces. Namely in $\mathbb{R}$ if $\|x\| = \log \natural x \natural$, then $\|xy\| \leq \|x\| + \|y\|$.

In the following we give some examples of multiplicative metric space that are not multiplicative norm space

Example 1.3. Let $\star$ be a multiplicative space and $\alpha \neq 1$. A function ρ_α on $\mathcal{M} \times \mathcal{M}$ is defined by

$$\rho_\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x = y, \\ \alpha & \text{if} \quad x \neq y. \end{cases}$$

Then it is obvious that $(\mathcal{M}, \rho_\alpha)$ is a multiplicative metric space that is not a multiplicative norm space.

Example 1.4. [2] Let $d^* : \mathbb{R}_n^+ \times \mathbb{R}_n^+ \to \mathbb{R}$ be defined as follows

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{R}_n^+$ and $|.|^* : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows

$$|a|^* = \begin{cases} a & \text{if} \quad a \geq 1, \\ \dfrac{1}{a} & \text{if} \quad a < 1. \end{cases}$$

Then it is obvious that$(\mathbb{R}_n^+, d^*)$ is a multiplicative metric space. On the other hand since $\mathbb{R}_n^+$ is not a multiplicative space hence $\mathbb{R}_n^+$ is not a multiplicative norm space.

Definition 1.5. Let $\mathcal{M}$ be a $\star$-normed multiplicative space. A sequence $(x_n) \subseteq \mathcal{M}$ is said to be $\star$-convergent to $x \in \mathcal{M}$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \geq N$ we have $\natural x_n x^{-1} \natural < 1 + \epsilon$, hence $\natural x_n x^{-1} \natural \to 1$ as $n \to \infty$ and it denoted by $x_n \mapsto x$. On the other hand, a sequence $(x_n) \subseteq \mathcal{M}$ is said to be $\star$-Cauchy sequence in the $\star$-normed multiplicative space $\mathcal{M}$, if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $\natural x_n x_m^{-1} \natural < 1 + \epsilon$, hence $\natural x_n x_m^{-1} \natural \to 1$ as $n, m \to \infty$. Also, a $\star$-normed multiplicative space is said to be complete $\star$-normed multiplicative space (CNM space) if every $\star$-Cauchy sequence is a $\star$-convergent sequence.

In the following we give an example of $\star$-normed multiplicative space.

Example 1.6. The set $(0, \infty)$ is a $\star$-normed multiplicative space that is a CNM space. Because, Suppose (a_n) is a Cauchy sequence in $(0, \infty)$, i.e.

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N}; \quad \forall m, n \geq N, \quad a_n(a_m)^{-1} < e^\epsilon.$$

Therefore,

$$\ln(a_n(a_m)^{-1}) = \ln\left(\frac{a_n}{a_m}\right) = \ln(a_n) - \ln(a_m) < \epsilon.$$

Since $\mathbb{R}$ has been a complete space, there exist $a \geq 1$, such that

$$\ln(a_n) - \ln(a) < \epsilon,$$

and so

$$\ln\left(\frac{a_n}{a}\right) < \epsilon \quad \Rightarrow \quad a_n a^{-1} < e^\epsilon.$$

Thus $(0, \infty)$ is a $\star$-normed multiplicative space that is a CNM space.

Definition 1.7. Let $\mathcal{A}$ and $\mathcal{B}$ be two multiplicative spaces. The operator $\Phi : \mathcal{A} \to \mathcal{B}$ is a multiplicative operator if

$$\Phi(x^\alpha y^\beta) = \Phi(x)^\alpha \Phi(y)^\beta, \quad \forall x, y \in \mathcal{A}, \alpha, \beta \in \mathbb{R}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\star$-normed multiplicative space with $\star$-norms $\natural.\natural_{\mathcal{A}}$, $\natural.\natural_{\mathcal{B}}$ respectively on $\mathcal{A}$ and $\mathcal{B}$. Let

$$\mathcal{M}(\mathcal{A}, \mathcal{B}) = \{\Phi : \mathcal{A} \to \mathcal{B} |\ \Phi \text{ is a bounded multiplicative operator}\},$$

and

$$A^\star = \mathcal{M}(\mathcal{A}, \mathbb{R}^+).$$

In the following we give an example that it give a relation between a Banach algebra and a CNM space.

Example 1.8. Let $\mathcal{A}$ be a Banach algebra. Then $inv(\mathcal{A})$ is a CNM space.

Proof . Suppose $X = inv(\mathcal{A})$. Put

$$\natural x \natural = \begin{cases} \|x^{-1}\| & x \in B_X \\ \\ \|x\| & x \notin B_X. \end{cases}$$

We show that

$$\natural xy \natural \leq \natural x \natural \ \natural y \natural, \quad \forall x, y \in X.$$

The proof is in three parts: (i) Let $x, y \in B_X$. Then $xy \in B_X$ and so

$$\begin{aligned} \natural xy \natural &= \|(xy)^{-1}\| = \|y^{-1}x^{-1}\| \\ &\leq \|y^{-1}\|\|x^{-1}\| \\ &= \natural x \natural \natural y \natural. \end{aligned}$$

(ii) Let $x \in B_X$ and $y \notin B_X$. Then, if $xy \in B_X$, hence we have

$$\begin{aligned} \natural xy \natural &= \|(xy)^{-1}\| = \|y^{-1}x^{-1}\| \\ &\leq \|y^{-1}\|\|x^{-1}\| \\ &\leq \natural x \natural \\ &\leq \natural x \natural \natural y \natural, \end{aligned}$$

else, if $xy \notin B_X$, hence we have

$$\begin{aligned} \natural xy \natural &= \|xy\| \\ &\leq \|x\|\|y\| \\ &\leq \natural y \natural \\ &\leq \natural x \natural \natural y \natural. \end{aligned}$$

(iii) Let $x, y \notin B_X$. Then $xy \notin B_X$ and so

$$\begin{aligned} \natural xy \natural &= \|xy\| \\ &\leq \|x\|\|y\| \\ &= \natural x \natural \natural y \natural. \end{aligned}$$

Therefore $(inv(\mathcal{A}), \natural.\natural)$ is a $\star$-normed multiplicative space. □

We show that the space $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is a $\star$-normed multiplicative space. Hence, in the following we define three norm on $\mathcal{M}(\mathcal{A}, \mathcal{B})$.

Example 1.9. We define a $\star$-norm on $\mathcal{M}(\mathcal{A}, \mathcal{B})$ as follows

$$\natural \Phi \natural_1 = \sup_{x \in \mathcal{A}} \left\{ \natural \Phi x \natural^{\frac{1}{\natural x \natural}} \right\}.$$

If $\Phi = 1_{\mathcal{M}(\mathcal{A},\mathcal{B})}$, it follows that $\Phi x = 1 \in \mathcal{B}$ for all $x \in \mathcal{A}$ and so $\natural \Phi x \natural = 1$ which implies

$$\natural \Phi \natural_1 = \sup_{x \in \mathcal{A}} \natural \Phi x \natural^{\frac{1}{\natural x \natural}} = 1.$$

If $\natural \Phi \natural_1 = 1$, it follows that $\natural \Phi x \natural = 1$, for all $x \in \mathcal{A}$(since $\natural \Phi x \natural \geq 1$). As $\natural . \natural$ is a $\star$-norm, so $\Phi x = 1$ for all $x \in \mathcal{A}$, that is $\Phi = 1_{\mathcal{M}(\mathcal{A},\mathcal{B})}$. For $\alpha \in \mathbb{R}$ and $\Phi \in \mathcal{M}(\mathcal{A}, \mathcal{B})$, we consider,

$$\Phi^{\alpha}(f) = (\Phi f)^{\alpha}, \qquad \text{for all} \quad f \in \mathcal{A}, \qquad \text{for all} \quad \alpha \in \mathbb{R}.$$

Since for $\alpha \in \mathbb{R}$, $\Phi^\alpha x = (\Phi x)^\alpha$ and $\natural(\Phi x)^\alpha\natural = \natural(\Phi x)\natural^{|\alpha|}$ and so

$$\begin{aligned}\natural\Phi^\alpha\natural_1 &= \sup_{x\in\mathcal{A}} \natural\Phi^\alpha x\natural^{\frac{1}{\natural x\natural}} \\ &= \sup_{x\in\mathcal{A}} (\natural\Phi x\natural)^{\frac{|\alpha|}{\natural x\natural}} \\ &= \left(\sup_{x\in\mathcal{A}} \natural\Phi x\natural^{\frac{1}{\natural x\natural}}\right)^{|\alpha|} \\ &= \natural\Phi\natural_1^{|\alpha|}.\end{aligned}$$

Therefore, (ii) of Definition 1.2 is satisfied. For condition (iii) of Definition 1.2, let $\Phi, \Psi \in \mathcal{M}(\mathcal{A},\mathcal{B})$

$$\natural\Phi\Psi\natural_1 = \sup_{x\in\mathcal{A}} \natural\Phi x\Psi x\natural^{\frac{1}{\natural x\natural}} \leq \natural\Phi\natural_1\natural\Psi\natural_1.$$

The last inequality follows by $\natural\Phi\Psi\natural \leq \natural\Phi\natural\natural\Psi\natural$. In this example, we prove that $\natural.\natural_1$ on $\mathcal{M}(\mathcal{A},\mathcal{B})$ is a $\star$-norm.

Example 1.10. On $\mathcal{M}(\mathcal{A},\mathcal{B})$, we define a $\star$-norm as follows

$$\natural\Phi\natural_2 = \sup_{x\in\mathcal{A}} \left\{\natural\Phi x\natural^{-\natural x\natural} : \text{ for all } \ T \in \mathcal{M}(\mathcal{A},\mathcal{B})\right\}.$$

As in the previous example, it proves that $\natural\Phi\natural_2$ is a $\star$-norm on $\mathcal{M}(\mathcal{A},\mathcal{B})$.

For $\natural\Phi\natural_2$, we have

$$\natural\Phi x\natural \leq \natural\Phi\natural_2^{-\frac{1}{\natural x\natural}}, \qquad \text{for all } x \in \mathcal{A}.$$

Since

$$\natural\Phi\natural_2 = \sup\left\{\natural\Phi x\natural^{-\natural x\natural} : \qquad x \in \mathcal{M}(\mathcal{A},\mathbb{R})\right\},$$

we have

$$\natural\Phi x\natural^{-\natural x\natural} \leq \natural\Phi\natural_2, \qquad \text{for all } x \in \mathcal{A},$$

and so

$$\natural\Phi x\natural \leq \natural\Phi\natural_2^{-\frac{1}{\natural x\natural}}.$$

In the following we present an example that is suitable for the extension of the Hahn-Banach theorem

Example 1.11. We define a $\star$-norm on $\mathcal{M}(\mathcal{A},\mathcal{B})$, as follows

$$\natural\Phi\natural_3 = e^{\sup\left\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural} :\ x\neq 1,\ x\in\mathcal{A}\right\}}.$$

Clearly, condition (i) of Definition 1.2 is satisfied. For condition (ii),

$$\begin{aligned}\natural\Phi^\alpha\natural_3 &= e^{\sup\left\{\frac{\ln\natural\Phi^\alpha x\natural}{\ln\natural x\natural} :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\ &= e^{\sup\left\{\frac{\ln\natural\Phi x\natural^{|\alpha|}}{\ln\natural x\natural} :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\ &= e^{\sup\left\{\frac{|\alpha|\ln\natural\Phi x\natural}{\ln\natural x\natural} :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\ &= \left(e^{\sup\left\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural} :\ x\neq 1,\ x\in\mathcal{A}\right\}}\right)^{|\alpha|} \\ &= \natural\Phi\natural_3^{|\alpha|}.\end{aligned}$$

Finally the triangle inequality follows by

$$\begin{aligned}
\natural\Phi\Psi\natural_3 &= e^{\sup\left\{\frac{\ln\natural\Phi x\Psi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\
&\leq e^{\sup\left\{\frac{\ln(\natural\Phi x\natural\natural\Psi x\natural)}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\
&= e^{\sup\left\{\frac{\ln\natural\Phi x\natural+\ln\natural\Psi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{A}\right\}} \\
&\leq e^{\sup\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{M}(\mathcal{A},\mathbb{R})\}+\sup\{\frac{\ln\natural\Psi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{A}\}} \\
&= e^{\sup\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{M}(\mathcal{A},\mathbb{R})\}}.e^{\sup\{\frac{\ln\natural\Psi x\natural}{\ln\natural x\natural}\ :\ x\neq 1,\ x\in\mathcal{A}\}} \\
&= \natural\Phi\natural_3.\natural\Psi\natural_3.
\end{aligned}$$

For $\natural\Phi\natural_3$, we have $\natural\Phi x\natural \leq \natural\Phi\natural_3^{\ln\natural x\natural}$, for all $x\neq 1,\ x\in\mathcal{A}$ and $T\in\mathcal{M}(\mathcal{A},\mathcal{B})$. Since

$$\natural\Phi\natural_3 = e^{\sup\left\{\frac{\ln\natural\Phi x\natural}{\natural x\natural}:\quad x\neq 1,\ x\in\mathcal{A}\right\}},$$

it follows that $\ln\natural\Phi\natural_3 = \sup\left\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural}:\quad x\neq 1,\ x\in\mathcal{A}\right\}$, and so, for all $x\neq 1,\ x\in\mathcal{A}$

$$\frac{\ln\natural\Phi\natural}{\ln\natural x\natural}\leq \ln\natural\Phi\natural_3.$$

Therefore

$$\ln\natural\Phi x\natural \leq \ln\natural\Phi\natural_3^{\ln\natural x\natural}.$$

2 Main Results

We investigate, in what conditions, $\mathcal{M}(\mathcal{A},\mathcal{B})$ will be a CNM space? This is a central question, which is answered in the following theorem.

Theorem 2.1. If $\mathcal{B}$ is a CNM space, then $\mathcal{M}(\mathcal{A},\mathcal{B})$ is a CNM space with respect to $\natural.\natural_3$.

Proof . Let (Φ_n) be a $\star$-Cauchy sequence in $\mathcal{M}(\mathcal{A},\mathcal{B})$. So we have

$$\forall\epsilon>1,\quad \exists N\in\mathbb{N},\quad \forall m,n\geq N,\quad \natural\Phi_n(\Phi_m)^{-1}\natural_3 < e^{\epsilon}.$$

Which implies that

$$\natural\Phi_n(\Phi_m)^{-1}\natural_3 = e^{\sup\left\{\frac{\ln\natural\Phi_n x(\Phi_m x)^{-1}\natural}{\ln\natural x\natural}:\quad x\neq 1,\ x\in\mathcal{A}\right\}} < e^{\epsilon},$$

and so

$$\sup\left\{\frac{\ln\natural\Phi_n x(\Phi_m x)^{-1}\natural}{\ln\natural x\natural}:\quad x\neq 1,\ x\in\mathcal{A}\right\}<\epsilon,$$

and we have

$$\frac{\ln\natural\Phi_n x(\Phi_m x)^{-1}\natural}{\ln\natural x\natural}<\epsilon,\quad \text{for all}\quad x\neq 1,\ x\in\mathcal{A}.$$

Fixed $x\in\mathcal{A}$, we have

$$\begin{aligned}
\ln\natural\Phi_n x(\Phi_m x)^{-1}\natural &< \epsilon', \\
\natural\Phi_n x(\Phi_m x)^{-1}\natural &< \epsilon''.
\end{aligned}$$

So (Φ_n) is a $\star$-Cauchy sequence in $\mathcal{B}$. As $\mathcal{B}$ is a CNM space, there is $\Phi x\in\mathcal{B}$ such that $\Phi_n x\mapsto\Phi x$. We define $\Phi\in\mathcal{M}(\mathcal{A},\mathcal{B})$ as $T:\mathcal{A}\to\mathcal{B}$ for $x\in\mathcal{A}$, Φx is the limit of the sequence $(\Phi_n x)$, that is, $\Phi_n x\mapsto\Phi x$. Since (Φ_n) is a $\star$-Cauchy sequence in $\mathcal{M}(\mathcal{A},\mathcal{B})$, we have

$$\forall\epsilon>1,\quad \exists N\in\mathbb{N},\quad \forall m,n\geq N,\quad \natural\Phi_n(\Phi_m)^{-1}\natural_3<\epsilon,$$

which implies that for all $x \in \mathcal{M}(\mathcal{A}, \mathbb{R})$

$$\natural \Phi_n x(\Phi_m x)^{-1} \natural \leq \natural \Phi_n (\Phi_m)^{-1} \natural_3^{\ln \natural x \natural} < \epsilon^{\ln \natural x \natural}.$$

Let $m \to \infty$, we have for all $x \neq 1,\ x \in \mathcal{A}$

$$\natural \Phi_n x(\Phi x)^{-1} \natural < \epsilon^{\ln \natural x \natural},$$

and so

$$\ln \natural \Phi_n x(\Phi x)^{-1} \natural < \ln \natural x \natural \ln \epsilon,$$

which implies

$$\frac{\ln \natural \Phi_n x(\Phi x)^{-1} \natural}{\ln \natural x \natural} < \ln \epsilon.$$

The supremum being taken over all $x \in \mathcal{A}$, and so

$$\sup \left\{ \frac{\ln \natural \Phi_n x(\Phi x)^{-1} \natural}{\ln \natural x \natural} : \quad x \neq 1,\ x \in \mathcal{A} \right\} < \ln \epsilon,$$

therefore,

$$\natural \Phi_n \Phi^{-1} \natural_3 = e^{\sup \left\{ \frac{\ln \natural \Phi_n x(\Phi x)^{-1} \natural}{\ln \natural x \natural} : \quad x \neq 1,\ x \in \mathcal{A} \right\}} < \epsilon.$$

It follows that $\Phi_n \mapsto \Phi$ in $\mathcal{M}(\mathcal{A}, \mathcal{B})$. This shows that $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is a CNM space. $\Box$

Theorem 2.2. If $\mathcal{B}$ is a CNM space, then $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is a CNM space with respect to $\natural . \natural_2$.

Proof . Let (Φ_n) be a $\star$-Cauchy sequence in $\mathcal{M}(\mathcal{A}, \mathcal{B})$. So we have

$$\forall \epsilon > 1, \quad \exists N \in \mathbb{N}, \quad \forall m, n \geq N, \quad \natural \Phi_n (\Phi_m)^{-1} \natural_2 < e^{\epsilon}.$$

Which follows that

$$\sup \left\{ \natural \Phi_n x(\Phi_m x)^{-1} \natural^{-\natural x \natural} : \quad x \in \mathcal{M}(\mathcal{A}, \mathbb{R}) \right\} < \epsilon,$$

and so for any $x \in \mathcal{A}$, we have

$$\natural \Phi_n x(\Phi_m x)^{-1} \natural^{-\natural x \natural} < \epsilon.$$

Thus for fixed $x \in \mathcal{A}$, we have

$$\natural \Phi_n x(\Phi_m x)^{-1} \natural < \epsilon'.$$

That is, $\{\Phi_n x\}$ is a $\star$-Cauchy sequence in $\mathcal{B}$, since $\mathcal{B}$ is a CNM space, there exists $\Phi x \in \mathcal{B}$ such that $\Phi_n x \mapsto \Phi x$. Which implies that for all $x \in \mathcal{A}$ we have

$$\natural \Phi_n x(\Phi_m x)^{-1} \natural \leq \natural \Phi_n (\Phi_m)^{-1} \natural_2^{-\frac{1}{\natural x \natural}} < e^{-\frac{\epsilon}{\natural x \natural}}.$$

Let $m \to \infty$, we have for all $x \in \mathcal{A}$

$$\natural \Phi_n x(\Phi x)^{-1} \natural < e^{-\frac{\epsilon}{\natural x \natural}},$$

and so

$$\natural \Phi_n x(\Phi x)^{-1} \natural^{-\natural x \natural} < e^{\epsilon}.$$

The supremum being taken over all $x \in \mathcal{A}$, and so

$$\natural \Phi_n \Phi^{-1} \natural_2 = \sup \left\{ \natural \Phi_n x(\Phi x)^{-1} \natural^{-\natural x \natural} : \quad x \in \mathcal{A} \right\} < e^{\epsilon}.$$

It follows that $\Phi_n \mapsto \Phi$ in $\mathcal{M}(\mathcal{A}, \mathcal{B})$. This shows that $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is CNM. $\Box$

3 Applications in the Hahn Bananch theorem

Suppose that $\mathcal{M}$ and $\mathcal{Q}$ are multiplicative spaces and $\mathcal{M}$ is a multiplicative subspace of P. It is important to know whether a bounded multiplicative operator from $\mathcal{M}$ into $\mathcal{Q}$ can be extended to a bounded multiplicative operator of $\mathcal{M}$ into $\mathcal{Q}$. When this is possible, it can also be important to know whether it can be done without increasing the multiplicative $\star$-normed of the multiplicative operator. When $\mathcal{Q}$ is a CNM space and $\mathcal{M}$ is dense in $\mathcal{M}$, there is nothing to do. We obtain the following theorem for $\natural.\natural_3$.

Theorem 3.1. Let $\mathcal{M}$ be a dense subspace of a multiplicative space $\mathcal{M}$, and $\mathcal{Q}$ be a CNM space, and that $\Phi_0 : \mathcal{M} \to \mathcal{Q}$ is a bounded multiplicative operator. Then there is a unique extension bounded multiplicative operator $\Phi : \mathcal{M} \to \mathcal{Q}$ that $\Phi\mid_{\mathcal{M}} = \Phi_0$ and $\natural\Phi\natural = \natural\Phi_0\natural$.

Proof . Let $x \in \mathcal{M}$ be arbitrary. Since $\mathcal{M}$ is a dense subspace of $\mathcal{M}$, there is a sequence (x_n) in $\mathcal{M}$ which converges to x. So (x_n) is a $\star$-Cauchy sequence in $\mathcal{M}$. As we have

$$\natural\Phi_0 x_n(\Phi_0 x_m)^{-1}\natural \leq \natural\Phi_0\natural_3^{\ln \natural x_n x_m^{-1}\natural},$$

thus $(\Phi_0 x_n)$ is a $\star$-Cauchy sequence in $\mathcal{Q}$, since $\mathcal{Q}$ is a CNM space, the sequence $(\Phi_0 x_n)$ is a $\star$-convergent sequence in $\mathcal{Q}$. We define the operator $\Phi : \mathcal{M} \to \mathcal{Q}$ as follows, for any $x \in \mathcal{M}$, there exists a sequence (x_n) in $\mathcal{M}$ such that (x_n) is $\star$-converge to x, we define $\Phi x = \lim_{n\to\infty} \Phi_0 x_n$ which is a $\star$-convergent sequence in $\mathcal{Q}$. Now we show that Φ is a well defined. Let (y_n) and (x_n) be two $\star$-convergent sequence in $\mathcal{M}$, such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, which implies that $\lim_{n\to\infty} \Phi_0 x_n = \lim_{n\to\infty} \Phi_0 y_n$.

For any $x \in \mathcal{M}$, we consider the constant sequence (x_n) with for each $n \in \mathbb{N}$, $x_n = x$. So we have $\Phi x = \lim_{n\to\infty} \Phi_0 x_n = \Phi_0 x$, that is Φ and Φ_0 equals on $\mathcal{M}$.

Let $\alpha \in \mathbb{R}$, and (x_n) and (y_n) be two convergent sequences in $\mathcal{M}$, which is $\star$-converging to x and y respectively, then $x_n^\alpha y_n \to x^\alpha y$, which follows that

$$\begin{aligned}\Phi(x^\alpha y) &= \lim_{n\to\infty} \Phi_0(x_n^\alpha y_n),\\ &= \lim_{n\to\infty} \Phi_0(x_n)^\alpha \lim_{n\to\infty} \Phi_0(y_n),\\ &= \Phi(x)^\alpha \Phi(y).\end{aligned}$$

That is $\Phi : \mathcal{M} \to \mathcal{Q}$ is a multiplicative operator. For a sequence (x_n) in $\mathcal{M}$, and $x \in \mathcal{M}$, as $x_n \mapsto x$ implies $\Phi x = \lim_{n\to\infty} \Phi_0 x_n$ and so $\natural\Phi x\natural = \lim_{n\to\infty} \natural\Phi_0 x_n\natural$, and $\mathcal{M}$ is dense in $\mathcal{M}$, we have

$$\begin{aligned}\natural\Phi\natural_3 &= e^{\sup\left\{\frac{\ln\natural\Phi x\natural}{\ln\natural x\natural},\quad x\neq 1,\ x\in\mathcal{M}\right\}},\\ &= e^{\sup\left\{\frac{\ln\natural\Phi_0 x\natural}{\ln\natural x\natural},\quad x\neq 1,\ x\in\mathcal{M}\right\}},\\ &= \natural\Phi_0\natural_3.\end{aligned}$$

Since Φ_0 is bounded and $\natural\Phi\natural_3 = \natural\Phi_0\natural_3$, it follows that Φ is a bounded multiplicative operator. The uniqueness of Φ obtained straight. $\square$

The main purpose of this section is to show that a bounded multiplicative functions on a multiplicative subspace of a multiplicative space can always be extended to a bounded multiplicative functional on the entire multiplicative space without increasing its multiplicative $\star$-normed. The plan is to prove this for real multiplicative spaces. In this manner we need some preliminary result about real multiplicative functionals on multiplicative spaces.

Definition 3.1. Let p be a real valued function on a multiplicative space $\mathcal{M}$. Then p is called multiplicative positive homogeneous if $p(x^t) = p(x)^t$ for $t > 0$ and $x \in \mathcal{M}$, and is called multiplicative subproduct if $p(xy) \leq p(x)p(y)$ whenever $x, y \in \mathcal{M}$. If p has both properties, then it is said to be a submultiplicative functional.

In the following we give an important extension of the Hahn Banach theorem to nonlinear functionals.

Theorem 3.2. Suppose that $p : \mathcal{M} \to [0,\infty)$ is a submultiplicative functional on a multiplicative space $\mathcal{M}$ and that $f_0 : \mathcal{Q} \to [0,\infty)$ is a multiplicative functional on a multiplicative subspace $\mathcal{Q}$ of $\mathcal{M}$ such that $f_0(y) \leq p(y)$ for $y \in \mathcal{Q}$. Then there is a multiplicative functional f on $\mathcal{M}$ such that the restriction of f to $\mathcal{Q}$ is f_0 and $f(x) \leq p(x)$ whenever $x \in \mathcal{M}$.

Proof . First we show that if $\mathcal{Q} \neq \mathcal{M}$, then there is a multiplicative extension f_1 of f_0 to a multiplicative subspace of $\mathcal{M}$ larger than $\mathcal{Q}$ such that f_1 is still dominated by $\star$ on this subspace. Let $x_1 \in \mathcal{M} \setminus \mathcal{Q}$ and $\mathcal{Q}_1 = \langle \mathcal{Q} \cup \{x_1\} \rangle$, where $\langle \{x\} \rangle = \{x^\alpha : \quad x \in \mathcal{M}, \alpha \in \mathbb{R}\}$, a multiplicative subspace of $\mathcal{M}$ that properly includes $\mathcal{Q}$. Notice that if $yx_1^t = y'x_1^{t'}$, where $y, y' \in \mathcal{Q}$ and $t, t' \in \mathbb{R}$, then $x_1^t x_1^{-t'} = x_1^{t-t'} = y'y^{-1} \in \mathcal{Q}$, and so $t = t'$ and $y = y'$. Thus, each member of $\mathcal{Q}_1$ has unique representation in the form yx_1^t, where $y \in \mathcal{Q}$ and $t \in \mathbb{R}$. Whenever $y_1, y_2 \in \mathcal{Q}$, since f_0 is a multiplicative functional, we have

$$\begin{aligned} f_0(y_1)f_0(y_2) &= f_0(y_1y_2), \\ &\leq p(y_1x_1^{-1}x_1y_2), \\ &\leq p(y_1x_1^{-1})p(x_1y_2). \end{aligned}$$

and so

$$f_0(y_1)p(y_1x_1^{-1})^{-1} \leq p(x_1y_2)f_0(y_2)^{-1}.$$

It follows that

$$\sup\{f_0(y)p(yx_1^{-1})^{-1} : y \in \mathcal{Q}\} \leq \inf\{p(x_1y)f_0(y)^{-1} : y \in \mathcal{Q}\},$$

so there is a real number t_1 such that

$$\sup\{f_0(y)p(yx_1^{-1})^{-1} : y \in \mathcal{Q}\} \leq t_1 \leq \inf\{p(x_1y)f_0(y)^{-1} : y \in \mathcal{Q}\}.$$

Let $f_1(yx_1^t) = f_0(y)t_1^t$ for each $y \in \mathcal{Q}$ and $t \in \mathbb{R}$. We show that f_1 is a multiplicative functional on $\mathcal{Q}_1$. Let $y, y' \in \mathcal{Q}$ and $t, t' \in \mathbb{R}$. We have

$$\begin{aligned} f_1((yx_1^t)^\alpha(y'x_1^{t'})) &= f_1(y^\alpha y'(x_1^{t\alpha+t'})), \\ &= f_0(y^\alpha y')t_1^{t\alpha+t'}, \\ &= (f_0(y))^\alpha f_0(y')t_1^{t\alpha}t_1^{t'}, \\ &= (f_0(y)t_1^t)^\alpha f_0(y')t_1^{t'}, \\ &= (f_1(yx_1^t))^\alpha f_1(y'x_1^{t'}). \end{aligned}$$

For $y \in \mathcal{Q}$, we have $f_1(y) = f_1(yx^0) = f_0(y)\Phi_1^0 = f_0(y)$, that is restriction of f_1 on $\mathcal{Q}$ agrees f_0. It follows from the definition of t_1 that for any $y \in \mathcal{Q}$ and any positive t, we have

$$\begin{aligned} f_1(yx_1^t) = f_0(y)t_1^t &= (f_0(y^{\frac{1}{t}})t_1)^t, \\ &= (f_0(x_1y^{\frac{1}{t}}))^t, \\ &\leq (p(x_1y^{\frac{1}{t}}))^t, \\ &= p(yx_1^t), \end{aligned}$$

and

$$\begin{aligned} f_1(yx_1^{-t}) = f_0(y)t_1^{-t} &= (f_0(y^{\frac{1}{t}})t_1^{-1})^t, \\ &= (f_0(y^{\frac{1}{t}}x_1^{-1}))^t, \\ &\leq (p(y^{\frac{1}{t}}x_1^{-1}))^t, \\ &= p(yx_1^{-t}), \end{aligned}$$

that is, for $x \in \mathcal{Q}_1$, we have $f_1(x) \leq p(x)$. Now, Let $\mathcal{F}$ be the collection of all multiplicative functionals $g : \mathcal{Q}' \to \mathbb{R}$, where $\mathcal{Q}'$ is a multiplicative subspace of $\mathcal{M}$ that includes $\mathcal{Q}$ and the restriction of $g\mid_{\mathcal{Q}} = f_0$ and g is dominated by $\star$ on $\mathcal{Q}'$. We define a partial ordering $\preceq$ on $\mathcal{F}$ by declaring that for $g_1, g_2 \in \mathcal{F}$, we have $g_1 \preceq g_2$, whenever Dom $g_1 \subseteq$ Dom g_2 and g_2 restricted to Dom g_1 agrees g_1.

Each nonempty chain $\mathcal{C}$ in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Consider the multiplicative functional f with domain D_f is equal to union of the domains of the members in chain $\mathcal{C}$, and for all $t \in D_f$, $f(t)$ agrees with evrey member of chain $\mathcal{C}$ that is defined at t. By Zorn's lemma, $\mathcal{F}$ has a maximal element f_0. We must have Dom $f_0 = \mathcal{M}$, because if Dom $f_0 \neq \mathcal{M}$, we apply the first part and get h in $\mathcal{F}$ such that $f_0 \preceq h$ but $f_0 \neq h$, which contradict the maximality of f_0. $\square$ We obtain the following theorem for $\natural.\natural_3$.

Theorem 3.3. Let $\mathcal{Q}$ be a multiplicative subspace of a multiplicative $\star$-normed space $\mathcal{M}$ and $\Phi_0 : \mathcal{Q} \to [0,\infty)$ be a bounded multiplicative functional on $\mathcal{Q}$. Then Φ_0 can be extended to a bounded multiplicative functional T defined on $\mathcal{M}$ with the same $\star$-norm, i.e. $\natural\Phi\natural = \natural\Phi_0\natural$.

Proof . Let $p(x) = e^{\ln \natural\Phi_0\natural \ln \natural x\natural}$, for any $x \in \mathcal{M}$. For $t > 0$ and $x, y \in \mathcal{M}$, we have

$$p(x^t) = e^{\ln \natural\Phi_0\natural \ln \natural x^t\natural} = e^{\ln \natural\Phi_0\natural \ln \natural x\natural^t} = e^{t \ln \natural\Phi_0\natural \ln \natural x\natural} = p(x)^t,$$

and

$$p(xy) = e^{\ln \natural\Phi_0\natural \ln \natural xy\natural} \le e^{\ln \natural\Phi_0\natural \ln(\natural x\natural\natural y\natural)} = e^{\ln \natural\Phi_0\natural(\ln \natural x\natural + \ln \natural y\natural)} = p(x)p(y).$$

Thus p is a submultiplicative functional on $\mathcal{M}$. On the other hand, by definition $\natural.\natural_3$, we have

$$\natural\Phi_0\natural = e^{\sup\left\{\frac{\ln|\Phi_0 x|}{\ln \natural x\natural}: \quad x\in\mathcal{Q},\ x\neq 1\right\}},$$

and so

$$e^{\frac{\ln|\Phi_0 x|}{\ln \natural x\natural}} \le \natural\Phi_0\natural, \qquad \text{for all } x \in \mathcal{Q},\ x \neq 1,$$
$$\frac{\ln|\Phi_0 x|}{\ln \natural x\natural} \le \ln \natural\Phi_0\natural, \qquad \text{for all } x \in \mathcal{Q},\ x \neq 1,$$
$$\ln|\Phi_0 x| \le \ln \natural\Phi_0\natural \ln \natural x\natural, \qquad \text{for all } x \in \mathcal{Q},\ x \neq 1,$$

which follows that

$$\Phi_0 x = |\Phi_0 x| \le e^{\ln \natural\Phi_0\natural \ln \natural x\natural} = p(x), \qquad \text{for all } x \in \mathcal{Q},\ x \neq 1.$$

That is, $\Phi_0 x \le p(x)$ for all $x \in \mathcal{Q}$. By Theorem 3.2 and its proof, there is a real positive extension T of Φ_0 defined on $\mathcal{M}$ such that for all $x \in \mathcal{M}$,

$$\Phi x \le p(x),$$

and so for all $x \in \mathcal{M}$, we have

$$|\Phi x| \le e^{\ln \natural\Phi_0\natural \ln \natural x\natural},$$

and so for all $x \in \mathcal{M}$ with $x \neq 1$, we have

$$\frac{\ln|\Phi x|}{\ln \natural x\natural} \le \ln \natural\Phi_0\natural,$$

which implies

$$\natural\Phi\natural = e^{\sup\left\{\frac{\ln|\Phi x|}{\ln \natural x\natural}: \quad x\in\mathcal{M},\ x\neq 1\right\}} \le \natural\Phi_0\natural.$$

Also, we have

$$\begin{aligned}\natural\Phi\natural = e^{\sup\left\{\frac{\ln|\Phi x|}{\ln \natural x\natural}: \quad x\in\mathcal{M},\ x\neq 1\right\}} &\ge e^{\sup\left\{\frac{\ln|\Phi x|}{\ln \natural x\natural}: \quad x\in\mathcal{Q},\ x\neq 1\right\}} \\ &= e^{\sup\left\{\frac{\ln|\Phi_0 x|}{\ln \natural x\natural}: \quad x\in\mathcal{Q},\ x\neq 1\right\}} \\ &= \natural\Phi_0\natural.\end{aligned}$$

Therefore it implies that $\natural\Phi\natural = \natural\Phi_0\natural$. Since Φ_0 is bounded and $\natural\Phi\natural = \natural\Phi_0\natural$, it follows the boundedness of T. □

Theorem 3.4. Let $\mathcal{Q}$ be a closed subspace of a multiplicative normed space $\mathcal{A}$. Suppose that $x \in \mathcal{A} \setminus \mathcal{Q}$. Then there is a $f \in \mathcal{A}^\star$ such that $\natural f\natural = e$, $f(x) = d(x, \mathcal{Q})$, and $f|_{\mathcal{Q}} = \{1\}$.

Proof . Let $f_0(yx^\alpha) = d(x, \mathcal{Q})^\alpha$ for each y in $\mathcal{Q}$ and each scalar α. Then f_0 a multiplicative functional on $\mathcal{Q}(span\{x\})$, such that $f_0(x) = d(x, \mathcal{Q})$ and $f_0(y) = 1$ for each y in $\mathcal{Q}$. Whenever $y \in \mathcal{Q}$ and $\alpha \neq 0$, $\lfloor f_0(y)\rfloor \le \natural y\natural_3$, and so, $\ln\lfloor f_0(y)\rfloor \le \ln \natural y\natural_3$. Then we have

$$\sup \frac{\ln\lfloor f_0(y)\rfloor}{\ln \natural y\natural_3} \le 1,$$

so f_0 is bounded and $\natural f_0\natural_3 \le e$. Also, we have

$$e^{\ln \natural xy^{-1}\natural \ln \natural f_0\natural_3} \ge \lfloor f_0(xy^{-1})\rfloor = d(x, \mathcal{Q}).$$

We get the infinite on the left. As a result

$$e^{\ln d(x,\mathcal{Q}) \ln \natural f_0 \natural_3} \geq d(x,\mathcal{Q}),$$

and so,

$$\ln d(x,\mathcal{Q}) \ln \natural f_0 \natural_3 \geq d(x,\mathcal{Q}).$$

Since, $\ln \natural f_0 \natural_3 \geq 1$, it follows that $\natural f_0 \natural_3 \geq e$, and so $\natural f_0 \natural_3 = e$. To finish, let f be any extension of f_0 to $\mathcal{A}$ by Theorem 3.3. □

Some of the immediate consequences of Theorem 3.4 are as follows:

Corollary 3.5. Let x be a nonzero element of a multiplicative normed space $\mathcal{A}$. Then there exists $f \in \mathcal{A}^\star$ such that $f(x) = \natural x \natural$ and $\natural f \natural_3 = e$.

Corollary 3.6. If x is a nonzero element of a multiplicative normed space $\mathcal{A}$, then there is a $f \in \mathcal{A}^\star$ such that $\natural f \natural_3 = e$ and $f(x) = \natural x \natural$.

Corollary 3.7. If x and y are different elements of a multiplicative normed space $\mathcal{A}$, then there is a $f \in \mathcal{A}^\star$ such that $f(x) \neq f(y)$.

Definition 3.2. A subset $\mathcal{Q}$ of a multiplicative space $\mathcal{A}$ is said to be multiplicative convex or $\star$-convex if $y^t z^{(1-t)} \in \mathcal{Q}$ whenever $y, z \in \mathcal{Q}$ and $0 < t < 1$.

Definition 3.3. A CMN space $\mathcal{A}$ is said to be strictly convex if

$$\natural x \natural = e, \natural y \natural = e \quad \text{with} \quad x \neq y \Rightarrow \natural x^{(1-\lambda)} y^{\lambda} \natural < e \text{ for all } \lambda \in (0,1).$$

For example $\mathbb{R}^+$ with norm $\natural x \natural$ defined by

$$\natural x \natural = \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ x & x > 1. \end{cases}$$

is strictly convex.

Proposition 3.8. Let $\mathcal{A}$ be a strictly convex CMN space and $\mathcal{Q}$ a nonempty convex subset of $\mathcal{A}$. Then there is at most one point x in $\mathcal{Q}$ such that $\natural x \natural = \inf\{\natural z \natural \ : \ z \in \mathcal{Q}\}$.

Proof . Suppose, there exist two points $x, y \in \mathcal{Q}$, $x \neq y$ such that

$$\natural x \natural = \natural y \natural = \inf\{\natural z \natural \ : \ z \in \mathcal{Q}\} = d(say).$$

If $t \in (0,1)$, then by strict convexity of $\mathcal{A}$ we have that

$$\natural x^{(1-t)} y^t \natural < d,$$

which is a contradiction, as $x^{(1-t)} y^t \in \mathcal{Q}$ by the convexity of $\mathcal{Q}$. □

Theorem 3.9. Let $\mathcal{A}$ be a multiplicative norm space. Then the following are equivalent:

(i) $\mathcal{A}$ is strictly convex.

(ii) For each $f \in \mathcal{A}^\star \setminus \{1\}$, there exists at most one point x in $\mathcal{A}$ such that $\natural x \natural = e$ and $f(x) = \natural f \natural_3$.

Proof . $(i) \Rightarrow (ii)$. Let $\mathcal{A}$ be a strictly convex CNM space and f an element in $\mathcal{A}^*$. Suppose there exist two distinct points x, y in $\mathcal{A}$ with $\natural x \natural = \natural y \natural = e$ such that $f(x) = f(y) = \natural f \natural_3$. If $t \in (0,1)$, then

$$\begin{aligned} \natural f \natural_3 &= f(x)^t f(y)^{(1-t)} \qquad (as f(x) = f(y) = \natural f \natural_3) \\ &= f(x^t y^{(1-t)}) \\ &\leq \natural f \natural_3 \natural x^t y^{(1-t)} \natural \\ &\leq \natural f \natural_3, \end{aligned}$$

which is a contradiction. Therefore, there exists at most one point x in $\mathcal{A}$ with $\natural x \natural = e$ such that $f(x) = \natural f \natural_3$.
$(ii) \Rightarrow (i)$. Suppose $\natural x \natural = e$ and $\natural y \natural = e$ with $x \neq y$ such that $\natural (xy)^{\frac{1}{2}} \natural = e$. By Corollary 3.5, there exists a functional $f \in \mathcal{A}^*$ such that

$$\natural f \natural_3 = e \qquad and \qquad f((xy)^{\frac{1}{2}}) = \natural (xy)^{\frac{1}{2}} \natural.$$

Because $f(x) \leq e$ and $f(y) \leq e$, we have $f(x) = f(y)$. This implies, by hypothesis, that $x = y$. Therefore, $(b) \Rightarrow (a)$ is proved. □

Theorem 3.10. Let $\mathcal{A}$ be a strictly convex multiplicative norm space. If $\natural xy \natural = \natural x \natural \natural y \natural$ for $1 \neq x \in \mathcal{A}$ and $y \in \mathcal{A}$, then there exists $t \geq 0$ such that $y = x^t$.

Proof . Let $x, y \in \mathcal{A} \backslash \{1\}$ be such that $\natural xy \natural = \natural x \natural \natural y \natural$. From Corollary 3.5, there exists $f \in \mathcal{A}^\star$ such that

$$f(xy) = \natural xy \natural \qquad \text{and} \qquad \natural f \natural_3 = e.$$

Because $f(x) \leq \natural x \natural$ and $f(y) \leq \natural y \natural$, we must have $f(x) = \natural x \natural$ and $f(y) = \natural y \natural$. This means that $e^{\ln(\natural x \natural) \ln(\natural f \natural)} = e$ and since $\natural f \natural_3 = e$, we have $\natural x \natural = e$. Similarly $\natural y \natural = e$. Therefore by strict convexity of $\mathcal{A}$, it follows from Theorem 3.9 , result holds. □

References

[1] M. Abbas, B. Ali and Y.I. Suleiman, *Common fixed points of locally contractive mappings in multiplicative metric spaces with application*, Int. J. Math. Math. Sci. **2015** (2015), Article ID 218683.

[2] A.E. Bashirov, E. Misirli, Y. Tandogdu and A. Ozyapici, *On modeling with multiplicative differential equations*, J. Chinese Univer. **26** (2011), 425–438.

[3] A Bashirov, G Bashirova, *Dynamics of literary texts and diffusion*, Online J. Commun. Media Technol. **1** (2011), 60–82.

[4] A.E. Bashirov, E. Misirli Kurpinar and A. Ozyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (2008), 36–48.

[5] A. Ben Amar, S.Chouayekh and A. Jeribi, *Fixed point theory in a new class of Banach algebras and application,* Afr. Mat. **24** (2013), 705–724.

[6] L. Florack and H.V. Assen, *Multiplicative calculus in biomedical image analysis*, J. Math. Imag. Vision **42** (2012), 64–75.

[7] R.D. Gill and S. Johansen, *A survey of product-integration with a view toward application in survival analysis,* Ann. Stat. **18** (1990), 1501–1555.

[8] M. Grossman and R. Katz. *Non-Newtonian Calculus*, Pigeon Cove, Lee Press, Massachusats, 1972.

[9] G. Metakides, A. Nerode and R.A. Shore, *Recursive limits on the Hahn-Banach theorem*, Errett Bishop: Reflections on him and his Res. **39** (1985), 85–91.

[10] H. Xiaoju, M. Song and D. Chen, *Common fixed points for weak commutative mappings on a multiplicative metric space*, Fixed Point Theory Appl. **2014** (2014), Article ID 48.

[11] D. Stanley, *A multiplicative calculus*, Primus **9** (1999), no. 4, 310–326.