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Dynamics of the modified Halley's method

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Abstract

In this work, the dynamics of the Modified Halley's method to multiple roots are established. We find the fixed and critical points. The stable and unstable behaviors are studied. The parameter space associated with the method is studied and finally, some dynamical planes that show different aspects of the dynamics of this method are presented.

Keywords: nonlinear equations, Modified Halley's method, dynamics, multiple roots
2020 MSC: 65H05

1 Introduction

Iterative methods are necessary usually for solving nonlinear equations $f(x) = 0$. Several good methods exist in the literature when the roots are simple: Newton, Halley and Chebyshev methods between others (see [12] and[44] and reference therein). The classic way to analyze the behavior of these methods is by doing the dynamic study in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Some of the works related to this type of study can be seen in [1]-[8], [10]-[11], [13]-[14] and [18]-[43]. In [17] a new approach is presented to carry out the dynamic study of the applied methods in the case that the two roots are multiple and with different multiplicity. This approach is used in this paper to study the complex dynamics of the modified Halley's method (see [16]).

The rest of the paper is organized as follows. In section 2 the basic preliminaries are presented. Then, in section 3 the dynamics of the method under study is described in detail, taking into account the conjugacy classes, the study of the stability of the fixed points, the behavior of the critical points and the study of the parameter space, presenting to finish this section some dynamic planes and basins of attraction. In section 4 we conclude by presenting the most important results and a discussion on the topic treated.

2 Basic preliminaries

In this section some definitions necessary for the development of the points mentioned above are presented. Usually they are found in most of the papers that deal with complex dynamics, however they are presented with the purpose of being self-contained.

Given a rational function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ is the Riemann sphere

Definition 2.1. For $z \in \widehat{\mathbb{C}}$ we define its orbit as the set

$$orb(z) = \left\{z, R(z), R^2(z), \cdots, R^n(z), \cdots\right\}$$

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Definition 2.2. A periodic point z_0 of period $m > 1$ is a point such that $R^m(z_0) = z_0$ y $R^k(z_0) \neq z_0$, for $k < m$.

Definition 2.3. A pre-periodic point is a point z_0 that is not periodic but exist a $k > 0$ such that R^k is periodic.

Definition 2.4. A point z_0 is a fixed point of R if $R(z_0) = z_0$.

Definition 2.5. A critical point z_{cr} is a point such that $R'(z_{cr}) = 0$.

Definition 2.6. A fixed point z_0 is called attractor if $|R'(z_0)| < 1$, repulsive if $|R'(z_0)| > 1$, and parabolic or neutral if $|R'(z_0)| = 1$. If $|R'(z_0)| = 0$ then the fixed point is called superattractor. A fixed point superattractor is also a critical point.

Definition 2.7. A fixed point z_0 that is not associated to the roots of the function $f(z)$ is called strange fixed point.

Definition 2.8. The basin of attraction of a attractor $\alpha \in \widehat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α.

3 Dynamical behavior of the rational function associated with the method in study

Aim of this paper is to study the dynamics of the rational map obtained by applying the modified Halley's method given by the iteration equation

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \frac{2m}{(m+1-mL_f(z_n))} \quad with \quad L_f(z) = \frac{f(z)f''(z)}{[f'(z)]^2} \tag{3.1}$$

where m is the multiplicity of one of the roots of f.

In the next subsections the conjugacy class is established and the analytical expressions for the fixed and critical points of this method in terms of the parameter K (which is defined in the next subsection) is obtained. Then study of the fixed points, critical points and parameter space are presented. To finish this section several dynamical planes to different values of K selected from parameter space are shown.

3.1 Conjugacy classes

Throughout the remainder of this paper we study the dynamics of the rational map R arising from the method (3.1)

$$R_f = z - \frac{f(z)}{f'(z)} \frac{2m}{(m+1-mL_f(z))} \quad with \quad L_f(z) = \frac{f(z)f''(z)}{[f'(z)]^2}$$

applied to a generic polynomial $p(z) = (z-a)^m(z-b)^n, a \neq b$ with $m = Kn$. Let us first recall the definition of analytic conjugacy classes.

Definition 3.1. [9]. Let f and g be two maps from the Riemann sphere into itself. An analytic conjugacy between f and g is an analytic diffeomorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$.

R_f has the following property for an analytic function f.

Theorem 3.2. (The Scaling Theorem). Let $f(z)$ be an analytical function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, $\alpha \neq 0$, be an affine map. If $g(z) = f \circ T(z)$, then $T \circ R_g \circ T^{-1} = R_f(z)$. That is, R_f is analytically conjugate to R_g by T.

Proof . With the iteration function $R(z)$, we have

$$R_g(T^{-1}(z)) = T^{-1}(z) - \frac{g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)} \left(\frac{2m}{m+1-mL_g\left(T^{-1}(z)\right)}\right)$$

Since $\alpha T^{-1}(z)+\beta = z$, $g\circ T^{-1}(z) = f(z)$ and $\left(g\circ T^{-1}\right)'(z) = \frac{1}{\alpha}g'\left(T^{-1}(z)\right)$, we get $g'\left(T^{-1}(z)\right) = \alpha\left(g\circ T^{-1}\right)'(z) = \alpha f'(z)$, $g''\left(T^{-1}(z)\right) = \alpha^2 f''(z)$. We therefore have

$$
\begin{aligned}
T\circ R_g\circ T^{-1}(z) &= T\left(R_g(T^{-1}(z))\right) = \alpha R_g\left(T^{-1}(z)\right)+\beta \\
&= \alpha T^{-1}(z) - \frac{\alpha g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)}\left(\frac{2m}{m+1-m\frac{g(T^{-1}(z))g''(T^{-1}(z))}{(g'(T^{-1}(z)))^2}}\right)+\beta \\
&= z-\frac{f(z)}{f'(z)}\left(\frac{2m}{m+1-m\frac{f(z)f''(z)}{(f'(z))^2}}\right) = R_f(z)
\end{aligned}
$$

This completing the proof □

The above scaling theorem indicates that up to a suitable change of coordinates the study of the dynamics of the iteration equation (3.1) for polynomials can be reduced to the study of the dynamics of the same iteration function for simpler polynomials.

Definition 3.3. [35]. We say that a one-point iterative root-finding algorithm $p\to T_p$ has a universal Julia set (for polynomials of degree d) if there exists a rational map S such that for every degree d polynomial p, $J(T_p)$ is conjugate by a Möbius transformation to $J(S)$

The following theorem establishes a universal Julia set for polynomials with two multiple roots with known multiplicity m and n for the method (3.1).

Theorem 3.4. For a rational map $R_p(z)$ arising from the method (3.1) applied to $p(z) = (z-a)^m(z-b)^n, a\neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to

$$S(z) = \frac{(K+1)z^3}{(1-K)z^2+2K(1-K)z+2K^2}$$

where $m = Kn$

Proof . Let $p(z) = (z-a)^m(z-b)^n, a\neq b$ with $m = Kn$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse given by $M^{-1}(u) = \frac{bu-a}{u-1}$, which may be considered as a map from $\mathbb{C}\cup\{\infty\}$. We then have

$$M\circ R_p\circ M^{-1}(u) = M\circ R_p\left(\frac{bu-a}{u-1}\right) = \frac{u^3(K+1)}{(1-K)u^2+2K(1-K)u+2K^2}.$$

□

We observe that parameters a and b have been obviated in $S(z)$, as an effect of the Scaling Theorem that is verified by this method.

3.2 Study of the fixed points

The fixed points of $S(z)$ are the roots of the equation $S(z) = z$, that is, $z = 0$, $z = \infty$, $z = 1$ and $z = -K$. In order to study the stability of the fixed points, we calculate the first derivative of $S(z)$,

$$S'(z) = \frac{(K+1)z^2\left[(1-K)z^2+4K(1-K)z+6K^2\right]}{\left[(1-K)z^2+2K(1-K)z+2K^2\right]^2} \tag{3.2}$$

Is obvious from (3.2) that $z = 0$ and $z = \infty$ are superatractive fixed points. The stability of the others fixed points changes depending on the values of the parameter K. The operator $S'(z)$ in $z = 1$ gives

$$|S'(1)| = |2K+1| \tag{3.3}$$

In the following result we present the stability of the fixed point $z = 1$.

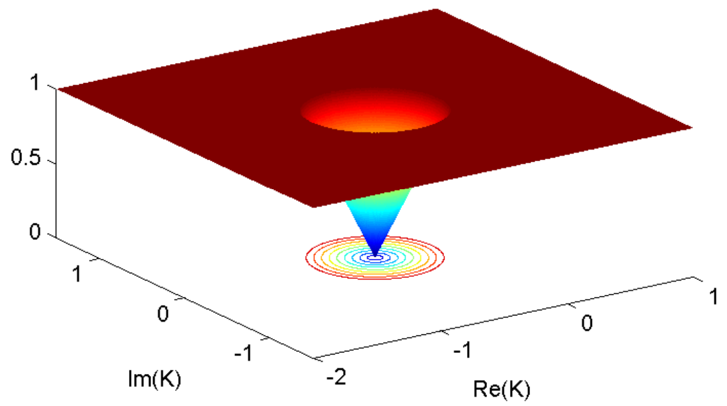

Figure 1: Stability function.

Theorem 3.5. The strange fixed point $z = 1$ satisfies the following statements:
1. If $|K + 1/2| < 1/2$, then $z = 1$ is an attractor and if $K = -1/2$ is a superattactor
2. If $|K + 1/2| = 1/2$, then $z = 1$ is a parabolic fixed point
3. If $|K + 1/2| > 1/2$, then $z = 1$ is a repulsive fixed point.

Proof . From (3.3),

$$|S'(1)| = |2K + 1| < 1 \iff |K + 1/2| < 1/2.$$

In similar form for the other two items. □

The operator $S'(z)$ in $z = -K$ gives

$$|S'(-K)| = 3 > 1 \quad \forall K \in \widehat{\mathbb{C}}$$

Therefore $z = -K$ is a repulsive fixed point.

3.3 Study of the critical points

Critical points of $S(z)$ satisfy $S'(z) = 0$, that is, $z = 0$, $z = \infty$ and

$$\begin{aligned} C_1 &= \frac{K[2(1-K) + \sqrt{4K^2 - 2K - 2}]}{K-1} \\ C_2 &= \frac{K[2(1-K) - \sqrt{4K^2 - 2K - 2}]}{K-1} \end{aligned}$$

It's easy to check that $C_2 = \frac{6K^2}{(1-K)C_1}$. When $K \in \mathbb{R}$ and $K > 1$ criticals points are reals and these are complex if $0 < K < 1$. If $K \to \infty$ then $C_1 \to \frac{3}{2}$ and $C_2 \to -4(K + 1)$.

In Figure 2, the behavior of the critical points for real values of K between 1 and 10 is represented.

3.4 Study of parameter space

In order to find the best members of the family in terms of stability, the parameter spaces associated to the free critical points C_1 and C_2 will be shown. It is well known that there is at least one critical point associated with each invariant Fatou component. The parameter plane is obtained by iterating the selecting critical point; each point of the parameter plane is associated with a complex value of K. Here, we using a mesh of 1000×1000 points, a maximum of 50 iterations and a tolerance of 10^{-2}. All the figures presented below were made with some modifications of the codes presented in [15].

Green color in Figure 3 means that the critical point is in the basin of attraction of $z = 0$, if this is red is in the basin of attraction of $z = \infty$, while the black color indicates that the critical point generates iterations that do not converge to the roots.

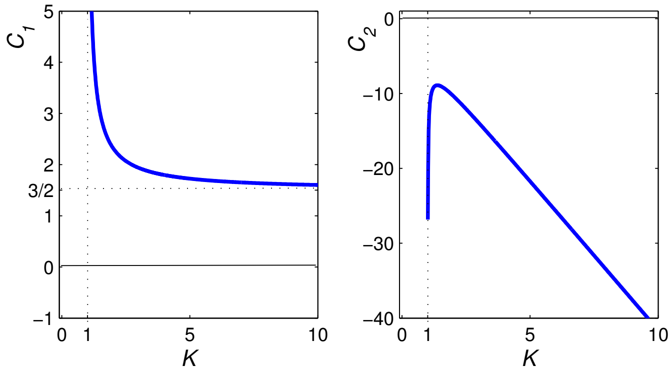

Figure 2: Critical point. Left: C_1, Right:C_2

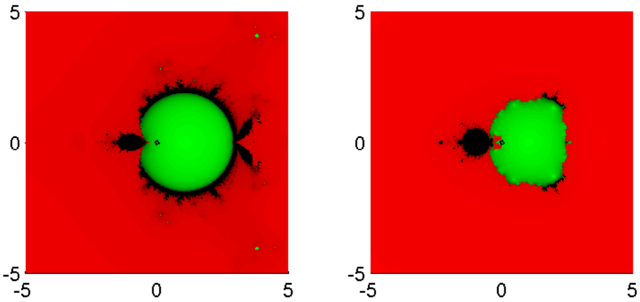

Figure 3: Parameter spaces associated to the critical point C_1 and C_2

3.5 Dynamical Planes

Then, focussing the attention in the regions shown in Figure 3 it is evident that there exist members of the family with complicated behavior. In Figures 4-5 diverse stable dynamical planes are shown. In these dynamical planes the convergence to 0 appear red, in green it appears the convergence to ∞, in black the zones with no convergence to the roots.

3.6 Basins of attraction

To complete this work, the dynamic study of the function $g(x) = (x-1)^m(x+1)^n$ is carried out, calculating the fixed points and critical points when applying the method under study and thus proceed to make the graphical representation of the attraction basins of the two roots $z = -1$ y $z = 1$.

Applying the modified Halley method to the function g gives the iteration equation

$$x_{n+1} = \frac{(K+1)x_n^3 + 2(K-1)(K+1)x_n^2 + (4K^2+K+1)x_n + 2K(K-1)}{(K+1)(2K+1)x_n^2 + 2(K-1)(2K+1)x_n + 2K^2 - K + 1}$$

So, the iteration function in the complex plane is given by

$$G(z) = \frac{(K+1)z^3 + 2(K-1)(K+1)z^2 + (4K^2+K+1)z + 2K(K-1)}{(K+1)(2K+1)z^2 + 2(K-1)(2K+1)z + 2K^2 - K + 1} \tag{3.4}$$

Figures 6 and 7 present the dynamic planes of the attraction basins obtained by selecting initial values within the square $[0,3] \times [0,3]$.

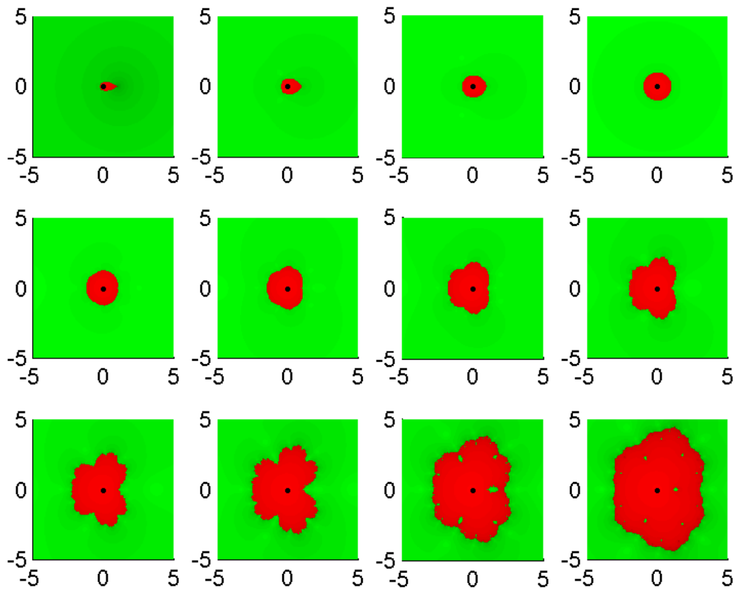

Figure 4: Dynamical planes. $K = 0.25 : 0.25 : 3$

3.6.1 Stability of the fixed points

To calculate the fixed points of G, $G(z) = z$ is used, obtaining $z = 1$, $z = -1$ and $z = \frac{1-K}{K+1}$. It is clear that $z = 1$ is a superatractor fixed point, since it is the root of g with multiplicity m. It is necessary to study the stability of the fixed points $z = -1$ and $z = \frac{1-K}{K+1}$. To do this, we differentiate G from (3.4)

$$G'(z) = \frac{(K+1)(z-1)^2[(K+1)(2K+1)z^2 + 2(6K^2+K-1)z + 10K^2 - 5K + 1]}{[(K+1)(2K+1)z^2 + 2(K-1)(2K+1)z + 2K^2 - K + 1]^2}$$

and evaluate these values:

$$G'\left(\frac{1-K}{K+1}\right) = 3 > 1$$

then strange fixed point $z = \frac{1-K}{K+1}$ is repulsive. The operator $G'(z)$ in $z = -1$ gives

$$|G'(-1)| = \left|\frac{K-1}{K+1}\right| \tag{3.5}$$

If we analyze this function, we obtain an horizontal asymptote in $|G'(-1)| = 1$ when $K \to \pm\infty$, and a vertical asymptote in $K = -1$.

In the following result we present the stability of the fixed point $z = -1$.

Theorem 3.6. The strange fixed point $z = -1$ satisfies the following statements:

1. If $Re\{K\} > 0$, then $z = -1$ is an attractor and is a superattractor for $K = 1$.
2. If $Re\{K\} = 0$, then $z = -1$ is a parabolic fixed point.
3. If $Re\{K\} < 0$, then $z = -1$ is a repulsive fixed point.

Proof . From (3.5),

$$|G'(-1)| = \left|\frac{K-1}{K+1}\right| < 1 \Rightarrow |K-1| < |K+1|$$

Let $K = \alpha + i\beta$ be an arbitrary complex number. Then,

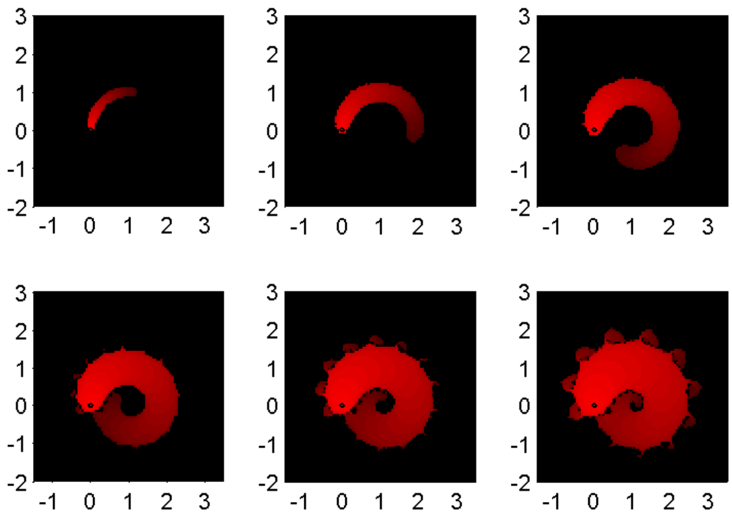

Figure 5: Dynamical planes. $K = 0.05I : 0.05I : 0.3I$

$$|K-1|^2 = (\alpha-1)^2 + \beta^2$$

and

$$|K+1|^2 = (\alpha+1)^2 + \beta^2$$

So

$$4\alpha > 0$$

Therefore,

$$|G'(-1)| < 1 \Leftrightarrow Re\{K\} > 0$$

Finally, if $Re\{K\} \leq 0$, then $|G'(-1)| \geq 1$. $\square$

3.6.2 Critical points

Critical points of $G(z)$ satisfy $G'(z) = 0$, that is, $z = 1$ with multiplicity two and

$$C_{1,2} = \frac{1 - K - 6K^2 \pm 2K\sqrt{2(K-1)(2K+1)}}{(K+1)(2K+1)}$$

4 Result and discussion

In this article we present the study of complex dynamics for the modified Halley method for the polynomial class with two roots with known multiplicities. For this, initially the scaling theorem and conjugation mapping for this method were established, then the fixed points and critical points of the rational operator under study were calculated. The parameter space is also analyzed, selecting different values of this parameter to plot the respective dynamical planes. Finally, the attraction basins of a particular example are presented, emphasizing the stability of the fixed points. It is clear that more studies are needed on the dynamics of this family.

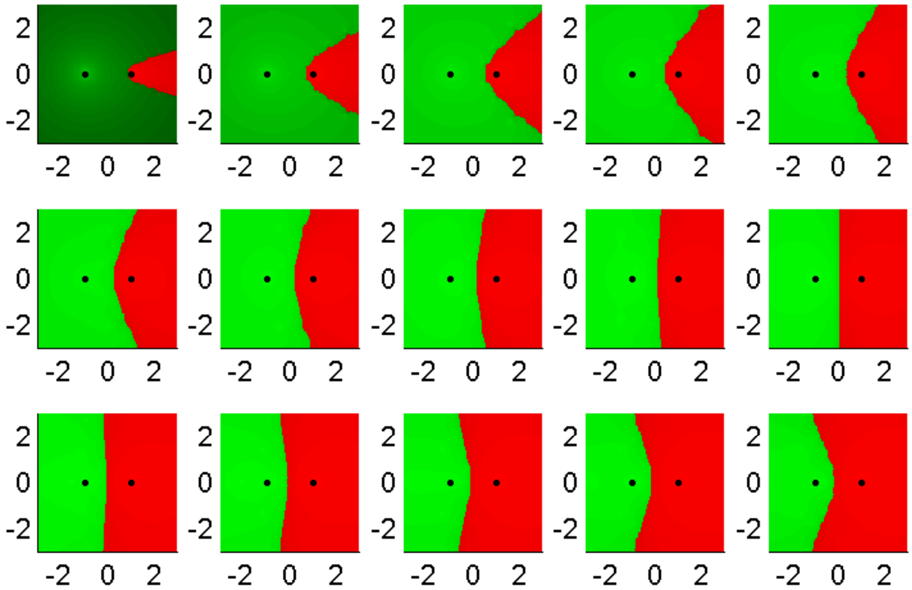

Figure 6: Dynamical planes. $f(x) = (x-1)^m(x+1)^n$, $m = Kn, K = 0.1 : 0.1 : 1.5$

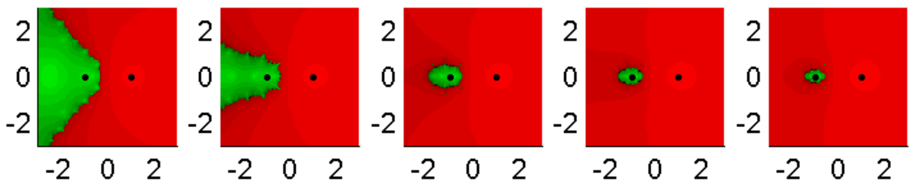

Figure 7: Dynamical planes. $f(x) = (x-1)^m(x+1)^n$, $m = Kn, K = 2 : 0.5 : 4$

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