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Order improvement for the sequence of α-Bernstein-Păltănea operators

Jaspreet Kaur, Meenu Goyal*

Thapar Institute of Engineering and Technology, Patiala, 147004, India

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Abstract

In the present paper, we give the modifications of $\alpha-$Bernstein-Păltănea operators with better approximation properties. We present three modifications of these operators having linear, quadratic and cubic order of approximation whereas the classical operators are of linear order. By increasing the order of approximation of these operators, the speed of the convergence will be increased. We establish some approximation results concerning the rate of convergence, error estimation and Voronovskaja type formulas for the new modifications. Also, we verify our analytical results with the help of MAPLE algorithms.

Keywords: Modulus of continuity, Convergence of series and sequences, Rate of convergence, Approximation by positive operators, Asymptotic approximations
2010 MSC: 26A15, 40A05, 41A10, 41A25, 41A35, 41A60

1 Introduction

S. N. Bernstein [11] established Weierstrass' approximation theorem by adopting a probabilistic technique, which he defined for $g \in B[0,1]$, where $B[0,1]$ denotes the space of all bounded functions on $[0,1]$, as:

$$\mathcal{B}_n(g;x) = \sum_{j=0}^{n} p_{n,j}(x) g\left(\frac{j}{n}\right),$$

where $p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$. These operators have slow rate of convergence, in order to make it more attractive, from the computational point of view, several modifications and improvements have been investigated by many authors (see [4, 12, 19, 26, 29, 31]). In [13], Chen et al. gave generalization of Bernstein operators depending upon a real parameter α, where $0 \leq \alpha \leq 1$ as:

$$\mathcal{T}_n^{\alpha}(g;x) = \sum_{j=0}^{n} p_{n,j}^{\alpha}(x) g\left(\frac{j}{n}\right), \tag{1.1}$$

*Corresponding author
Email addresses: jazzbagri3@gmail.com (Jaspreet Kaur), meenu_rani@thapar.edu (Meenu Goyal)

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where $g(x) \in C[0,1]$ and $n \in \mathbb{N}$. Also, the $\alpha-$Bernstein polynomial basis $p^{\alpha}_{n,j}(x)$ of degree n is given by $p^{\alpha}_{1,0}(x) = 1-x, \quad p^{\alpha}_{1,1}(x) = x$ and

$$p^{\alpha}_{n,j}(x) = \left[\binom{n-2}{j}x(1-\alpha) + \binom{n-2}{j-2}(1-x)(1-\alpha) + \binom{n}{j}\alpha x(1-x)\right] x^{j-1}(1-x)^{n-j-1}, \quad n \geq 2.$$

For $\alpha = 1$, (1.1) reduce to original Bernstein operators. These operators have certain elementary properties, which give significant contribution in uniform convergence of functions without depending on the parameter α. Similarly, Ansari and Usta [10] generalized the Szász-Mirakyan operators with parameter α.
Since positive linear summation operators are useful for the convergence of only continuous functions, therefore, in 1967, Durrmeyer [15] modified the Bernstein operators to approximate Lebesgue integrable functions on $[0,1]$. Due to its usefulness, the Durrmeyer variant of Bernstein operators attracted attention of several authors (see [2, 3, 7, 14, 18, 21, 27]). Acar et al. [1] defined the Durrmeyer vaiant for the mobile interval of $[0,1]$ and presented its local and global approximation properties. Gal and Gupta [16] defined the Durrmeyer operators to approximate analytic functions. Recently, Kajla and Acar [22] introduced the Durrmeyer modification of the summation operators (1.1) and studied the rate of convergence and some approximation properties. Păltănea [30] generalized the Durrmeyer type operators with the help of a parameter $\rho > 0$. Ansari et al. [8] studied the approximation and error estimation properties by modified Păltănea operators for Gould-Hopper polynomials. From the applications of parametric generalizations many other researchers have worked on this basis function (see [5, 9, 28]). Most recently, Kajla and Goyal [24] modified these Durrmeyer operators by using Păltănea basis function in an integral depending on a parameter $\rho > 0$, as:

$$Q^{\alpha}_{n,\rho}(g;x) = \sum_{j=0}^{n} p^{\alpha}_{n,j}(x) \int_0^1 \mu^{\rho}_{n,j}(t)g(t)dt, \tag{1.2}$$

where

$$\mu^{\rho}_{n,j}(t) = \frac{t^{j\rho}(1-t)^{(n-j)\rho}}{B(j\rho+1,(n-j)\rho+1)},$$

and $B(i,j)$ is beta function. They have studied the approximation properties, asymptotic behavior and the order of convergence of these operators.
Here, our main motive is to improve approximation behavior and order of convergence for the operators (1.2). In [25], Khosravian-Arab et al. modified the well known Bernstein operators by using a new technique to improve their degree of approximation. Following this, Acu et al. [6] have applied this approach on the Bernstein-Durrmeyer operators. In another paper [20], same authors have put it on the Bernstein-Kantorovich operators too. Similarly, Kajla and Acar [23] have modified the $\alpha-$Bernstein summation operators. The inspiration of getting better approximation results for positive linear operators leads us to modify the $\alpha-$Bernstein-Păltănea operators which are defined in (1.2). In the present paper, we apply an approach to represent the modifications of these operators which give better convergence than the classical one.
Our work is organized as follows: In section 2, we define $\alpha-$Bernstein-Păltănea operators of first order. In section 3 and 4, we introduce $\alpha-$Bernstein-Păltănea operators of second and third order respectively which possess better order of approximation than the operators (1.2). In section 5, we verify the theoretical results obtained in section 2-4 numerically using Maple algorithms.
Through out this paper, we denote $J^{\alpha,i}_{n,\rho}(g;x)$, $i = 1,2,3$ as three modifications of ith order of approximation.

2 $\alpha-$Bernstein-Păltănea operators of first order

In the present section, we define $\alpha-$Bernstein-Păltănea operators of first order as:

$$J^{\alpha,1}_{n,\rho}(g;x) = \sum_{j=0}^{n} p^{\alpha,1}_{n,j}(x) \int_0^1 \mu^{\rho}_{n,j}(t)g(t)\,dt, \quad x \in [0,1] \tag{2.1}$$

$$\text{where} \quad p^{\alpha,1}_{n,j}(x) = a(x,n)\,p^{\alpha}_{n-1,j}(x) + a(1-x,n)\,p^{\alpha}_{n-1,j-1}(x), \quad 0 \leq j \leq n-1, \tag{2.2}$$

where $\alpha \in [0,1]$, $\rho > 0$ and $a(x,n) = a_1(n)x + a_0(n)$, such that $a_0(n)$ and $a_1(n)$ are two unknown sequences, which can be determined to satisfy our purposes. For $a_0(n) = 1$ and $a_1(n) = -1$, (2.1) reduces to the operators (1.2).
Now, we compute some preliminary results which will be useful to study the uniform convergence and asymptotic results. For this we assume $e_k = x^k, k = 0,1,2,\cdots$.

Lemma 2.1. For the operators (2.1), we have:

$$\begin{aligned}
J^{\alpha,1}_{n,\rho}(e_0;x) &= (2a_0(n)+a_1(n));\\
J^{\alpha,1}_{n,\rho}(e_1;x) &= (2a_0(n)+a_1(n))x+\frac{1}{n\rho+2}\left[(1-2x)(a_0(n)(\rho+2)+a_1(n)(\rho+1))\right];\\
J^{\alpha,1}_{n,\rho}(e_2;x) &= (2a_0(n)+a_1(n))x^2+\frac{1}{(n\rho+2)(n\rho+3)}\left[n\left\{\rho x(3-5x)(2a_0(n)+a_1(n))\right.\right.\\
&\left.+\rho^2x(a_0(n)(4-6x)+a_1(n)(3-5x))\right\}+(-6x^2+2\alpha\rho^2x^2-2\alpha\rho^2x+2)(2a_0(n)+a_1(n))\\
&\left.+\rho(\rho+3-6x)a_0(n)+\rho(2\rho x^2-6x-2\rho x+\rho+3)a_1(n)\right].
\end{aligned}$$

Through out the paper, we denote $\phi^k_x(t)=(t-x)^k, k=1,2,3,\cdots$.

Lemma 2.2. For the operators (2.1), we have the central moments as:

$$\begin{aligned}
J^{\alpha,1}_{n,\rho}(\phi^1_x(t);x) &= \frac{1}{n\rho+2}\left[(1-2x)(a_0(n)(\rho+2)+a_1(n)(\rho+1))\right];\\
J^{\alpha,1}_{n,\rho}(\phi^2_x(t);x) &= \frac{1}{(n\rho+2)(n\rho+3)}\left[x(1-x)\rho(1+\rho)(2a_0(n)+a_1(n))n\right.\\
&-x(1-x)\left(4a_0(n)(3+3\rho+\alpha\rho^2)+2a_1(n)(3+6\rho+\rho^2(1+\alpha))\right)\\
&\left.+a_0(n)(4+3\rho+\rho^2)+a_1(n)(2+3\rho+\rho^2)\right];\\
J^{\alpha,1}_{n,\rho}(\phi^4_x(t);x) &= \frac{1}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)}\left[3\rho^2(1+\rho)^2x^2(1-x)^2(2a_0(n)+a_1(n))n^2\right]+O\left(\frac{1}{n^3}\right).
\end{aligned}$$

To obtain uniform convergence of the operators (2.1), throughout this paper, the sequences $a_i(n),\ i=0,1$ will satisfy the condition:

$$2a_0(n)+a_1(n)=1. \tag{2.3}$$

Depending on the choices of the sequences, we get two cases which are given by:
Case 1. Let

$$a_0(n)\geq 0 \quad\text{and}\quad a_0(n)+a_1(n)\geq 0. \tag{2.4}$$

From this, we get $0\leq a_0(n)\leq 1$ and $-1\leq a_1(n)\leq 1$. So, both sequences are bounded. Also, the operators (2.1) are positive for this case.
Case 2. Let

$$a_0(n)<0 \quad\text{or}\quad a_0(n)+a_1(n)<0. \tag{2.5}$$

If $a_0(n)<0$, then $a_0(n)+a_1(n)>1$ and if $a_0(n)+a_1(n)<0$, then $a_0(n)>1$. In this case the operators (2.1) are not positive.
Firstly, we prove the basic convergence and asymptotic results for case 1.

Theorem 1. Let $g\in C[0,1]$. If $a_0(n),a_1(n)$ satisfy both the equations (2.3) and (2.4), then

$$\lim_{n\to\infty}J^{\alpha,1}_{n,\rho}(g;x)=g(x),$$

uniformly on [0,1].

Proof . From the conditions on $a_i(n), i=0,1$ the operators (2.1) are positive. So, by Korovkin theorem and Lemma 2.1, we can find the uniform convergence of the operators. □

Theorem 2. Let $a_i(n), i=0,1$ are convergent sequences satisfying the conditions (2.3)-(2.4) and $l_i = \lim\limits_{n\to\infty} a_i(n)$. If $g'' \in C[0,1]$, then:

$$\lim_{n\to\infty} n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) = \frac{(1-2x)((\rho+2)l_0 + (\rho+1)l_1)}{\rho} g'(x) + \frac{x(1-x)(1+\rho)(2l_0+l_1)}{2\rho} g''(x),$$

uniformly on [0,1].

Proof . By the Taylor's formula, we have:

$$g(t) = g(x) + \phi^1_x(t)g'(x) + \frac{1}{2}\phi^2_x(t)g''(x) + \Theta(t,x)\phi^2_x(t).$$

where $\Theta(t,x) \in C[0,1]$ with $\lim\limits_{t\to x}\Theta(t,x) = 0$. Apply the operators $J^{\alpha,1}_{n,\rho}(.,x)$ on Taylor's formula, we get:

$$n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) = nJ^{\alpha,1}_{n,\rho}(\phi^1_x(t);x)g'(x) + \frac{n}{2}J^{\alpha,1}_{n,\rho}(\phi^2_x(t);x)g''(x) + nJ^{\alpha,1}_{n,\rho}(\Theta(t,x)\phi^2_x(t);x).$$

Using Cauchy-Schwarz inequality on the last term of the above equation, we obtain

$$nJ^{\alpha,1}_{n,\rho}(\Theta(t,x)\phi^2_x(t);x) \le n\sqrt{J^{\alpha,1}_{n,\rho}(\Theta^2(t,x);x)}\sqrt{J^{\alpha,1}_{n,\rho}(\phi^4_x(t);x)}. \tag{2.6}$$

Since $\Theta^2(x,x) = 0$, $\Theta^2(t,x) \in C[0,1]$ and $J^{\alpha,1}_{n,\rho}(g;x) \to g(x)$, we have $\lim\limits_{n\to\infty} J^{\alpha,1}_{n,\rho}(\Theta^2(t,x);x) = 0$ uniformly on [0,1]. Hence, from Lemma 2.2, the above inequality (2.6) reduces to

$$\lim_{n\to\infty} nJ^{\alpha,1}_{n,\rho}(\Theta(t,x)\phi^2_x(t);x) = 0,$$

which gives the required result. □

Now, we study the convergence and asymptotic results for the case 2.

Theorem 3. Let $g \in C[0,1]$ and $a_i(n), i=0,1$ be convergent sequences which satisfy the conditions (2.3) and (2.5). Then:

$$\lim_{n\to\infty} J^{\alpha,1}_{n,\rho}(g;x) = g(x),$$

uniformly on [0,1].

Proof . We can rewrite the operators (2.1) as:

$$J^{\alpha,1}_{n,\rho}(g;x) = K^{\alpha,1}_{n,\rho}(g;x) - L^{\alpha,1}_{n,\rho}(g;x) \tag{2.7}$$

where

$$\begin{aligned} K^{\alpha,1}_{n,\rho}(g;x) &= \sum_{j=0}^{n}[a_1(n)\ x\ p^{\alpha}_{n-1,j}(x) + a_1(n)\ p^{\alpha}_{n-1,j-1}(x)]\int_0^1 \mu^{\rho}_{n,j}(t)g(t)dt, \\ L^{\alpha,1}_{n,\rho}(g;x) &= \sum_{j=0}^{n}[-a_0(n)\ p^{\alpha}_{n-1,j}(x) + (a_1(n)x - a_0(n))\ p^{\alpha}_{n-1,j-1}(x)]\int_0^1 \mu^{\rho}_{n,j}(t)g(t)dt. \end{aligned}$$

As both the operators i.e. $K^{\alpha,1}_{n,\rho}(g;x)$ and $L^{\alpha,1}_{n,\rho}(g;x)$ are positive, so we can apply extended Korovkin theorem ([25], see page 122) on it. The moments of these operators are given below:

$$\begin{aligned}
K_{n,\rho}^{\alpha,1}(e_0;x) &= a_1(n)(1+x);\\
K_{n,\rho}^{\alpha,1}(e_1;x) &= a_1(n)(1+x)\left[\frac{(n-1)\rho x}{n\rho+2}\right]+\frac{a_1(n)x}{n\rho+2}+\frac{a_1(n)(\rho+1)}{n\rho+2};\\
K_{n,\rho}^{\alpha,1}(e_2;x) &= \frac{a_1(n)(1+x)}{(n\rho+2)(n\rho+3)}\left[\rho^2(n-1)^2\left(x^2+\frac{n-1+2(1-\alpha)}{(n-1)^2}x(1-x)\right)+3\rho(n-1)x+2\right]\\
&\quad+\frac{a_1(n)}{(n\rho+2)(n\rho+3)}\left[2\rho^2(n-1)x+\rho^2+3\rho\right],\\
L_{n,\rho}^{\alpha,1}(e_0;x) &= a_1(n)x-2a_0(n);\\
L_{n,\rho}^{\alpha,1}(e_1;x) &= (a_1(n)x-2a_0(n))\left[\frac{(n-1)\rho x}{n\rho+2}\right]-\frac{a_0(n)(\rho+2)}{n\rho+2}+\frac{a_1(n)x(\rho+1)}{n\rho+2};\\
L_{n,\rho}^{\alpha,1}(e_2;x) &= \frac{a_1(n)x-2a_0(n)}{(n\rho+2)(n\rho+3)}[\rho^2(n-1)^2x^2+\rho^2(n-1+2(1-\alpha))x(1-x)+3\rho(n-1)x+2]\\
&\quad+\frac{a_1(n)x-a_0(n)}{(n\rho+2)(n\rho+3)}[2\rho^2(n-1)x+\rho^2+3\rho].
\end{aligned}$$

Since $a_1(n)$ is convergent, $\lim\limits_{n\to\infty} a_1(n)=l_1(say)$, we obtain

$$\lim_{n\to\infty} K_{n,\rho}^{\alpha,1}(g;x)=l_1(1+x)g(x) \text{ uniformly on } [0,1],$$

$$\lim_{n\to\infty} L_{n,\rho}^{\alpha,1}(g;x)=[l_1(1+x)-1]g(x) \text{ uniformly on } [0,1].$$

By using both the above limits and equation (2.7), we get the required result. □

Theorem 4. Let $a_i(n), i=0,1$ are convergent sequences satisfying the conditions (2.3), (2.5) and $l_i=\lim\limits_{n\to\infty} a_i(n)$. If $g''\in C[0,1]$, then:

$$\lim_{n\to\infty} n(J_{n,\rho}^{\alpha,1}(g;x)-g(x))=\frac{(1-2x)((\rho+2)l_0+(\rho+1)l_1)}{\rho}g'(x)+\frac{x(1-x)(1+\rho)(2l_0+l_1)}{2\rho}g''(x),$$

uniformly on [0,1].

Proof . Similar to the proof of Theorem 2, it is enough to prove that:

$$\lim_{n\to\infty} nJ_{n,\rho}^{\alpha,1}(\Theta(t,x)\phi_x^2(t);x)=0.$$

We can rewrite operators (2.1) in the following way:

$$J_{n,\rho}^{\alpha,1}(g;x)=\sum_{j=0}^{n-1}p_{n-1,j}^{\alpha}(x)\left(a(x,n)\int_0^1\mu_{n,j}^{\rho}(t)g(t)dt+a(1-x,n)\int_0^1\mu_{n,j+1}^{\rho}(t)g(t)dt\right). \tag{2.8}$$

For $\epsilon>0$, there exists $delta>0$ such that $|t-x|<\delta$, then $|\Theta(t,x)|<\epsilon$. Divide the interval $[0,1]$ into two parts as below:

$$I_1=(x-\delta,x+\delta)\cap[0,1], \qquad I_2=[0,1]\setminus(x-\delta,x+\delta).$$

Since $a_i(n), i=0,1$ are convergent, so are bounded. Thus, there exists $C>0$ such that $|a(x,n)|<C$. Now,

$n|J^{\alpha,1}_{n,\rho}(\Theta(t,x)\phi_x^2(t);x)|$

$$\begin{aligned}
&\leq nC\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\left(\int_0^1\mu^{\rho}_{n,j}(t)|\Theta(t,x)|\phi_x^2(t)dt+\int_0^1\mu^{\rho}_{n,j+1}(t)|\Theta(t,x)|\phi_x^2(t)dt\right)\\
&< nC\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\left[\epsilon\left(\int_{I_1}\mu^{\rho}_{n,j}(t)\phi_x^2(t)dt+\int_{I_1}\mu^{\rho}_{n,j+1}(t)\phi_x^2(t)dt\right)\right.\\
&\quad\left.+\frac{M}{\delta^2}\left(\int_{I_2}\mu^{\rho}_{n,j}(t)\phi_x^4(t)dt+\int_{I_2}\mu^{\rho}_{n,j+1}(t)\phi_x^4(t)dt\right)\right],\ \text{where } M=\sup_{0\leq t\leq 1}|\Theta(t,x)|\\
&\leq n\epsilon C\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\left[\int_0^1\mu^{\rho}_{n,j}(t)\phi_x^2(t)dt+\int_0^1\mu^{\rho}_{n,j+1}(t)\phi_x^2(t)dt\right]\\
&\quad+\frac{nMC}{\delta^2}\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\left[\int_0^1\mu^{\rho}_{n,j}(t)\phi_x^4(t)dt+\int_0^1\mu^{\rho}_{n,j+1}(t)\phi_x^4(t)dt\right]\\
&\leq \epsilon C_1(x,\rho,\alpha)+O\left(\frac{1}{n}\right).
\end{aligned}$$

Thus, from the last inequality, we get the proof. □

Theorem 5. Let g be bounded for $x\in[0,1]$, $a_0(n)$ is a bounded sequence and $a_i(n), i=0,1$ satisfy the condition (2.3) then

$$||J^{\alpha,1}_{n,\rho}g-g||\leq(1+3|a_0(n)|)C_3\,\omega\left(g;\frac{1}{\sqrt{n}}\right),$$

where $||.||$ is the uniform norm over $[0,1]$, $\omega(g;\sigma)$ is the first order modulus of continuity and $C_3>0$ is a constant.

Proof . From the definition of our operators (2.8) and using relation $\omega(g;\lambda\sigma)\leq(1+\lambda)\,\omega(g;\sigma)$ for $\lambda>0$, (by taking $\lambda=\sqrt{n}\,|t-x|, \sigma=\frac{1}{\sqrt{n}}$), we get

$$\begin{aligned}
|J^{\alpha,1}_{n,\rho}(g;x)-g(x)| &\leq |a(x,n)|\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j}(t)|g(t)-g(x)|dt\\
&\quad+|a(1-x,n)|\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j+1}(t)|g(t)-g(x)|dt\\
&\leq |a(x,n)|\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j}(t)\,\omega(g;|\phi_x^1(t)|)dt\\
&\quad+|a(1-x,n)|\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j+1}(t)\,\omega(g;|\phi_x^1(t)|)dt\\
&\leq |a(x,n)|\,\omega\left(g;\frac{1}{\sqrt{n}}\right)\left[1+\sqrt{n}\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j}(t)|\phi_x^1(t)|dt\right]\\
&\quad+|a(1-x,n)|\,\omega\left(g;\frac{1}{\sqrt{n}}\right)\left[1+\sqrt{n}\sum_{j=0}^{n-1}p^{\alpha}_{n-1,j}(x)\int_0^1\mu^{\rho}_{n,j+1}(t)|\phi_x^1(t)|dt\right].
\end{aligned}$$

Now, by using Holder's inequality, we get

$$\begin{aligned}\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j}^{\rho}(t)|\phi_x^1(t)|dt &\leq \left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\right]^{\frac{1}{2}}\left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^2(t)dt\right]^{\frac{1}{2}}\\ &= \sqrt{\frac{n(-\rho^2x^2-\rho x^2+\rho^2+3\rho x)+(6x^2+6\rho x^2+\rho^2 x-3\rho x+2)}{(n\rho+2)(n\rho+3)}}.\end{aligned}$$

Therefore,

$$\sqrt{n}\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j}^{\rho}(t)|\phi_x^1(t)|dt \leq \frac{C_1\sqrt{n}}{\sqrt{n\rho+2}} \leq C_1. \tag{2.9}$$

Similarly,

$$\sqrt{n}\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j+1}^{\rho}(t)|\phi_x^1(t)|dt \leq \frac{C_2\sqrt{n}}{\sqrt{n\rho+2}} \leq C_2. \tag{2.10}$$

Using the inequalities (2.9) and (2.10), we get the following relation:

$$|J_{n,\rho}^{\alpha,1}(g;x)-g(x)| \leq \omega\left(g;\frac{1}{\sqrt{n}}\right)\left[|a(x,n)|(1+C_1)+|a(1-x,n)|(1+C_2)\right]. \tag{2.11}$$

From equation (2.3), we find

$$|a(x,n)| = |a_1(n)x+a_0(n)| \leq 1+3|a_0(n)| \quad \text{and} \quad |a(1-x,n)| \leq 1+3|a_0(n)|.$$

Now, using these inequalities and equation (2.11), our proof is completed. □

Corollary 2.3. (i) If we assume $g \in C[0,1]$ in Theorem 5, then $\lim_{n\to\infty} \omega\left(g;\frac{1}{\sqrt{n}}\right) = 0$, which gives another proof of the Theorems 1 and 3.
(ii) If $g \in Lip_M(\tau)$ on $[0,1]$, then result obtained in Theorem 5 reduces to:

$$|J_{n,\rho}^{\alpha,1}(g;x)-g(x)| \leq M.C_3\left(1+3|a_0(n)|\right)n^{-\tau/2},$$

where C_3 is same constant as in Theorem 5.

Now, we find the errors in respect of modulus of continuity in asymptotic formula for our operators (2.1) for which $J_{n,\rho}^{\alpha,1}(e_i;x) = e_i, i = 0,1$. Thus, we have the conditions:

$$2a_0(n)+a_1(n) = 1, a_0(n)(\rho+2)+a_1(n)(\rho+1) = 0.$$

By solving these equations, we get $a_0(n) = \frac{\rho+1}{\rho}$ and $a_1(n) = -\frac{\rho+2}{\rho}$.

Theorem 6. Let $g \in C^2[0,1], x \in [0,1]$ is fixed. Then

$$\left|J_{n,\rho}^{\alpha,1}(g;x)-g(x)-\frac{1}{2}J_{n,\rho}^{\alpha,1}(\phi_x^2(t);x)g''(x)\right| \leq C\frac{1}{n}\omega\left(g'';\frac{1}{n}\right)$$

where $C > 0$ is a constant restrained from n, x.

Proof . For $g \in C^2[0,1]$. By using the Taylor's formula and apply $J_{n,\rho}^{\alpha,1}(.;x)$, we get:

$$|J_{n,\rho}^{\alpha,1}(g;x) - g(x) - \frac{1}{2}J_{n,\rho}^{\alpha,1}(\phi_x^2(t);x)g''(x)| = \frac{1}{2}|J_{n,\rho}^{\alpha,1}(\Theta(t,x)\phi_x^2(t);x)|,$$

where $\Theta(t,x) = g''(\xi_x) - g''(x)$ and ξ_x lies between t and x. From modulus of continuity, we have:

$$|\Theta(t,x)| = |g''(\xi_x) - g''(x)| \leq \omega(g''; |\phi_x^1(t)|) \leq (1 + \sqrt{n}\,|\phi_x^1(t)|)\,\omega\left(g''; \frac{1}{\sqrt{n}}\right).$$

Also, $|a(x,n)| = |a_1(n)x + a_0(n)| \leq \dfrac{\rho+1}{\rho}$, and $|a(1-x,n)| \leq \dfrac{\rho+1}{\rho}$, for $x \in [0,1]$ and using the operators (2.8), we obtain:

$$|J_{n,\rho}^{\alpha,1}(\Theta(t,x)\phi_x^2(t);x)| \leq \frac{\rho+1}{\rho}\omega\left(g''; \frac{1}{\sqrt{n}}\right)(J_1 + \sqrt{n}J_2), \tag{2.12}$$

$$\begin{aligned}
\text{where} \quad J_1 &= \sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\left[\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^2(t)dt + \int_0^1 \mu_{n,j+1}^{\rho}(t)\phi_x^2(t)dt\right] \\
&= \frac{1}{(n\rho+2)(n\rho+3)}\left[n(-2\rho^2x^2 - 2\rho x^2 + 2\rho^2 x + 4\rho x)\right. \\
&\quad \left. + (12x^2 + 12\rho x^2 - 12\rho x + \rho^2 + 3\rho + 4 - 4\alpha\rho^2 x(1-x))\right] \leq \frac{(\rho+2)^2(n+2)}{(n\rho+2)(n\rho+3)}. \\
J_2 &= \sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\left[\int_0^1 \mu_{n,j}^{\rho}(t)|\phi_x^1(t)|\phi_x^2(t)dt + \int_0^1 \mu_{n,j+1}^{\rho}(t)|\phi_x^1(t)|\phi_x^2(t)dt\right] \\
&\leq \sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\left[\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^2(t)dt\right]^{\frac{1}{2}}\left[\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^4(t)dt\right]^{\frac{1}{2}} \\
&\quad + \sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\left[\int_0^1 \mu_{n,j+1}^{\rho}(t)\phi_x^2(t)dt\right]^{\frac{1}{2}}\left[\int_0^1 \mu_{n,j+1}^{\rho}(t)\phi_x^4(t)dt\right]^{\frac{1}{2}} \\
&\leq \left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^2(t)dt\right]^{\frac{1}{2}} \times \left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j}^{\rho}(t)\phi_x^4(t)dt\right]^{\frac{1}{2}} \\
&\quad + \left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j+1}^{\rho}(t)\phi_x^2(t)dt\right]^{\frac{1}{2}} \times \left[\sum_{j=0}^{n-1} p_{n-1,j}^{\alpha}(x)\int_0^1 \mu_{n,j+1}^{\rho}(t)\phi_x^4(t)dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Using the results of Lemma 2.2, we have:

$$J_2 \leq \frac{P}{n\sqrt{n}}, \text{ where } P(>0) \text{ is unrestrained from } n.$$

Now replacing the values of J_1 and J_2 in the relation (2.12), we can find a constant $C > 0$ which is unrestrained from n and x such that

$$|J_{n,\rho}^{\alpha,1}(\Theta(t,x)\phi_x^2(t);x)| \leq C\frac{1}{n}\omega\left(g''; \frac{1}{\sqrt{n}}\right). \tag{2.13}$$

Hence, we get the required result. □

Now, we find error estimation of the operators $J_{n,\rho}^{\alpha,1}(g;x)$ with regard to second order modulus of continuity which gives better results that we have found in Theorem 5.

Theorem 7. Let $g \in C[0,1]$, $a_0(n) = \dfrac{\rho+1}{\rho}$, $a_1(n) = -\dfrac{\rho+2}{\rho}$, then

$$||J^{\alpha,1}_{n,\rho}(g;.) - g|| \leq C\omega_2\left(g;\frac{1}{\sqrt{n}}\right).$$

Proof . From the definition of our operators (2.1), we can write:

$$||J^{\alpha,1}_{n,\rho}g|| \leq (|a_0(n)| + |a_1(n)|)||g||.$$

Let $f \in C^2[0,1]$, then from Theorem 6, we obtain:

$$|J^{\alpha,1}_{n,\rho}(f;x) - f(x)| \leq \frac{1}{2}J^{\alpha,1}_{n,\rho}(\phi^2_x(t);x)|f''(x)| + \frac{C_1}{n}\omega\left(f'';\frac{1}{\sqrt{n}}\right),$$

where $C_1 > 0$ is a constant unrestrained from x and n. Using Lemma 2.2 and the property of modulus of continuity i.e. $\omega(h,\delta) \leq 2\|h\|$, we get:

$$||J^{\alpha,1}_{n,\rho}(f;.) - f|| \leq \frac{C_2}{n}||f''||,$$

where $C_2 > 0$ is a constant unrestrained from x and n. Thus, for $g \in C[0,1]$, we have:

$$\begin{aligned} ||J^{\alpha,1}_{n,\rho}(g;.) - g|| &\leq ||J^{\alpha,1}_{n,\rho}(g-f) - (g-f)|| + ||J^{\alpha,1}_{n,\rho}(f) - f|| \\ &\leq C_3||g-f|| + \frac{C_2}{n}||f''|| \leq C\left[||g-f|| + \frac{1}{n}||f''||\right], \end{aligned} \tag{2.14}$$

where C and C_3 are some positive constants unrestrained from n and x. Now, keeping in mind the equivalence of second order modulus of continuity $\omega_2(g,t)$ and $K-$functional $K_2(g,t^2) := \inf_{f\in C^2[0,1]}\{||g-f||+t^2||f''||\}$ i.e. $K_2(g,t^2) \leq \frac{7}{2}\omega_2(g,t), 0 \leq t \leq 1, g \in C[0,1]$, (see [[17], Corollary 2.7]) and then by taking the infimum over all $f \in C^2[0,1]$ to (2.14), we get the required result. □

3 $\alpha-$Bernstein Păltănea operators of second order

Similarly, we can define the second order modification of the operators $Q^{\alpha}_{n,\rho}(g;x)$ which is given by:

$$J^{\alpha,2}_{n,\rho}(g;x) = \sum_{j=0}^{n} p^{\alpha,2}_{n,j}(x)\int_0^1 \mu^{\rho}_{n,j}(t)g(t)dt, \tag{3.1}$$

where
$$\begin{aligned} p^{\alpha,2}_{n,j}(x) &= a(x,n)p^{\alpha}_{n-2,j}(x) + b(x,n)p^{\alpha}_{n-2,j-1}(x) + a(1-x,n)p^{\alpha}_{n-2,j-2}(x), \\ \mu^{\rho}_{n,j}(t) &= \frac{t^{j\rho}(1-t)^{(n-j)\rho}}{B(j\rho+1,(n-j)\rho+1)}, \end{aligned}$$

and $a(x,n) = a_2(n)x^2 + a_1(n)x + a_0(n)$, $b(x,n) = b_0(n)x(1-x)$.
If $a_2(n) = 1, a_1(n) = -2, a_0(n) = 1$ and $b_0(n) = 2$, then we will get our original operators (1.2).

Lemma 3.1. For the operators (3.1), we have:

$$\begin{aligned} J^{\alpha,2}_{n,\rho}(e_0;x) &= x^2(2a_2(n) - b_0(n)) + x(b_0(n) - 2a_2(n)) + (2a_0(n) + a_1(n) + a_2(n)); \\ J^{\alpha,2}_{n,\rho}(e_1;x) &= \frac{1}{(n\rho+2)}\left[n\{x^3(2a_2(n) - b_0(n))\rho + x^2(-2a_2(n) + b_0(n))\rho\right. \\ &\quad +x(2a_0(n) + a_1(n) + a_2(n))\rho\} + \{x^3(-4a_2(n) + 2b_0(n))\rho + x^2(2a_2(n)(3\rho+1) - b_0(n)(3\rho+1)) \\ &\quad +x(-2a_2(n)(3\rho+1) - 4\rho a_1(n) - 4\rho a_0(n) + b_0(n)(\rho+1)) \\ &\quad \left.+ (a_2(n)(1+2\rho) + a_1(n)(1+2\rho) + 2a_0(n)(1+\rho))\}\right]; \end{aligned}$$

$$\begin{aligned}
J^{\alpha,2}_{n,\rho}(e_2;x) &= \frac{1}{(n\rho+2)(n\rho+3)}\left[n^2\rho^2\big\{x^4(2a_2(n)-b_0(n))+x^3(-2a_2(n)+b_0(n))\right.\\
&\quad +x^2(2a_0(n)+a_1(n)+a_2(n))\big\}+n\big\{-5x^4\rho^2(2a_2(n)-b_0(n))\\
&\quad +x^3\rho((16\rho+6)a_2(n)-(7\rho+4)b_0(n))\\
&\quad +x^2\rho(-3(5\rho+2)a_2(n)-9\rho a_1(n)-10\rho a_0(n)+3(\rho+1)b_0(n))\\
&\quad +x\rho(6(\rho+1)a_0(n)+(3+5\rho)(a_1(n)+a_2(n)))\big\}+\big\{2x^4\rho^2(2+\alpha)(2a_2(n)-b_0(n))\\
&\quad -2x^3\rho(2\alpha\rho+4\rho+3)(2a_2(n)-b_0(n))\\
&\quad +x^2\left(\rho^2(6(4+\alpha)a_2(n)+2(6+\alpha)a_1(n)+4(2+\alpha)a_0(n)-(5+2\alpha)b_0(n))\right.\\
&\quad \left.+(9\rho+2)(2a_2(n)-b_0(n))\right)\\
&\quad +x\left(-2(2a_2(n)-b_0(n))-3\rho(6a_2(n)+4(a_1(n)+a_0(n))-b_0(n))\right.\\
&\quad \left.-\rho^2(2(8+\alpha)a_2(n)+2(6+\alpha)a_1(n)+4(2+\alpha)a_0(n)-b_0(n))\right)\\
&\quad +4\rho^2(a_2(n)+a_1(n)+a_0(n))+6\rho(a_2(n)+a_1(n)+a_0(n))\\
&\quad \left.+2(a_2(n)+a_1(n)+2a_0(n))\big\}\right].
\end{aligned}$$

To study the uniform convergence of these operators, we take $J^{\alpha,2}_{n,\rho}(e_0,x)=1$, which give the following conditions:

$$2a_2(n)-b_0(n)=0,\quad 2a_0(n)+a_1(n)+a_2(n)=1.$$

With both of these conditions, other moments reduce to:

$$\begin{aligned}
J^{\alpha,2}_{n,\rho}(e_1;x) &= x+\frac{1}{n\rho+2}[(1+2\rho-2\rho a_0(n))-2x(1+2\rho-2\rho a_0(n))];\\
J^{\alpha,2}_{n,\rho}(e_2;x) &= x^2-\frac{1}{(n\rho+2)(n\rho+3)}\left[(2a_2(n)\rho^2+2\alpha\rho^2-n\rho^2-4\rho^2-n\rho-8\rho-6)x(1-x)\right.\\
&\quad \left.+(1+2\rho)\right].
\end{aligned}$$

In order to have $\lim_{n\to\infty} J^{\alpha,2}_{n,\rho}(e_i;x)=x^i, i=0,1,2,$ we choose undetermined coefficients as:

$$a_0(n)=\frac{1+2\rho}{2\rho}, a_2(n)=\frac{n(\rho+1)}{2\rho},\ b_0(n)=\frac{n(\rho+1)}{\rho}, a_1(n)=\frac{-(n+2)(\rho+1)}{2\rho}.$$

Thus, our operators become:

$$\overline{J}^{\alpha,2}_{n,\rho}(g;x)=\sum_{j=0}^{n}\overline{p}^{\alpha,2}_{n,j}(x)\int_0^1\mu^{\rho}_{n,j}(t)g(t)dt, \tag{3.2}$$

where

$$\begin{aligned}
\overline{p}^{\alpha,2}_{n,j}(x) &= \left(\frac{n(\rho+1)}{2\rho}x^2-\frac{(n+2)(\rho+1)}{2\rho}x+\frac{1+2\rho}{2\rho}\right)p^{\alpha}_{n-2,j}(x)+\frac{n(\rho+1)}{\rho}x(1-x)p^{\alpha}_{n-2,j-1}(x)\\
&\quad +\left(\frac{n(\rho+1)}{2\rho}x^2-\frac{(n-2)(\rho+1)}{2\rho}x-\frac{1}{2\rho}\right)p^{\alpha}_{n-2,j-2}(x).
\end{aligned}$$

Lemma 3.2. For the operators (3.2), we get:

$$\begin{aligned}
\overline{J}^{\alpha,2}_{n,\rho}(e_0;x) &= 1;\\
\overline{J}^{\alpha,2}_{n,\rho}(e_1;x) &= x;\\
\overline{J}^{\alpha,2}_{n,\rho}(e_2;x) &= x^2+\frac{1}{(n\rho+2)(n\rho+3)}\left[-1-2\rho+(6+8\rho+4\rho^2-2\alpha\rho^2)x(1-x)\right].
\end{aligned}$$

Lemma 3.3. For the operators (3.2), we get the central moments as:

$$\begin{aligned}
\overline{J}_{n,\rho}^{\alpha,2}(\phi_x^2(t);x) &= \frac{1}{(n\rho+2)(n\rho+3)}\left[-1-2\rho+(6+8\rho+4\rho^2-2\alpha\rho^2)x(1-x)\right];\\
\overline{J}_{n,\rho}^{\alpha,2}(\phi_x^3(t);x) &= \frac{2x(1-x)(2x-1)\rho(1+\rho)(2+\rho)n}{(n\rho+2)(n\rho+3)(n\rho+4)}+O\left(\frac{1}{n^3}\right);\\
\overline{J}_{n,\rho}^{\alpha,2}(\phi_x^4(t);x) &= -\frac{3x^2(1-x)^2\rho^2(1+\rho)^2n^2}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)}+O\left(\frac{1}{n^3}\right);\\
\overline{J}_{n,\rho}^{\alpha,2}(\phi_x^5(t);x) &= \frac{30x^2(1-x)^2(2x-1)\rho^2(1+\rho)^2(2+\rho))n^2}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)(n\rho+6)}+O\left(\frac{1}{n^4}\right);\\
\overline{J}_{n,\rho}^{\alpha,2}(\phi_x^6(t);x) &= -\frac{30x^3(1-x)^3\rho^3(1+\rho)^3n^3}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)(n\rho+6)(n\rho+7)}+O\left(\frac{1}{n^4}\right).
\end{aligned}$$

Theorem 8. If $g\in C^6[0,1]$ and $x\in[0,1]$, then for sufficiently large n, we have:

$$\overline{J}_{n,\rho}^{\alpha,2}(g;x)-g(x)=O\left(\frac{1}{n^2}\right).$$

Proof . Applying the operators $\overline{J}_{n,\rho}^{\alpha,2}(.;x)$ to Taylor's formula, we get:

$$\overline{J}_{n,\rho}^{\alpha,2}(g;x)=g(x)+\sum_{j=1}^{6}g^{(j)}(x)\overline{J}_{n,\rho}^{\alpha,2}\left(\phi_x^j(t);x\right)+\overline{J}_{n,\rho}^{\alpha,2}\left(\Theta(t,x)\phi_x^6(t);x\right),$$

where $\lim_{t\to x}\Theta(t,x)=0$. We can easily see from Lemma 3.3 that it is enough to prove that $\overline{J}_{n,\rho}^{\alpha,2}\left(\Theta(t,x)\phi_x^6(t);x\right)=0.$. Now,

$$\begin{aligned}
|\overline{J}_{n,\rho}^{\alpha,2}\left(\Theta(t,x)\phi_x^6(t);x\right) \leq\ & \left|-\frac{n(\rho+1)}{2\rho}x(1-x)-\frac{2(\rho+1)}{2\rho}x+\frac{1+2\rho}{2\rho}\right|\sum_{j=0}^{n-2}p_{n-2,j}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\mid\Theta(t,x)\mid\phi_x^6(t)dt\\
&+\left|\frac{n(\rho+1)}{\rho}x(1-x)\right|\sum_{j=1}^{n-1}p_{n-2,j-1}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\mid\Theta(t,x)\mid\phi_x^6(t)dt\\
&+\left|-\frac{n(\rho+1)}{2\rho}x(1-x)+\frac{2(\rho+1)}{2\rho}x-\frac{1}{2\rho}\right|\sum_{j=2}^{n}p_{n-2,j-2}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\mid\Theta(t,x)\mid\phi_x^6(t)dt.
\end{aligned}$$

Let $M=\sup_{t\in[0,1]}|\Theta(t,x)|$, then we have:

$$\begin{aligned}
&|\overline{J}_{n,\rho}^{\alpha,2}\left(\Theta(t,x)\phi_x^6(t);x\right)|<M\left[\frac{n(\rho+1)}{8\rho}+\frac{1+2\rho}{2\rho}\right]\sum_{j=0}^{n-2}p_{n-2,j}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt\\
&+M\left[\frac{n(\rho+1)}{4\rho}\right]\sum_{j=1}^{n-1}p_{n-2,j-1}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt+M\left[\frac{n(\rho+1)}{8\rho}+\frac{1+2\rho}{2\rho}\right]\sum_{j=2}^{n}p_{n-2,j-2}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt\\
&=\frac{Mn(\rho+1)}{8\rho}\left[\sum_{j=0}^{n-2}p_{n-2,j}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt+2\sum_{j=1}^{n-1}p_{n-2,j-1}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt+\sum_{j=2}^{n}p_{n-2,j-2}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt\right]\\
&+\frac{1+2\rho}{2\rho}M\left[\sum_{j=0}^{n-2}p_{n-2,j}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt+\sum_{j=2}^{n}p_{n-2,j-2}^{\alpha}(x)\int_0^1\mu_{n,j}^{\rho}(t)\phi_x^6(t)dt\right]
\end{aligned}$$

$$=\frac{Mn(\rho+1)}{8\rho}\left\{\frac{15x^3\rho^3(4(1+\rho)^3(1-3x+3x^2)-x^3(4+4\rho(3+3\rho(3+\rho))))n^3}{\prod_{k=2}^{7}(n\rho+k)}+O\left(\frac{1}{n^4}\right)\right\}$$

$$+\frac{1+2\rho}{2\rho}M\left\{\frac{15x^3\rho^3((1+\rho)^3(1-3x+3x^2)-x^3(1+\rho(3+\rho(3+\rho))))n^3}{\prod_{k=2}^{7}(n\rho+k)}+\frac{15x^3(1-x)^3\rho^3(1+\rho)^3n^3}{\prod_{k=2}^{7}(n\rho+k)}+O\left(\frac{1}{n^4}\right)\right\}$$

$$=O\left(\frac{1}{n^2}\right).$$

Hence, the proof is completed. □

For our comparison result, let us recall the asymptotic formula for $Q^{\alpha}_{n,\rho}(g;x)$ proved in [24].

Theorem 9. Let $f\in C[0,1]$. If f'' exists at a point $x\in[0,1]$, then we have

$$\lim_{n\to\infty} n(Q^{\alpha}_{n,\rho}(f;x)-f(x))=\frac{1-2x}{\rho}f'(x)+\frac{(1+\rho)x(1-x)}{2\rho}f''(x).$$

Theorem 10. Let $g\in C^6[0,1]$. If there exists an $n_0\in\mathbb{N}$ such that

$$g(x)\le \overline{J}^{\alpha,2}_{n,\rho}(g;x)\le Q^{\alpha}_{n,\rho}(g;x),\quad \forall\, n\ge n_0,\quad x\in[0,1]$$

then $(1-2x)g'(x)+(1+\rho)x(1-x)g''(x)\ge 0$, $x\in[0,1]$.

Proof . Let us consider

$$g(x)\ \le\ \overline{J}^{\alpha,2}_{n,\rho}(g;x)\le Q^{\alpha}_{n,\rho}(g;x).$$

Then $0\le n(\overline{J}^{\alpha,2}_{n,\rho}(g;x)-g(x))\le n(Q^{\alpha}_{n,\rho}(g;x)-g(x))$. From Theorems 8 and 9, we get the result. □

4 $\alpha-$Bernstein Păltănea operators of third order

Continuing in the same way as above, we can modify operators to obtain third order approximation operators, given as:

$$J^{\alpha,3}_{n,\rho}(g;x)=\sum_{j=0}^{n}p^{\alpha,3}_{n,j}(x)\int_0^1\mu^{\rho}_{n,j}(t)g(t)dt, \tag{4.1}$$

where

$$\begin{aligned}p^{\alpha,3}_{n,j}(x) &= a(x,n)p^{\alpha}_{n-4,j}(x)+b(x,n)p^{\alpha}_{n-4,j-1}(x)+d(x,n)p^{\alpha}_{n-4,j-2}(x)\\ &\quad +b(1-x,n)p^{\alpha}_{n-4,j-3}(x)+a(1-x,n)p^{\alpha}_{n-4,j-4}(x),\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}a(x,n) &= a_4(n)x^4+a_3(n)x^3+a_2(n)x^2+a_1(n)x+a_0(n),\\ b(x,n) &= b_4(n)x^4+b_3(n)x^3+b_2(n)x^2+b_1(n)x+b_0(n),\\ d(x,n) &= d_0(n)x^2(1-x)^2,\end{aligned}$$

here $a_j(n), b_j(n), j = 0, 1, \cdots, 4$ and $d_0(n)$ are the sequences to be determined in such that the operators (4.1) reduce to new operators $\tilde{J}_{n,\rho}^{\alpha,3}(g;x)\,(say)$ with third order approximation. In order to get $J_{n,\rho}^{\alpha,3}(e_i,x) = e_i, i = 0, 1, 2, 3$, we find unknown sequences which are given by:

$$\begin{aligned}
\tilde{a}_0(n) &= \frac{12\rho^3+19\rho^2+8\rho+1}{12\rho^3}, \tilde{a_1}(n) = -\frac{7\rho^2+11\rho+4}{12\rho^2}n - \frac{(29+5\alpha)\rho^3+(48+3\alpha)\rho^2+30\rho+7}{6\rho^3},\\
\tilde{a}_2(n) &= \frac{(1+\rho)^2}{8\rho^2}n^2 + \frac{17\rho^2+29\rho+12}{12\rho^2}n + \frac{(41-11\alpha)\rho^3+(71-9\alpha)\rho^2+60\rho+18}{6\rho^3},\\
\tilde{a}_3(n) &= -\frac{(1+\rho)^2}{4\rho^2}n^2 - \frac{5\rho^2+9\rho+4}{6\rho^2}n - \frac{(3-\alpha)\rho^3+(7-\alpha)\rho^2+6\rho+2}{\rho^3},\\
\tilde{a}_4(n) &= \frac{(1+\rho)^2}{8\rho^2}n^2, \tilde{b_0}(n) = -\frac{12\rho^2+7\rho+1}{6\rho^3},\\
\tilde{b}_1(n) &= \frac{(3\rho^2+5\rho+2)}{3\rho^2}n + \frac{(16-4\alpha)\rho^3+(37-3\alpha)\rho^2+27\rho+7}{3\rho^3},\\
\tilde{b}_2(n) &= -\frac{(1+\rho)^2}{2\rho^2}n^2 - \frac{8\rho^2+14\rho+6}{3\rho^2}n - \frac{(34-10\alpha)\rho^3+(71-9\alpha)\rho^2+57\rho+18}{3\rho^3},\\
\tilde{b}_3(n) &= \frac{(1+\rho)^2}{\rho^2}n^2 + \frac{5\rho^2+9\rho+4}{3\rho}n + \frac{2((3-\alpha)\rho^3+(7-\alpha)\rho^2+6\rho+2)}{\rho^3},\\
\tilde{b}_4(n) &= -\frac{(1+\rho)^2}{2\rho^2}n^2, \tilde{d_0}(n) = \frac{3(1+\rho)^2}{4\rho^2}n^2.
\end{aligned}$$

Lemma 4.1. For the operators $\tilde{J}_{n,\rho}^{\alpha,3}(g;x)$, we get the following central moments:

$$\begin{aligned}
\tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^1(t);x) &= \tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^2(t);x) = \tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^3(t);x) = 0;\\
\tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^4(t);x) &= \frac{x(1-x)\rho(1+\rho)[11+27\rho+12\rho^2-x(1-x)\{2\rho^2(6\alpha-29)+58(1+2\rho)\}]n}{\prod_{k=2}^{5}(n\rho+k)} + O\left(\frac{1}{n^4}\right);\\
\tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^5(t);x) &= \frac{5x^2(1-x)^2(2x-1)\rho^2(1+\rho)^2(5+4\rho)n^2}{\prod_{k=2}^{6}(n\rho+k)} + O\left(\frac{1}{n^4}\right);\\
\tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^6(t);x) &= \frac{15x^3(1-x)^3\rho^3(1+\rho)^3n^3}{\prod_{k=2}^{7}(n\rho+k)} + O\left(\frac{1}{n^4}\right);\\
\tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^7(t);x) &= \tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^8(t);x) = O\left(\frac{1}{n^4}\right); \tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^9(t);x) = \tilde{J}_{n,\rho}^{\alpha,3}(\phi_x^{10}(t);x) = O\left(\frac{1}{n^5}\right).
\end{aligned}$$

To prove the asymptotic order of approximation of the operators $\tilde{J}_{n,\rho}^{\alpha,3}(g;x)$, we require $g \in C^{10}[0,1]$ in a similar way, as in Theorem 8, which is given as follows:

Theorem 11. If $g \in C^{10}[0,1]$ and $x \in [0,1]$, then for sufficiently large n, we have

$$\tilde{J}_{n,\rho}^{\alpha,3}(g;x) - g(x) = O\left(\frac{1}{n^3}\right).$$

5 Numerical Results

In the present section, we give the numerical examples to validate our theoretical results and error estimation by using maple algorithms:

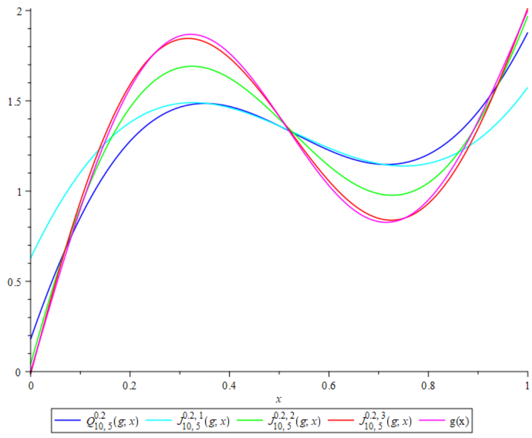

Figure 1: Approximation process

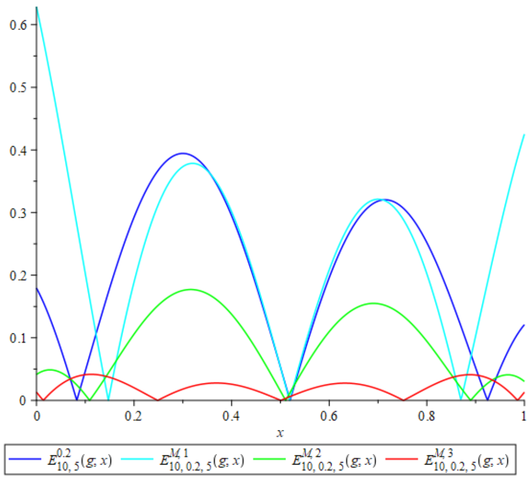

Figure 2: Error estimation

Example 5.1. Consider $g(x) = 2sin\left(\frac{\pi x}{2}\right) + sin(2\pi x)$, $n = 10, \rho = 5, \alpha = 0.2, a_0(n) = \frac{n-1}{2n}$ and $a_1(n) = \frac{1}{n}$. The comparison of convergence of the $\alpha-$Bernstein-Păltănea operators and its above modifications of orders one, two and three to $g(x)$ is given in Fig. 1.

Let $E^{\alpha}_{n,\rho}(g;x) = |g(x) - Q^{\alpha}_{n,\rho}(g;x)|$ and $E^{\alpha,i}_{n,\rho}(g;x) = |g(x) - J^{\alpha,i}_{n,\rho}(g;x)|$, $i = 1, 2, 3$ be error function of classical operators and its modifications respectively. The error of approximation of these operators are given in Fig. 2.
From both the figures, we can conclude that our modified operators are converging faster than original $\alpha-$Bernstein Păltănea operators. Also, we have given error of approximation at some certain points in Table 1.

Example 5.2. Let us choose $g(x) = x\cos(2\pi x)$, $\rho = 4, \alpha = 0.3, a_0(n) = \frac{n-1}{2n}$ and $a_1(n) = \frac{1}{n}$. The behavior of $\alpha-$Bernstein-Păltănea operators $Q^{\alpha}_{n,\rho}(g;x)$ and its three modifications $J^{\alpha,i}_{n,\rho}(g;x)$ where $i = 1, 2, 3$ to $g(x)$ for $n = 10, 20$ is given in Figs. 3, 5. We can observe from figures that our modifications are converging to a function as we increase the value of n and also give better convergence than original operators.

The error of approximation of $\alpha-$Bernstein Păltănea operators and its modifications of order $1, 2, 3$ are given in Figs. 4, 6 for $n = 10, 20$ respectively. It can be easily seen that error estimation by our modifications are less than original $\alpha-$Bernstein Păltănea operators. Also, the error of operators and its modifications at some points are given in Tables 2, 3 at the values $n = 10, 20$ respectively.

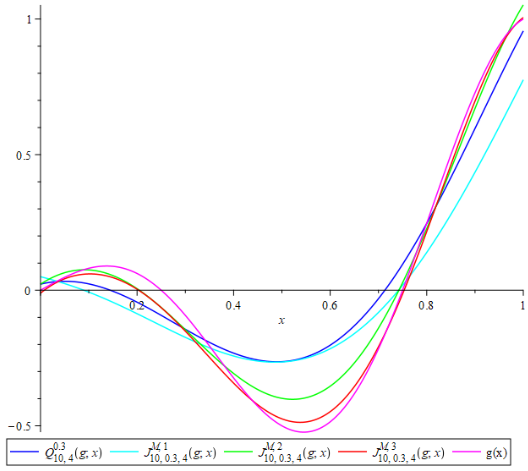

Figure 3: Approximation process

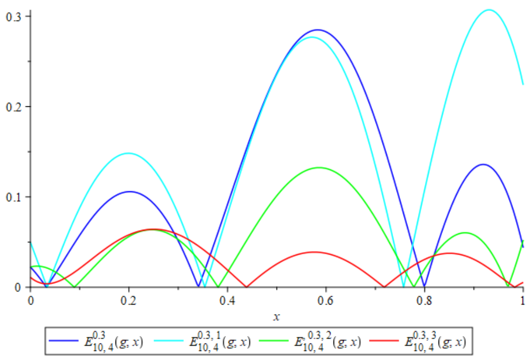

Figure 4: Error estimation

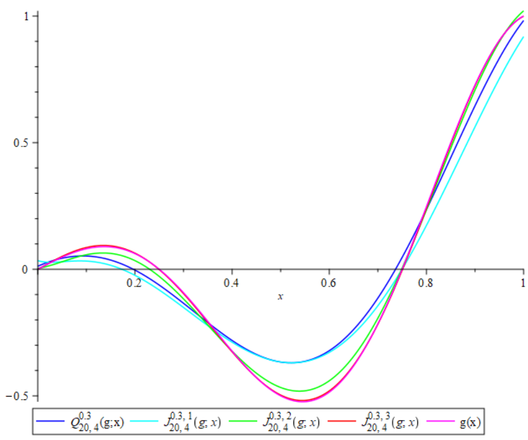

Figure 5: Approximation process

Table 1: Error of Approximation $E_{n,\rho}^{\alpha}$ and $E_{n,\rho}^{\alpha,i}, i = 1,2,3, n = 10, \rho = 5, \alpha = 0.2$

x	$E_{10,5}^{0.2}(g;x)$	$E_{10,5}^{0.2,1}(g;x)$	$E_{10,5}^{0.2,2}(g;x)$	$E_{10,5}^{0.2,3}(g;x)$
0.1	0.0465927917	0.2029191714	0.0090258508	0.0410711480
0.2	0.292607380	0.189033401	0.105857578	0.020163736
0.3	0.394424647	0.373376025	0.175570300	0.018015963
0.4	0.294804927	0.301231910	0.137423887	0.025812675
0.5	0.053236320	0.053236321	0.014926178	0.000239280
0.6	0.196044423	0.208534327	0.109839626	0.025493047
0.7	0.3182317026	0.3213021846	0.1544639662	0.0180914623
0.8	0.2521506342	0.2023484272	0.0944769302	0.0196894259
0.9	0.052294816	0.102854286	0.008980412	0.040540562

Table 2: Error of Approximation $E_{n,\rho}^{\alpha}$ and $E_{n,\rho}^{\alpha,i}, i = 1,2,3, n = 10, \rho = 4, \alpha = 0.3$

x	$E_{10,4}^{0.3}(g;x)$	$E_{10,4}^{0.3,1}(g;x)$	$E_{10,4}^{0.3,2}(g;x)$	$E_{10,4}^{0.3,3}(g;x)$
0.1	0.05755638690	0.07198182417	0.00593343135	0.02211119337
0.2	0.1058403144	0.1347020007	0.05618431396	0.00086323684
0.3	0.05131069544	0.07328503894	0.05274231364	0.01900023794
0.4	0.0916546188	0.0844657778	0.0166755487	0.0079394543
0.5	0.2366464156	0.2366464156	0.1013426174	0.0193312761
0.6	0.2827508318	0.2698804172	0.1314792584	0.0277669661
0.7	0.1851049397	0.1344774017	0.0765221240	0.0019799015
0.8	0.0010362418	0.1078472481	0.0202148003	0.0324610925
0.9	0.1316309131	0.2930201701	0.0579303733	0.0309826354

6 Conclusion

As the rate of convergence of $\alpha-$Bernstein-Durrmeyer operators by using Păltănea basis function is slow, so we have improved the rate of convergence of these operators by introducing three modifications which have better order of approximation. Moreover the convergence of these operators is independent of parameters involved in it. In order to validate our theoretical results, we have presented some numerical examples and their graphics by using MAPLE algorithms. With the similar procedure, we can also get higher order of approximation of these positive linear operators.

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Table 3: Error of Approximation $E_{n,\rho}^{\alpha}$ and $E_{n,\rho}^{\alpha,i}, i = 1,2,3, n = 20, \rho = 4, \alpha = 0.3$

x	$E_{20,4}^{0.3}(g;x)$	$E_{20,4}^{0.3,1}(g;x)$	$E_{20,4}^{0.3,2}(g;x)$	$E_{20,4}^{0.3,3}(g;x)$
0.1	0.02861514285	0.02978527157	0.00135692531	0.00493671473
0.2	0.06515132595	0.08083293178	0.01810232856	0.00184414784
0.3	0.04171602804	0.05754420964	0.02246595754	0.00339046874
0.4	0.0423780236	0.0356126371	0.0005113770	0.0023993240
0.5	0.1336280335	0.1336280335	0.0330134666	0.003147654
0.6	0.1637928626	0.1572063097	0.0448962830	0.0048952830
0.7	0.1024392756	0.0721778840	0.0230362411	0.0006129461
0.8	0.0113308695	0.0741312889	0.0121952799	0.0062328086
0.9	0.0822517165	0.1657103520	0.0201291953	0.0033482141

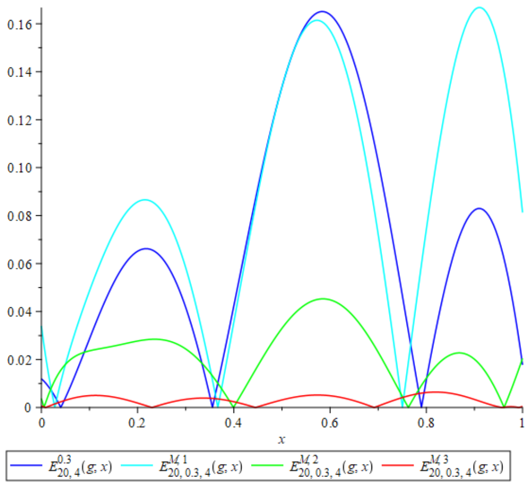

Figure 6: Error estimation

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