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Projective modules relative to a semiradical property

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Abstract

In this paper, we introduce a generalization of the projective modules. We show that for a module $M = M_1 \bigoplus M_2$. If M_2 is s.p-M_1-projective, then for every s.p-closed submodule A of M with $M = M_1 + A$, there exists a submodule K of A such that $M = M_1 \bigoplus K$.

Keywords: s.p-closed submodules, projective module, s.p-projective module
2020 MSC: 16E40

1 Introduction

Throughout article all rings are associative with identity and all modules are unitary left R-modules. Let N be a submodule of a module M. N is called an essential submodule of M (indicate by $N \leq_e M$) if $N \bigcap K \neq 0$, $\forall 0 \neq K \leq M$. A submodule B of M is called a closed submodule of M if B has no proper essential extension in M, see [6].

Let M be a module, recall that the socle of M (denoted by $Soc(M)$) is the sum of all simple submodules of M. A module M is said to be a semisimple module if $Soc(M) = M$, see [6, 8].

Let M be a module. Recall that the Jacobson radical of M (denoted by J(M)) is the intersection of all maximal submodules of M. If M has no maximal submodule, we write J(M)=M, see [13].

Let $m \in M$. Recall that $ann\ (m) = \{r \in R : rm = 0\}$. For a module M, the singular submodule is defined as follows $Z(M) = \{m \in M | ann(m) \leq_e R\}$ or equivalently, $Im = 0$ for some essential left ideal I of R. If $Z(M) = M$, then M is called a singular module. If $Z(M) = 0$, then M is called a nonsingular module. The second singular (or Goldie torsion) submodule of a module M (denoted by $Z_2(M)$) is defined as follows $Z(M/Z(M)) = Z_2(M)/Z(M)$, see [6].

Let R be a ring. An element $x \in R$ is said to be regular if there exists an element $r \in R$ such that $x = xrx$. R is called regular if every element in R is regular. A module B is called F-regular if for all $0 \neq x \in B, R/ann(x)$ is regular, equivalently an R-module M is F-regular if and only if for all $x \in B$ and $y \in R$, there exists $r \in R$ such that $ryrx = rx$, see [4].

Let N be a module and $M(N) = \sum_{\substack{K \leq N \\ K \text{ is regular}}} N$. Then N is F-regular if and only if $M(N) = N$, see [7]. Let A be a module, a module M is called A-projective if for every submodule B of A, any homomorphism g from M to A/B can be lifted to a homomorphism h from M to A. It is known that a module M is projective if M is A-projective, for every module A, see [5, 8, 9].

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Let S be a semiradical property. A submodule N of a module is said to be s.p-closed submodule of M (denoted by $N \leq_{s.p-c} M$) if $N \leq K \leq M$ and K/N has S implies that $N = K$. Equivalent A is s.p-closed submodule of M if and only if $S(M/A) = 0$, see [1].

In this paper we introduce the concept of projective modules relative to a semiradical property. Let S be a semiradical property. A property S is said to be a radical property if:

1. for each module M, there exists a submodule (denoted by S(M)) such that
   (a) S(M) has S.
   (b) $B \leq S(M)$, for any submodule B of M such that B has S.
2. If $f : M \longrightarrow L$ is an epimorphism and M has S, then L has S.
3. $S(M/S(M)) = 0$ for each module M, see [7].

A property S is said to be a semiradical property if it satisfies the following conditions 1 and 2, see [7]. It's known that each of the following two properties is a radical property, see [7].

1. $S = Z_2$. For a module M, $S(M) = Z_2(M)$, the second singular of M.
2. $S = Snr$. For a module M, $Snr(M)$ is a submodule of M s.t.
   (a) $J(Snr(M)) = Snr(M)$ {i.e. $Snr(M)$ has no maximal submodule}.
   (b) $A \leq Snr(M)$, for every submodule A of M such that $J(A) = A$, see [7].

While each the following two properties is a semiradical property (but not radical property), see [7].

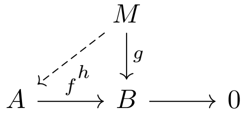
1. $S = Z$. For a module M, $S(M) = Z(M)$, the singular submodule of M.
2. $S = Soc$. For a module M, $S(M) = Soc(M) = \sum_{\substack{A \leq M \\ A \text{ is simple}}} A$.
3. $S = M$. For a module M, $S(B) = M(B) = \sum_{\substack{A \leq M \\ A \text{ is regular}}} A$, the unique maximal regular submodule of B{$M(B)$ is called semi Broun-McCoy radical}.

Let S be a semiradical property. It's known that

1. M has $S \Longleftrightarrow S(M) = M$.
2. $S(S(M)) = S(M)$.
3. If $M = \bigoplus_{i \in I} N_i$, then $S(M) = \bigoplus_{i \in I} S(N_i)$, where i is any index set.
4. If $S(M) = 0$, then $S(A) = 0$, $\forall A \leq M$.
5. For any $s.e.s. 0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$, if $S(M) = 0$ and $S(K) = 0$, then $S(N) = 0$, see [7].

Recall that a semiradical property S is called hereditary if S is closed under submodules, see [7]. In this paper, S is a semiradical algebraic property, unless otherwise stated.

Definition 1.1. Let M and A be R-modules. We say that M is s.p-A-projective, if for any epimorphism $f : A \longrightarrow B$, where B is any R-modules such that $S(B) = 0$ and for any homomorphism $g : M \longrightarrow B$, there exists a homomorphism $h : M \longrightarrow A$ such that $f \circ h = g$.



We say that a module M is s.p-projective if M is s.p-A-projective, for any module A. Clearly that every projective module is s.p-projective.

Remark 1.2. Every module has S is s.p-projective.

Proof . Suppose that $f : A \longrightarrow C$ be an epimorphism with $S(C) = 0$ and $\alpha : M \to C$ be a homomorphism. Since $S(M) = M$, $\alpha = 0$, by [7]. Hence α can be lifted to a homomorphism $0 = \beta : M \to A$ s.t. $f \circ \beta = \alpha$. □

Let S be a semiradical property. Recall that S is called a cohereditary property, if $S(M) = 0$ is closed under homomorphic images of M for every module M, see [7].

Remark 1.3. Let S be a cohereditary property and let M and K be modules such that $S(K) = 0$. Then M is K-projective $\Leftrightarrow$ M is s.p-K-projective.

Proof . $\Rightarrow$) clear.

$\Leftarrow$) Assume that $f : K \to K_1$ be an epimorphism and $g : M \to K_1$ be a homomorphism. Since $S(K) = 0$ and S is cohereditary property, then $S(K_1) = 0$. But M is s.p-K-projective, so there exists a homomorphism $h : M \to K$ s.t. $f \circ h = g$. Thus M is K-projective. □

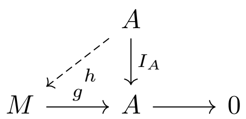
Remark 1.4. Let M and A be modules and $f : M \to B$ be any epimorphism s.t. $S(B) = 0$. Then M is A-projective $\Leftrightarrow$ M is s.p-A-projective.

Proof . $\Rightarrow$) clear.

$\Leftarrow$) Assume that $f : A \to B$ be an epimorphism such that $S(B) = 0$ and let $g : M \to B$ be a homomorphism. But M is s.p-A-projective, therefore there exists a homomorphism $h : M \to A$ s.t. $f \circ h = g$. Thus M is A-projective. □

Proposition 1.5. Let M and A be modules. If $S(A) = 0$ and A is s.p- projective, then every short exact sequence: $0 \to V \xrightarrow{f} M \xrightarrow{g} A \to 0$ is split.

Proof . Look the following graph:

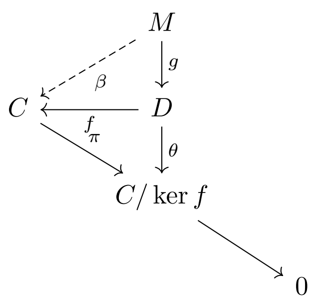


Since A is s.p- projective and $S(A) = 0$, there exists a homomorphism $h : A \to M$ such that $g \circ h = I_A$. Hence g has a right inverse. Thus by [8], the sequence is split. □

Theorem 1.6. Let M and C be modules. Then M is s.p-C-projective $\Leftrightarrow$ for any epimorphism $f : C \to D$, where Kerf is s.p-closed submodule of C and $\beta : M \to C$ be any homomorphism, there exists $g : M \to D$ be a homomorphism s.t. $f \circ g = \beta$.

Proof . $\Leftarrow$) clear.

$\Rightarrow$) Let M be s.p-C-projective and $f : C \to D$ be an epimorphism such that Kerf is s.p-closed submodule of C. By the first isomorphism theorem, $C/Kerf \cong D$, then there exists an isomorphism $\theta : D \to C/Kerf$ define as follows $\theta(d) = c + Kerf$, where $d \in D$ such that $f(c) = d$. Now look the following graph:



where π is the natural epimorphism. Since $\ker f$ is s.p-closed submodule of C, $S(C/\ker f) = 0$, so $S(D) = 0$. But M is s.p-C-projective, therefore there exists a homomorphism $\beta : M \to C$ such that $\pi \circ \beta = \theta \circ g$. Claim that $f \circ \beta = \theta \circ g$. To show that, let $x \in M$, then $\pi \circ \beta(x) = \beta(x) + \ker f = \theta \circ g(x) = c + \ker f$, where $c \in C$ such that $f(c) = g(x)$. Implies that $\beta(x) - c \in \ker f$, so $f(\beta(x) - c) = 0$. Hence $f(\beta(x)) = f(c) = g(x)$. Thus $f \circ h = g$. □

Example 1.7. 1. Let $S = Snr$, consider the module Q as Z-module. Since $S(Q) = Q$, by rem 1.2, Q is s.p-projective. But Z is a PID and Q is not a free Z-module, then Q is not projective.

2. Let $S = Snr$, consider Z/nZ as Z-module. Now consider the short exact sequence:

$$0 \longrightarrow nZ \xrightarrow{i} Z \xrightarrow{\pi} Z/nZ \longrightarrow 0$$

where i is the inclusion map and π is the natural epimorphism. Since Z is indecomposable module, $nZ \nleq \bigoplus Z, \forall n \geq 2$. So the sequence is not split. Hence by [8], Z/nZ is not Z-projective. But $S(Z/nZ) \cong S(Z_n)$ and Z_n is finitely generated, so $J(Z_n) \neq Z_n$. Then $S(Z/nZ) \cong S(Zn) = 0$. Thus by rem. 1.4, Z/nZ is not s.p-Z-projective.
3. Let $S = M$, consider Z_{P^∞} as Z- module. Let $f : Z_{P^\infty} \to Z_{P^\infty}$ be a map defined by $f(\frac{n}{Pm} + Z) = \frac{n}{Pm-1} + Z = p(\frac{n}{pm}+Z)$. Claim that f is an epimorphism. Since for every $y = p(\frac{n}{pm}+Z) \in Z_{P^\infty}$, there exists $x = \frac{n}{pm}+Z \in Z_{P^\infty}$ such that $f(x) = y$.

Now let $Z_{P^\infty} = \bigcup_m Zpm$ and let $x \in Zpm$, since when $m \neq 1$, then pm is not devoid of square, so x is not regular, by [12]. Now if $m = 1$, then x is regular, by [12]. Hence $S(Z_{P^\infty}) = Z_p$, so $S(Z_{P^\infty}/\ker f) \cong S(Z_{P^\infty}) = Z_p$. Then $\ker f$ is not s.p-closed submodule of Z_{P^∞}. Thus by Theorem 1.6, Z_{P^∞} is not s.p-Z-projective.

Proposition 1.8. Let M be a module. If A be a semisimple module, then M is s.p- A-projective.

Proof . Suppose that $f : A \to B$ be an epimorphism such that $S(B) = 0$ and $g : M \to B$ be a homomorphism. But A is semisimple, so $\ker f \leq \bigoplus A$. Hence f is split and so by [8], there exists $f_1 : B \to A$ such that $f \circ f_1 = I_B$. Let $h = f_1 \circ g : M \to A$. Cleary that $f \circ h = g$. Thus M is s.p-A-projective module. □

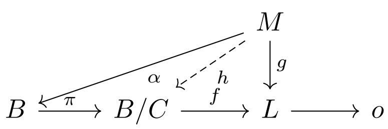
Corollary 1.9. Let S be a hereditary property and $M = A_1 \bigoplus A_2$ be a module such that A_1 has S and A_2 is semisimple. Then M is s.p-M-projective module.

Proof . Since $M = A_1 \bigoplus A_2$ be a module such that A_1 has S and A_2 is semisimple, then by [2], M is semisimple. Thus by prop. 1.8, M is s.p-M-projective module. □

Corollary 1.10. Let S be a hereditary property and M be a module. If $M = S(M) \bigoplus M_1$, where M_1 is semisimple, then M is s.p-M- projective module.

Proposition 1.11. Let M and B be modules and C be a submodule of a module B. If M is s.p-B-projective module, then M is s.p-B/C-projective.

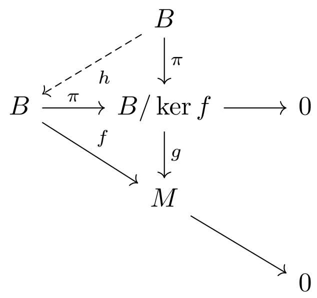
Proof . Let $f : B/C \to L$ be epimorphism such that $S(L) = 0$ and $g : M \to L$ be a homomorphism. Look the following graph:



where π is the natural epimorphism. Since M be s.p-B-projective, $f \circ \pi$ is an epiomorphism, then there exists a homomorphism $\alpha : M \to B$ s.t. $f \circ \pi \circ \alpha = g$. Let $h = \pi \circ \alpha : M \to B/C$. $f \circ h = f \circ \pi \circ \alpha = g$. Thus M is s.p-B/C-projective. □

Proposition 1.12. Let A be s.p-B-projective module and let $f : B \to M$ be an epimorphism such that $\ker f$ is s.p-closed submodule in B, then there exists a homomorphism $h \in End(B)$ s. t. $h(\ker(f)) \leq \ker(f)$.

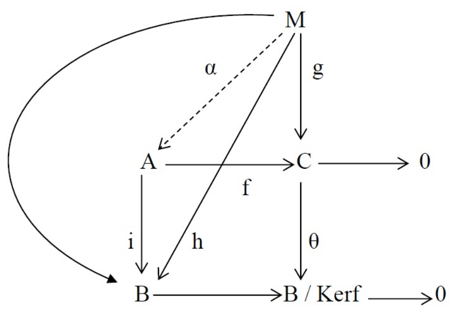
Proof . Suppose that $f : B \to M$ be an epimorphism such that $\ker f$ is s.p-closed submodule of B then by the first isomorphism theorem, $B/\ker f \cong M$. Consider the following diagram:



where π is the natural epimorphism and g is the isomorphism defined by $g(x + \ker(f)) = f(x)$ for all $x \in B$. Since A is s.p-B-projective module, $S(B/\ker f) = 0$. Hence there exists a homomorphism $h : B \to B$ such that $f \circ h = g \circ \pi$. To show that $h(\ker(f)) \leq \ker(f)$. Since $f \circ h(\ker f) = g \circ \pi(\ker f) = g(\pi(\ker f)) = g(0) = 0$, we have $f \circ h(\ker f) = 0$. Thus $h(\ker(f)) \leq \ker(f)$. □

Proposition 1.13. Let S be a cohereditary property and let M and B be modules such that $S(B) = 0$. If M is s.p-B-projective then for every submodule A of B, M is A-projective.

Proof . Let $f : A \to C$ be an epimorphism and $g : M \to C$ be a homomorphism. Look the following graph:



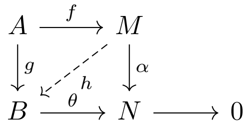
where i is the inclusion map and π is the natural epimorphism. Define $\theta : C \to B/\ker f$ as follows $\theta(c) = b + \ker f$ for each $c \in C$, $f(a) = c$. Now we want to show θ is well define, let c_1 and $c_2 \in C$ such that $c_1 = c_2$, then $f(a_1) = f(a_2) \Rightarrow f(a_1) - f(a_2) = 0 \Rightarrow f(a_1 - a_2) = 0 \Rightarrow a_1 - a_2 \in \ker f$, so $a_1 + \ker f = a_2 + \ker f$. Then $\theta(c_1) = \theta(c_2)$. Thus θ is well define. □

Now we want to show θ is homomorphism. Let $c_1, c_2 \in C$, then $\theta(c_1 + c_2) = a_1 + a_2 + \ker f = a_1 + \ker f + a_2 + \ker f = \theta(c_1) + \theta(c_2)$ and $\theta(rc) = ra + \ker f = r\theta(c)$. Since $S(B) = 0$ and S is cohereditary property, then $S(B/\ker f) = 0$. But M is s.p-B-projective, therefore there exists a homomorphism $h : M \to B$ s.t. $\pi \circ h = \theta \circ g$.

Claim that $h(M) \leq A$. Let $x \in h(M)$, then there exists $y \in M$ such $x = h(y)$. Since $\pi \circ h(y) = \theta \circ g(y) = \theta \circ f(a)$, for some $a \in A$. So $\pi \circ h(y) = a + \ker f \Rightarrow \pi(h(y)) = a + \ker f$. Hence, $\pi(x) = a + \ker f$. This means that $a + \ker f = x + \ker f$ and so, $a - x = a - h(y) \in \ker f$. This implies that $h(M) \leq A$. Define $\alpha : M \to A$ by $\alpha(m) = h(m)$, for each $m \in M$. Then $i \circ \alpha(m) = i(\alpha(m)) = \alpha(m) = h(m)$. Now we want to show $f \circ \alpha = g$. Since $\theta \circ f \circ \alpha(m) = \pi \circ i \circ \alpha(m) = \pi \circ \alpha(m) = \pi \circ h(m) = \theta \circ g(m)$. But θ is monomorphism, therefore $f \circ \alpha = g$. Thus M is A-projective.

Proposition 1.14. Let M, A and B be modules such that A is projective. Let $f : A \to M$ be an epimorphism. If for any homomorphism $g : A \to B$, there exists a homomorphism $h : M \to B$ such that $h \circ f = g$, then M is s.p-B-projective.

Proof . Let $\theta : B \in N$ be an epimorphism such that $S(N) = 0$ and $\alpha : M \to N$ be a homomorphism. Now look the following graph:

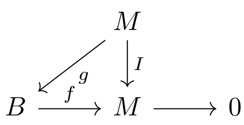


Since A is projective, there exists a homomorphism $g : A \to B$, such that $\theta \circ g = \alpha \circ f$. By assumption, there exists a homomorphism $h : M \to B$, such that $h \circ f = g$, implies that $\theta \circ h \circ f = \theta \circ g = \alpha \circ f$. □

Now, let $x \in M$, then $(\theta \circ h)(x) = \theta(h(x)) = \theta(h(f(y)))$, where $x = f(y)$, for some $y \in A$. Hence $(\theta \circ h)(x) = (\theta \circ h \circ f)(y) = (\theta \circ h)(f(y)) = (\theta \circ g)(y) = \alpha(f(y)) = \alpha(x) \Rightarrow \theta \circ h = \alpha$. Thus M is s.p-B-projective module.

Proposition 1.15. Let M and B be modules. if M is s.p-B-projective, then any epimorphism $f : B \to M$ with $\ker f$ is s.p-closed of B is split.

Proof . Suppose that M is a s.p-B-projective module and $f : B \to M$ be an epimorphism such that $\ker f$ is s.p-closed submodule of B. Look the following graph:



where I is the identity map. Then by th. 1.6, there exists a homomorphism $g : M \to B$ s.t. $f \circ g = I$. Hence f has a right inverse. Thus f is split by [8]. Then $\leq_{\bigoplus}$ of B. □

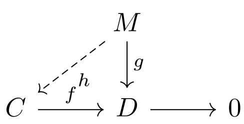
Proposition 1.16. Let M be a module. Then the following statements are equivalent:

1. M is s.p- projective module.
2. For any epimorphism $\theta : A \to B$ such that $S(B) = 0$, the homomorphism $Hom(I, \theta) : Hom(M, A) \to Hom(M, B)$ is an epimorphism.
3. For every epimorphism $\alpha : L \to K$ such that $S(K) = 0$, $\alpha \circ Hom(M, L) = Hom(M, K)$.

Proof . $1 \Rightarrow 2$) Let $\theta : A \to B$ be an epimorphism such that $S(B) = 0$ and $g \in Hom(M, B)$. Since M is s.p- projective, then there exists a homomorphism $\beta : M \to A$ such that $f \circ \beta = g$. So $Hom(I, \theta) \circ h = g$, hence $\beta \in Hom(M, A)$. Thus $Hom(I, \theta)$ is an epimorphism.

$2 \Rightarrow 3$) Let $\alpha : L \to K$ be an epimorphism such that $S(K) = 0$. By (2) $Hom(I, \theta) : Hom(M, L) \to Hom(M, K)$ is an epimorphism. Now we want to show that $\alpha \circ Hom(M, L) = Hom(M, K)$. Let $\zeta \in Hom(M, K)$, then there exists $\beta \in Hom(M, L)$ s.t. $Hom(I, \alpha) \circ \beta = \zeta$. Implies that $\alpha \circ \beta = \zeta$. Thus $\zeta \in \alpha \circ Hom(M, L)$, so $Hom(M, L) \leq \alpha \circ Hom(M, K)$. Clearly $\alpha \circ Hom(M, K) \leq Hom(M, L)$. Thus $\alpha \circ Hom(M, L) = Hom(M, K)$.

$3 \Rightarrow 1$) Let $f : C \to D$ be an epimorphism such that $S(D) = 0$ and $g : M \to D$ be a homomorphism. Look the following graph:



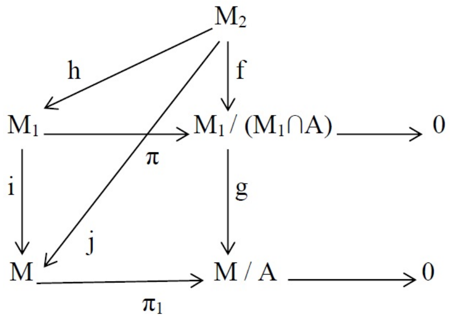
By (3), $f \circ Hom(M, C) = Hom(M, D)$ and $g \in Hom(M, D)$ there exists $h \in Hom(M, N)$ s. t. $f \circ h = g$ and hence $f \circ h = g$. Thus M is s.p- projective module. □

2 Characterization and the direct summand of s.p-projective modules

Theorem 2.1. Let $M = M_1 \bigoplus M_2$ be a module. If M_2 is s.p- M_1- projective. Then for every s.p- closed submodule A of M with $M = M_1 + A$, there exists a submodule K of A such that $M = M_1 \bigoplus K$.

Proof . Let $f : M_2 \to M_1/(M_1 \cap A)$ be a map defined as follows. Let $m_2 \in M_2, f(m_2) = x + (M_1 \cap A)$, where $m_2 = x + y, x \in M_1$ and $y \in A$.

Claim that f is well defined, to show that. Let $m_2 = m_2'$, where $m_2 = x + y$ and $m_2' = x_1 + y_1, x, x_1 \in M_1$ and $y, y_1 \in A$, then $x + y = x_1 + y_1$. So $x – x_1 = y_1 – y \in (M_1 \cap A)$.



Therefore $(x - x_1) \in (M_1 \cap A)$, then $(x - x_1) + (M_1 \cap A) = M_1 \cap A$. Hence $x + (M_1 \cap A) = x_1 + (M_1 \cap A)$. Then $f(m_2) = f(m_2')$. Thus f is well defined. By the second isomorphism theorem, $M/A = (M_1 + A)/A \cong M_1/(M_1 \cap A)$.

Let $g : M_1/(M_1 \cap A) \to M/A$ be the isomorphism defined by $g(m_1 + (M_1 \cap A)) = m_1 + A$. Now look the following graph:

where π and π_1 are the natural epimorphisms and i and j are the inclusion maps. Since A is s.p-closed submodule of M, $S(M/A) = 0$. But $M/A \cong M_1/(M_1 \cap A)$, so $S(M_1/(M_1 \cap A)) = 0$. Since M_2 is s.p-M_1-projective, there exists $h : M_2 \to M_1$ such that $\pi \circ h = f$. Since $(i \circ h + j)(M_2) = i \circ h(M_2) + j(M_2) = h(M_2) + M_2$. Now, we have $M = M_1 + M_2 = M_1 + h(M_2) + M_2 = M_1 + (i \circ h + j)(M_2)$. Let $x \in M_1 \cap (i \circ h + j)(M_2)$, $x = i \circ h(y)–j(y)$, for some $y \in M_2$. So, $x = h(y)–y$. Thus $h(y)–x = y \in M_1 \cap M_2 = 0$ and $h(y)–x = y = 0$. Hence, $x = 0$. Thus $M = M_1 \bigoplus (i \circ h–j)(M_2)$.

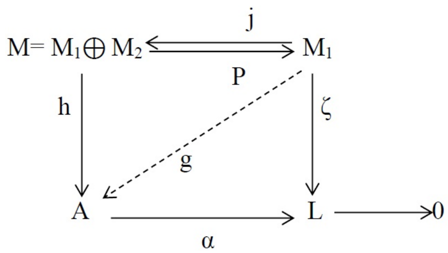
We claim that $(i \circ h–j)(M_2) \leq A$, to show that let $z \in M_2$, then $z = x + y$, where $x \in M_1$ and $y \in A$. So

$$\begin{aligned}(h(z)–z) + A &= \pi_1((i \circ h–j)(z)) \\ &= \pi_1 \circ i \circ h(z)–\pi_1 \circ j(z) \\ &= g \circ \pi \circ h(z)–\pi_1 \circ j(z) \\ &= g \circ f(z)–\pi_1 \circ j(z) \\ &= g(x + (M_1 \cap A)) - \pi_1 \circ j(z) \\ &= (x + A) - (z + A) \\ &= x - z + A \\ &= -y + A \\ &= A\end{aligned}$$

Hence, $h(z)–z \in A$, for every $z \in M_2$. Thus $(i \circ h - j)(M_2) \leq A$. □

Proposition 2.2. Every direct summand of s.p- projective module is s.p-projective.

Proof . Let $M = M_1 \bigoplus M_2$ is s.p- projective. Let $\alpha : A \to L$ be an epimorphism and let $\zeta : M_1 \to B$ be a homomorphism such that $S(L) = 0$. Look the following graph:

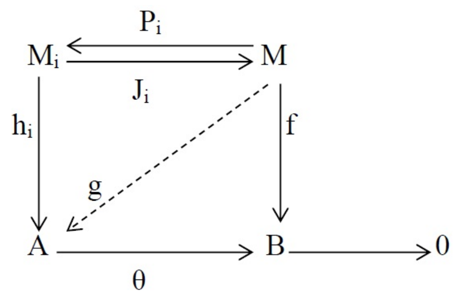


where j is the inclusion map and P is the projection map. Since M is s.p-projective, then there exists a homomorphism $h : M \to A$ s.t. $\alpha \circ h = \zeta \circ P$. Let $\zeta = h \circ j : M_1 \to A$.

Now $f \circ g = f \circ h \circ j = \theta \circ P \circ j = \theta \circ I = \theta$. Thus M_1 is s.p-projective. □

Proposition 2.3. Let $M = \bigoplus_{i \in I} M_i$ be a module. If M_i is s.p-projective for each $i \in I$, then M is s.p- projective module.

Proof . Let $\theta : C \to D$ be an epimorphism such that $S(D) = 0$ and $f : M \to D$ be a homomorphism. Look the following graph:



where J_i are the inclusions maps and P_i an the projections maps .Since M_i is s.p- projective, then $\forall i \in I$, there exists a homomorphism $h_i : M_i \to A$ such that $\theta \circ h_i = f \circ J_i$.

Define $g : M \to A$ by $g((m_i)i \in I) = \sum_{i \in I} h_i(m_i)$. Clearly that g is a homomorphism. Claim that $\theta \circ g = f$. To show that, let $(m_i)_{i \in I} \in M = \bigoplus_{i \in I} M_i$, then

$$\begin{aligned} \theta \circ g((m_i)_{i \in I}) &= \theta(\sum_{i \in I} h_i(m_i)) \\ &= \sum_{i \in I} \theta \circ h_i(m_i) \\ &= \sum_{i \in I} f \circ J_i(m_i) \\ &= f(\sum_{i \in I} J_i(m_i)) \\ &= f((m_i)_{i \in I}). \end{aligned}$$

Thus $\theta \circ g = f$. □

Proposition 2.4. Let M be s.p- projective module and L be s.p- closed submodule of M. If M/L is isomorphic to a direct summand B of M, then L is a direct summand of M.

Proof . Let $\pi : M \to M/L$ be the natural epimorphism and $\beta : M/L \to B$ be an isomorphism. Let $\beta = \pi \circ f : M \to B$. Clearly that f is an epimorphism and $ker\ h = L$. Then by Proposition 2.2, B is s.p-projective and hence by prop. 1.15, h is split. Thus ker $h = L \leq_{\bigoplus} M$. □

Let L be a submodule of a module M. L is called a fully invariant submodule of M if $f(L) \leq L$, for every homomorphism $f : M \to M$, see [8].

Corollary 2.5. If $M = A \bigoplus B$ is s.p- projective module, then A is s.p-B-projective and B is s.p-A-projective.

A module M is called have the (SIP) if the intersection of every two direct summands of M is a direct summand of M, see [11]. A module M is called duo module if every submodule of M is fully invariant, see [10].

Proposition 2.6. If a module M is duo, s.p- projective and has the SIP. Then for any two direct summands C and D of M, $C + D$ is s.p-projective module.

Proof . Let C and D be any direct summands of M, then $C \cap D$ is a direct summand of M. Let $M = (C \cap D) \bigoplus Z$, for some $Z \leq M$. Then $C = (C \cap D) \bigoplus (C \cap Z), D = (C \cap D) \bigoplus (D \cap Z)$, by modular law. Therefore $C + D = [(C \cap D) \bigoplus (C \cap Z)] + [(C \cap D) \bigoplus (D \cap Z)] = [(C \cap D) \bigoplus (C \cap Z)] + (D \cap Z)$. Since M is duo module, then $[(C \cap D) \bigoplus (C \cap Z)] \cap (D \cap Z) = ((C \cap D) \cap (D \cap Z)) \bigoplus ((C \cap Z) \cap (D \cap Z)) = 0$, by [3]. Hence $C + D = (C \cap D) \bigoplus (C \cap Z) \bigoplus (D \cap Z)$. Since M has SIP, $C \cap D, C \cap Z$ and $D \cap Z$ are direct summands of M. By prop. 2.2, $C \cap D, C \cap Z$ and $D \cap Z$ are s.p- projective. Thus $C + D$ is s.p-projective module, by prop. 2.3. □

Proposition 2.7. Let X and X_1 be a submodules of a module M such that X_1 is a direct summand of M. If $X + X_1$ is s.p-projective, then $(X + X_1)/X_1$ is s.p-projective.

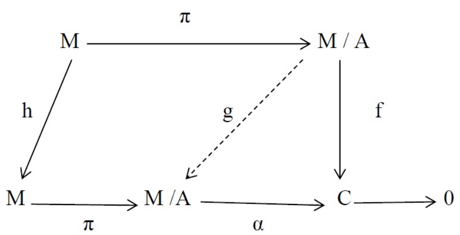
Proof . Let $M = X_1 \bigoplus Z$, for some submodule Z of M. Hence $X + X_1 = X_1 \bigoplus (X + X_1) \cap Z$, by modular law. Since $X + X_1$ is s.p-projective, $((X + X_1) \cap Z)$ is s.p-projective, by Proposition 2.2. But $(X + X_1)/X_1 \cong (X + X_1) \cap Z$, therefore $(X + X_1)/X_1$ is s.p- projective. □

Proposition 2.8. Let X and Y be submodules of a module M s.t. Y is a direct summand of M. If $X + Y$ is s.p-projective module and $X \cap Y$ is s.p-closed of M, then $X \cap Y$ is a direct summand of X.

Proof . Let $\pi : X \to X/(X \cap Y)$ is the natural epimorphisms. Since $(X + Y)/Y \cong X/(X \cap Y)$, by the second isomorphism theorem and Y is a summand of M, then $M = Y \bigoplus Z$ for a submodule Z of N. So $X + Y = Y \bigoplus ((X + Y) \cap Z)$, by modular law. Since $X + Y$ is s.p-projective, $(X + Y) \cap Z$ is s.p-projective, by Proposition 2.2. Hence $(X + Y)/Y$ is s.p-projective and so $X/(X \cap Y)$ is s.p-projective. Since $X \cap Y$ is s.p-closed of M, $S(M/(X \cap Y)) = 0$. Hence $S(X/(X \cap Y)) = 0$, by [7]. Thus $X \cap Y$ is s.p- closed of X. But $\pi : X \to X/(X \cap Y)$ is epimorphism and $\ker \pi = X \cap Y$, therefore $X \cap Y \leq_{\bigoplus} X$, by prop. 1.15. □

Proposition 2.9. Let M be s.p-M- projective module and let A be a fully invariant submodule of M. Then M/A is a s.p-M/A-projective module.

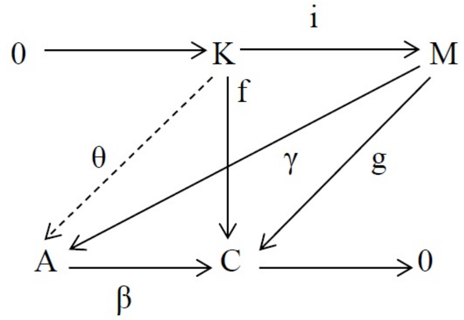
Proof . Let $\alpha : M/A \to C$ be an epimorphism such that $S(C) = 0$ and let $f : M/A \to C$ is a homomorphism. Look the following graph:



where π is the natural epimorphisms. Since M is s.p-projective, therefore there exists a homomorphism $h : M \to M$, such that $\alpha \circ \pi \circ h = f \circ \pi$. Let $g : M/A \to M/A$ define by $g(x + A) = h(x) + A$, for all $x \in M$. Claim that g is well defined. Let $x_1 + A = x_2 + A$, which implies that $x_1 - x_2 \in A$. Since A is a fully invariant submodule, thus $h(x_1 - x_2) \in h(A) \leq A$. Hence $h(x_1) + A = h(x_2) + A$. Clearly that g is a homomorphism. Now $\alpha \circ g(m_1 + A) = \alpha \circ \pi \circ h(m_1) = f \circ \pi(m_1) = f(m_1 + A)$. Thus M/A is s.p-M/A-projective module. □

Proposition 2.10. Let M and A be modules. If M is s.p- A- projective and every quotient of A is M-injective, then any submodule K of M is s.p-A-projective.

Proof . Suppose that $\beta : A \to C$ be an epimorphism s.t. $S(C) = 0$. Let $f : K \to B$ be a homomorphism. Look the following graph. Since B is M-injective, there exists a homomorphism $g : M \to B$ s.t. $g \circ i = f$. But M is s.p-A-projective, so there exists a homomorphism $\gamma : M \to A$ s.t. $\beta \circ \gamma = g$. Define $\theta = \gamma \circ i : K \to A$, now let $x \in K, (\beta \circ \theta)(x) = (\beta \circ \gamma \circ i)(x) = (g \circ i)(x) = g(x)$. Thus K is s.p-A-projective. □



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