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New subclasses of Ozaka's convex functions

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Abstract

Let $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ be the classes of functions f, analytic in the unit disc $\Delta = \{z : |z| < 1\}$, with the normalization $f(0) = f'(0) - 1 = 0$, which satisfies the conditions

$$\frac{zf'(z)}{f(z)} \prec (1+z)^\lambda \quad \text{and} \quad \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1+z)^\lambda \qquad (0 < \lambda \le 1),$$

where $\prec$ is the subordination relation, respectively. The classes $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ are subfamilies of the known classes of strongly starlike and convex functions of order λ. We consider the relations between $\mathcal{S}^*_L(\lambda)$, $\mathcal{CV}_L(\lambda)$ and other classes geometrically defined. Also, we obtain the sharp radius of convexity for functions belonging to $\mathcal{S}^*_L(\lambda)$ class. Furthermore, the norm of pre-Schwarzian derivatives and univalency of functions f which satisfy the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{\lambda}{2} \qquad (z \in \Delta),$$

are considered.

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1 Introduction and preliminary

Let $\mathcal{H}$ denote the class of *holomorphic functions* in the open unit disc $\Delta = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$, and let $\mathcal{A}$ denote the subclass of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta). \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of all *univalent* functions f in Δ, is denoted by $\mathcal{S}$. Robertson [14], Brannan and Kirwan [5], introduced the classes $\mathcal{ST}(\beta)$, $\mathcal{CV}(\beta)$, of *starlike and convex functions of order* $0 \le \beta < 1$, and $\mathcal{SS}^*(\alpha)$ and $\mathcal{CV}^*(\alpha)$ *strongly starlike and convex functions of order* $0 < \alpha \le 1$, respectively, which are defined by

$$\mathcal{ST}(\beta) := \left\{ f \in \mathcal{A} \colon \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in \Delta \right\},$$
$$\mathcal{CV}(\beta) := \left\{ f \in \mathcal{A} \colon \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \quad z \in \Delta \right\},$$

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and

$$\mathcal{SS}^*(\alpha) := \left\{ f \in \mathcal{A}\colon \left| \operatorname{Arg}\left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi\alpha}{2}, \quad z \in \Delta \right\},$$
$$\mathcal{CV}^*(\alpha) := \left\{ f \in \mathcal{A}\colon \left| \operatorname{Arg}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi\alpha}{2}, \quad z \in \Delta \right\}.$$

We also note that $\mathcal{SS}^*(1) = \mathcal{ST}(0) =: \mathcal{ST}$ and $\mathcal{CV}^*(1) = \mathcal{CV}(0) =: \mathcal{CV}$ are the well-known classes of all normalized starlike and convex functions in Δ, respectively. Let $\mathcal{S}(a, b)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the inequality

$$a < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < b \qquad (z \in \Delta),$$

for some real number a; $(0 \leq a < 1)$ and some real number b; $(b > 1)$ (See [7]). We define the norm of pre-Schwarzian derivatives $\|T_f\|$, as follows:

$$\|T_f\| = \sup_{z \in \Delta} \left(1 - |z|^2\right) \left| \frac{f''(z)}{f'(z)} \right|,$$

for function $f \in \mathcal{S}$.

Definition 1.1 ([6]). Let f and g be analytic in Δ. Then the function f is said to be *subordinate* to g in Δ, written by $f(z) \prec g(z)$, if there exists a function $\omega(z) \in \mathcal{B}$ such that $f(z) = g(\omega(z))$, $z \in \Delta$, where $\mathcal{B}$ is the family of all self-maps functions

$$\omega(z) = \sum_{n=1}^{\infty} w_n z^n \qquad (|\omega(z)| < 1, \ z \in \Delta). \tag{1.2}$$

From the definition of subordination, it is easy to show that the subordination $f(z) \prec g(z)$ implies that $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. In particular, if $g(z)$ is univalent in Δ, then the subordination $f(z) \prec g(z)$ is equivalent to the condition$f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let ϕ be an analytic function with positive real part in Δ, $\phi(0) = 1$, $\phi'(0) > 0$ and map Δ onto a region starlike with respect to $\phi(0) = 1$ and symmetric with respect to real axis. Ma and Minda [8] introduced the class $\mathcal{S}^*(\phi)$ defined by

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} \quad : \quad \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Delta \right\}, \tag{1.3}$$
$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{A} \quad : \quad \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \phi(z), \quad z \in \Delta \right\}. \tag{1.4}$$

Associated to classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$, a family $\mathcal{P}(\phi)$ to be introduced which consists of analytic functions p such that $p(0) = 1$ and $p(\mathbb{D}) \subset \phi(\mathbb{D})$, or equivalently $p \prec \phi$. The Carathéodory class

$$\mathcal{P} = \{p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \Re\, p(z) > 0, z \in \mathbb{D}\}$$

is a simply the class $\mathcal{P}((1 + z)/(1 - z))$.

Definition 1.2. A locally univalent function $f \in \mathcal{A}$ is said to belong to $\mathcal{G}(s)$ for some $s > 0$, if it satisfies the condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < 1 + \frac{s}{2} \qquad (z \in \Delta).$$

In [11], Ozaki introduced the class $\mathcal{G}(1)$ and proved that functions in the class $\mathcal{G}(1)$ are univalent. In [20], Umezawa generalized Ozaki's result for a version of the class $\mathcal{G}(1)$ (convex functions in one direction). A function $f \in \mathcal{A}$ is said to belong to $\mathcal{N}(s)$ for some $s > 0$, if it satisfies the condition

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{s}{2} \qquad (z \in \Delta).$$

It is easy to see that $f \in \mathcal{G}(s)$ if and only if $zf' \in \mathcal{N}(s)$.

Let us denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\Delta}\backslash\mathbf{E}(f)$, where $\mathbf{E}(f)=\{\zeta\colon \zeta\in\partial\Delta \quad \text{and} \quad \lim_{z\to\zeta}$ and are such that

$$f'(\zeta)\neq 0 \qquad \text{for} \quad \zeta\in\partial\Delta\setminus\mathbf{E}(f).$$

Lemma 1.3. [9, p.24] Let $q\in\mathcal{Q}$ with $q(0)=1$ and let $p(z)=1+p_1z+\cdots$ be analytic in Δ with $p(z)\neq 1$. If $p\not\prec q$ in Δ, then there exits points $z_0\in\Delta$ and $\zeta\in\partial\Delta\setminus\mathbf{E}(q)$ and there exits a real number $m\geq 1$ for which

$$p(|z|<|z_0|)\subset q(\Delta), \qquad p(z_0)=q(\zeta), \qquad z_0p'(z_0)=m\zeta q'(\zeta).$$

The purpose of this work is to define a new subfamily of $\mathcal{P}$ related to a domain bounded by *sinusoidal spiral*

$$\begin{aligned}\mathbb{LB}(\lambda)&=\left\{\rho e^{i\varphi}\colon \rho=\left(2\cos\frac{\varphi}{\lambda}\right)^{\lambda}, \quad -\frac{\lambda\pi}{2}<\varphi\leq\frac{\lambda\pi}{2}\right\}\\ &=\left\{w\in\mathbb{C}\colon \quad \Re\{w\}>0, \qquad \Re\left\{w^{-1/\lambda}\right\}=\frac{1}{2}\right\}\cup\{0\}.\end{aligned}$$

Since $\rho=\left(2\cos\frac{\varphi}{\lambda}\right)^{\lambda}$, we have

$$\rho^{1/\lambda}=\left(2\cos\frac{\varphi}{\lambda}\right) \qquad \text{or} \qquad \rho^{-1/\lambda}\cos\frac{\varphi}{\lambda}=\frac{1}{2} \quad \text{or} \quad \Re\left\{w^{-1/\lambda}\right\}=\frac{1}{2} \quad \text{when} \quad w=\rho e^{i\varphi}.$$

$\mathcal{P}\left((1+z)^{\lambda}\right)$, we present a new resolution to get the norm of pre-Schwarzian derivatives and univalence from class functions $\mathcal{G}(\lambda)$.

The remainder of the paper proceeds as follows. In sections 2, in order to express our original theorem, we introduce a family of functions and properties. The classes $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ are introduced and its properties and its relevance to other classes presented. In the sequel, we get the extremal functions of classes $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$. Furthermore, we obtain norm of pre-Schwarzian derivatives and univalency of functions f in class $\mathcal{G}(\lambda)$. Also, some examples are presented.

2 The classes $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ and its properties

This section provides a detailed exposition of an analytic function that maps the unit disk onto a domain bounded by a *sinusoidal spiral* and contained in a right half-plane. In fact, taking into account:

$$\mathfrak{q}_\lambda(z):=(1+z)^{\lambda}\equiv e^{\lambda\log(1+z)} \qquad (0<\lambda\leq 1),$$

where the branch of the power is chosen to be $\mathfrak{q}_\lambda(0)=1$, more explicitly,

$$\begin{aligned}\mathfrak{q}_\lambda(z)&=1+\sum_{k=1}^{\infty}\frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}z^k=1+\sum_{k=1}^{\infty}B_kz^k\\ &=1+\lambda z+\frac{\lambda(\lambda-1)}{2}z^2+\frac{\lambda(\lambda-1)(\lambda-2)}{6}z^3+\cdots \quad (z\in\Delta).\end{aligned} \tag{2.1}$$

The set $\mathfrak{q}_\lambda(\Delta)$ lies in the region bounded by the right loop of the *sinusoidal spiral* given by

$$\mathbb{LB}(\lambda)=\left\{\rho e^{i\varphi}\colon \rho=\left(2\cos\frac{\varphi}{\lambda}\right)^{\lambda}, \quad -\frac{\lambda\pi}{2}<\varphi\leq\frac{\lambda\pi}{2}\right\}.$$

To see this, note that writing $z=e^{i\theta}$, where $\theta\in(-\pi,\pi)$, we have

$$\mathfrak{q}_\lambda(e^{i\theta})=\left(1+e^{i\theta}\right)^{\lambda}=\left(2\cos\frac{\theta}{2}\right)^{\lambda}e^{i\frac{\lambda\theta}{2}}=\left(2\cos\frac{\theta}{2}\right)^{\lambda}\left(\cos\frac{\lambda\theta}{2}+i\sin\frac{\lambda\theta}{2}\right). \tag{2.2}$$

By (2.2), we have

$$\Re\{\mathfrak{q}_\lambda(e^{i\theta})\} = \left(2\cos\frac{\theta}{2}\right)^\lambda \cos\frac{\lambda\theta}{2} =: u(\theta) = u \qquad (-\pi < \theta < \pi),$$
$$\Im\{\mathfrak{q}_\lambda(e^{i\theta})\} = \left(2\cos\frac{\theta}{2}\right)^\lambda \sin\frac{\lambda\theta}{2} =: v(\theta) = v \qquad (-\pi < \theta < \pi).$$

So we can see that $u(\theta)$ and $v(\theta)$ are well defined also for $\theta = \pi$. The function $u(\theta)$ with $-\pi < \theta \leq \pi$ attains its minimal value when $\theta = \pi$, and maximum value when $\theta = 0$ and The function $v(\theta)$ with $-\pi < \theta \leq \pi$ attains its minimal value when $\theta = -\pi/(1+\lambda)$, and maximum value when $\theta = \pi/(1+\lambda)$. On the other hand for $-\pi < \theta \leq \pi$

$$0 \leq \Re\{\mathfrak{q}_\lambda(e^{i\theta})\} \leq 2^\lambda,$$
$$-\left(2\cos\frac{\pi}{2\lambda+2}\right)^\lambda \sin\frac{\pi\lambda}{2\lambda+2} \leq \Im\{\mathfrak{q}_\lambda(e^{i\theta})\} \leq \left(2\cos\frac{\pi}{2\lambda+2}\right)^\lambda \sin\frac{\pi\lambda}{2\lambda+2}.$$

If we take $\mathfrak{q}_\lambda(e^{i\theta}) = \rho e^{i\varphi}$, simple calculations show that $\varphi = \lambda\theta/2$ and $\rho = \left(2\cos\frac{\theta}{2}\right)^\lambda$. Therefore $\mathfrak{q}_\lambda(e^{i\theta})$ in the polar coordinates will be as follows

$$\mathfrak{q}_\lambda(e^{i\theta}) = \left\{w = \rho e^{i\varphi}: \quad \rho = \left(2\cos\frac{\varphi}{\lambda}\right)^\lambda, \quad -\frac{\lambda\pi}{2} < \varphi \leq \frac{\lambda\pi}{2}\right\}. \tag{2.3}$$

Thus from (2.3) we have $\left|\mathrm{Arg}\left\{\mathfrak{q}_\lambda(e^{i\theta})\right\}\right| < \lambda\pi/2$. Additionally, the right loop of the *sinusoidal spiral* $\mathbb{LB}(\lambda)$ is a boundary of the domain $\mathfrak{q}_\lambda(\Delta)$. Also note that

$$\mathfrak{q}_\lambda(\Delta) = \left\{w = \rho e^{i\varphi}: \quad \rho < \left(2\cos\frac{\varphi}{\lambda}\right)^\lambda, \quad -\frac{\lambda\pi}{2} < \varphi < \frac{\lambda\pi}{2}\right\}$$
$$= \left\{w \in \mathbb{C}: \quad \Re\{w\} > 0, \qquad \Re\left\{w^{-1/\lambda}\right\} > \frac{1}{2}\right\}.$$

is a domain which is symmetric about the real axis, starlike with respect to the point $\mathfrak{q}_\lambda(0) = 1$, and satisfies $\mathfrak{q}'_\lambda(0) = \lambda > 0$. Also, $\mathbb{LB}(\lambda)$ has tangential radial vector $\varphi = \pm\lambda\pi/2$.

Lemma 2.1. The functions $\mathfrak{q}_\lambda(z)$ are convex univalent in Δ for each $0 < \lambda \leq 1$. Moreover $g_\lambda(z) = (\mathfrak{q}_\lambda(z) - 1)/\lambda \in \mathcal{CV}((1+\lambda)/2)$ and $g_1(z) = \mathfrak{q}_1(z) - 1 = z \in \mathcal{CV}$. Also, if $|z| = r < 1$, then

$$\min_{|z|=r} |\mathfrak{q}_\lambda(z)| = \mathfrak{q}_\lambda(-r) \qquad \text{and} \qquad \max_{|z|=r} |\mathfrak{q}_\lambda(z)| = \mathfrak{q}_\lambda(r).$$

Proof . Let us consider

$$g_\lambda(z) = (\mathfrak{q}_\lambda(z) - 1)/\lambda \qquad (z \in \Delta).$$

Then, we have

$$\Re\left\{1 + \frac{z g''_\lambda(z)}{g'_\lambda(z)}\right\} = \Re\left\{\frac{1+\lambda z}{1+z}\right\} > \frac{\lambda+1}{2},$$

so $g_\lambda \in \mathcal{CV}((\lambda+1)/2) \subset \mathcal{ST}$. In order to prove the second part of lemma, if $\theta \in [0, 2\pi)$, then the function

$$Q(\theta) = \left|\mathfrak{q}_\lambda(re^{i\theta})\right| = \left|1 + re^{i\theta}\right|^\lambda = \left(1 + r^2 + 2r\cos\theta\right)^{\frac{\lambda}{2}} \quad (0 < r < 1)$$

attains its minimum at $\theta = \pi$ and maximum at $\theta = 0$. This ends the proof. □

The following theorem describes some properties of the functions that are in class

$$\mathcal{P}(\mathfrak{q}_\lambda) = \{p \in \mathcal{H} : p \prec \mathfrak{q}_\lambda\}.$$

Theorem 2.2. Let $\mathfrak{p} \in \mathcal{P}(\mathfrak{q}_\lambda)$. Then

$$|\mathrm{Arg}\,\{\mathfrak{p}(z)\}| < \frac{\lambda\pi}{2}, \quad 0 < \Re\{\mathfrak{p}(z)\} < 2^\lambda, \quad |\Im\{\mathfrak{p}(z)\}| < \left(2\cos\frac{\pi}{2\lambda+2}\right)^\lambda \sin\frac{\pi\lambda}{2\lambda+2} \tag{2.4a}$$

and

$$\left|\mathfrak{p}^{1/\lambda}(z) - 1\right| < 1, \tag{2.4b}$$

or

$$0 < \Re\left\{\mathfrak{p}^{1/\lambda}(z)\right\} < 2. \tag{2.4c}$$

Conversely, if $\mathfrak{p} \in \mathcal{P}$ with $|\mathrm{Arg}\,\{\mathfrak{p}(z)\}| < \lambda\pi/2$ and $\mathfrak{p}$ satisfies (2.4b), then $\mathfrak{p} \prec \mathfrak{q}_\lambda$ in Δ.

Proof . The subordination $\mathfrak{p} \prec \mathfrak{q}_\lambda$ with $\mathfrak{p}(0) = \mathfrak{q}_\lambda(0)$, and the geometric properties of $\mathfrak{q}_\lambda(\Delta)$ yield (2.4a). In order to prove the second part of theorem, since $\mathfrak{p} \in \mathcal{P}(\mathfrak{q}_\lambda)$, then

$$\mathfrak{p}(z) = (1+\omega(z))^\lambda \quad \text{or} \quad \omega(z) = \mathfrak{p}^{1/\lambda}(z) - 1, \quad |\omega(z)| < 1,$$

where $\omega \in \mathcal{B}$ and finally assertion (2.4b) as follows. For the prove (2.4c) we rewrite (2.4b) as

$$-1 < -\left|p^{1/\lambda}(z) - 1\right| \le \Re\{p^{1/\lambda}(z) - 1\} \le \left|p^{1/\lambda}(z) - 1\right| < 1,$$

that reduces to (2.4c). Conversely, it is enough to show that $\mathfrak{p}(\Delta) \subset \mathfrak{q}_\lambda(\Delta)$. To do this, let $w = \rho e^{i\varphi} \in \mathfrak{p}(\Delta)$. Since w satisfy the condition (2.4b), we conclude

$$\rho^{1/\lambda} < 2\cos\frac{\varphi}{\lambda}. \tag{2.4d}$$

Making use of $|\mathrm{Arg}\,\{w\}| < (\lambda\pi)/2$, we have $\Re\, w^{1/\lambda} > 0$ or, equivalently $\cos(\varphi/\lambda) > 0$. From (2.4d), we obtain $w \in \mathfrak{q}_\lambda(\Delta)$ and completes the proof. □

Using the same notation and the same reasoning as in the proof of Theorem 2.2, we get the following Theorem.

Theorem 2.3. Let $\mathfrak{p} \in \mathcal{P}(\mathfrak{q}_\lambda)$. Then

$$\Re\left\{\mathfrak{p}^{-1/\lambda}(z)\right\} > \frac{1}{2}, \tag{2.5a}$$

or

$$0 < \Re\left\{\mathfrak{p}^{1/\lambda}(z)\right\} < 2. \tag{2.5b}$$

Conversely, if $\mathfrak{p} \in \mathcal{P}$ and $\mathfrak{p}$ satisfies (2.5a), then $\mathfrak{p} \prec \mathfrak{q}_\lambda$ in Δ.

Definition 2.4. Let $\mathcal{S}^*_L(\lambda)$ denote the class of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{q}_\lambda(z) \qquad (z \in \Delta)\,. \tag{2.6a}$$

and $\mathcal{CV}_L(\lambda)$ denote the class of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \mathfrak{q}_\lambda(z) \qquad (z \in \Delta)\,. \tag{2.6b}$$

Geometrically, the condition (2.6a) and (2.6b) means that the quantities $zf'(z)/f(z)$ and $1 + zf''/f'$ lies in the region bounded by the right loop of the *sinusoidal spiral* $\mathbb{LB}(\lambda)$, respectively. Since a domain $\mathfrak{q}_\lambda(\Delta)$ is contained in a right half-plane, we deduce that $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ are proper subset of classes of a starlike functions $\mathcal{ST}$ and convex functions $\mathcal{CV}$, respectively. Now we turn to the relationship between the classes $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$ and the classes mentioned in the section 1. By Theorems 2.2 and 2.3 we get

$$\begin{aligned}\mathcal{S}^*_L(\lambda) &= \left\{ f \in \mathcal{SS}^*(\lambda)\colon \quad \left|\left[\frac{zf'(z)}{f(z)}\right]^{1/\lambda} - 1\right| < 1,\, z \in \Delta \right\} \\ &= \left\{ f \in \mathcal{ST}\colon \quad \Re\left\{\left[\frac{zf'(z)}{f(z)}\right]^{-1/\lambda}\right\} > \frac{1}{2},\, z \in \Delta \right\},\end{aligned}$$

$$\begin{aligned}\mathcal{CV}_L(\lambda) &= \left\{f \in \mathcal{CV}^*(\lambda)\colon \quad \left|\left[1+\frac{zf''(z)}{f'(z)}\right]^{1/\lambda} - 1\right| < 1,\, z \in \Delta\right\}\\ &= \left\{f \in \mathcal{CV}\colon \quad \Re\left\{\left[1+\frac{zf''(z)}{f'(z)}\right]^{-1/\lambda}\right\} > \frac{1}{2},\, z \in \Delta\right\},\end{aligned}$$

$$\mathcal{S}^*_L(\lambda) \subset \mathcal{SS}^*(\alpha) \quad \text{and} \quad \mathcal{CV}_L(\lambda) \subset \mathcal{CV}^*(\alpha) \quad \text{for} \quad \lambda \le \alpha \le 1,$$
$$\mathcal{S}^*_L(\lambda) \subset \mathcal{S}(0,b) \quad \text{and} \quad \mathcal{CV}_L(\lambda) \subset \mathcal{S}(0,b) \quad \text{for} \quad b \ge 2^\lambda,$$
$$\mathcal{S}^*_L(\lambda_1) \subset \mathcal{S}^*_L(\lambda_2) \quad \text{and} \quad \mathcal{CV}_L(\lambda_1) \subset \mathcal{CV}_L(\lambda_2) \quad \text{for} \quad \lambda_1 \le \lambda_2.$$

Applying the Lemma 2.1 and Theorem 2.2 and the Briot-Bouquet differential subordination [9, Theorem 3.2a], we can easily see that $\mathcal{CV}_L(\lambda) \subset \mathcal{S}^*_L(\lambda)$.

Lemma 2.5. Let $0 < \lambda \le 1$. If $M \ge 2^{\lambda-1}$, then

$$(1+z)^\lambda \prec \frac{M+Mz}{M-(M-1)\,z} =: P_M(z) \qquad (z \in \Delta)\,. \tag{2.7}$$

Proof . Since $q(0) = P_M(0) = 1$, from Definition 1.1 it is enough to prove $\mathfrak{q}_\lambda(\Delta) \subset P_M(\Delta)$. Since for $-\pi/2 < \varphi/\lambda < \pi/2$, we have

$$\left(2\cos\frac{\varphi}{\lambda}\right)^\lambda \le (2\cos\varphi)^\lambda \le 2^\lambda \cos\varphi. \tag{2.8}$$

Also, the function P_M is univalent in Δ, and maps the unit circle onto the circle

$$\left\{\rho = 2M\cos\varphi\colon\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\}.$$

For the establishment of relation $\mathfrak{q}_\lambda(\Delta) \subset P_M(\Delta)$, taking into account relation (2.8), we deduce $2^\lambda \le 2M$ and

$$\mathfrak{q}_\lambda(\Delta) \subset \left\{\rho e^{i\varphi}\colon \rho \le 2M\cos\varphi, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}, \quad M \ge 2^{\lambda-1}\right\} = P_M(\Delta).$$

Moreover

$$\mathcal{S}^*_L(\lambda) \subset \left\{f \in \mathcal{A}\colon \left|\frac{zf'(z)}{f(z)} - M\right| < M, \quad z \in \Delta, M \ge 2^{\lambda-1}\right\}.$$

□

The Relation (2.8) show that the image of the unit circle $|z| = 1$ under the functions $\mathfrak{q}_\lambda$ (The right-half of the lemniscate of Bernoulli $\gamma_1\colon \rho = \left(2\cos\frac{\varphi}{\lambda}\right)^\lambda$) and P_M (The circle $\gamma_2\colon \rho = 2^\lambda\cos\varphi$ with $-\pi/2 < \varphi < \pi/2$) for $\lambda = 1/5$ and $M = 1/\sqrt[5]{16}$, respectively.

By possessing a comprehensive form of functions p, i.e. $p \in \mathcal{P}(\mathfrak{q}_\lambda)$, we obtain by integration, the exhibition formula for the functions in $\mathcal{S}^*_L(\lambda)$ and $\mathcal{CV}_L(\lambda)$. Namely, $f \in \mathcal{S}^*_L(\lambda)$ if and only there exists a function $p \in \mathcal{P}(\mathfrak{q}_\lambda)$ such that

$$f(z) = z\exp\left(\int_0^z \frac{p(t)-1}{t}\,\mathrm{d}t\right) \qquad (z \in \Delta)\,, \tag{2.9a}$$

or, $f \in \mathcal{CV}_L(\lambda)$ if and only there exists a function $p \in \mathcal{P}(\mathfrak{q}_\lambda)$ such that

$$f(z) = \int_0^z \exp\left(\int_0^w \frac{p(t)-1}{t}\,\mathrm{d}t\right)\mathrm{d}w \qquad (z \in \Delta)\,. \tag{2.9b}$$

Let $g \in \mathcal{A}$ and let $zg'(z)/g(z) = p(z)$ $(1 + zg''(z)/g'(z) = p(z)$ resp.) with $p \in \mathcal{P}(\mathfrak{q}_\lambda)$, $z \in \Delta$. Clearly, $g \in \mathcal{S}^*_L(\lambda)$ ($\mathcal{CV}_L(\lambda)$ resp.) and g is extremal function in the class $\mathcal{S}^*_L(\lambda)$ ($\mathcal{CV}_L(\lambda)$ resp.). This representation gives many examples of functions in class $\mathcal{S}^*_L(\lambda)$ ($\mathcal{CV}_L(\lambda)$ resp.). To do this, by taking $p(z) = \mathfrak{q}_\lambda(z^n)$ with $n = 1, 2, 3, \ldots$, the function $F_{\lambda,n}$ with definition

$$\begin{aligned}F_{\lambda,n}(z) = z\exp\left(\int_0^z \frac{\mathfrak{q}_\lambda(t^n)-1}{t}\,\mathrm{d}t\right) &= z + \frac{\lambda}{n}z^{n+1} + \frac{\lambda^2(n+2)-n\lambda}{4n^2}z^{2n+1}\\ &+ \frac{\lambda\left(\left(2n^2+9n+6\right)\lambda^2 - \left(6n^2+9n\right)\lambda + 4n^2\right)}{36n^3}z^{3n+1} + \cdots \quad (z \in \Delta)\,,\end{aligned} \tag{2.9c}$$

is extremal function for several problems in the class $\mathcal{S}^*_L(\lambda)$. Especially for $n=1$ we have

$$F_\lambda(z) := F_{\lambda,1}(z) = z\exp\left(\int_0^z \frac{\mathfrak{q}_\lambda(t)-1}{t}\,\mathrm{d}t\right)$$
$$= z + \lambda z^2 + \left(\frac{3\lambda^2-\lambda}{4}\right)z^3 + \left(\frac{17\lambda^3-15\lambda^2+4\lambda}{36}\right)z^4+\cdots. \quad (2.9\text{d})$$

Also, by taking $p(z)=\mathfrak{q}_\lambda(z^n)$ with $n=1,2,3,\ldots$, the function $K_{\lambda,n}$ with definition

$$K_{\lambda,n}(z) = \int_0^z \exp\left(\int_0^w \frac{\mathfrak{q}_\lambda(t^n)-1}{t}\,\mathrm{d}t\right)\mathrm{d}w = z + \frac{\lambda}{n(n+1)}z^{n+1} + \frac{\lambda^2(n+2)-n\lambda}{4n^2(2n+1)}z^{2n+1}$$
$$+\frac{\lambda\left(\left(2n^2+9n+6\right)\lambda^2-\left(6n^2+9n\right)\lambda+4n^2\right)}{36n^3(3n+1)}z^{3n+1}+\cdots\ (z\in\Delta), \quad (2.10\text{a})$$

is extremal function for several problems in the class $\mathcal{CV}_L(\lambda)$. Especially for $n=1$ we have

$$K_\lambda(z) := K_{\lambda,1}(z) = \int_0^z \exp\left(\int_0^w \frac{\mathfrak{q}_\lambda(t)-1}{t}\,\mathrm{d}t\right)\mathrm{d}w$$
$$= z + \frac{\lambda}{2}z^2 + \left(\frac{3\lambda^2-\lambda}{12}\right)z^3 + \left(\frac{17\lambda^3-15\lambda^2+4\lambda}{144}\right)z^4+\cdots. \quad (2.10\text{b})$$

Theorem 2.6. Let p be an analytic function in the unit disk Δ, such that $p(0)=1$. If

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} < \frac{\lambda}{2} \qquad (0<\lambda\le 1,\ z\in\Delta), \quad (2.11)$$

then

$$p\in\mathcal{P}(\mathfrak{q}_\lambda).$$

Proof . From (2.11) it follows that $p(z)\neq 0$ for all $z\in\Delta$. Otherwise, suppose that p has a point zero of order m, $m\ge 1$ at the point ζ that satisfies $|\zeta|<1$. Then we have $p(z)=(z-\zeta)^m\,q(z)$, $q(z)\neq 0$ on Δ and

$$\frac{zp'(z)}{p(z)} = \frac{mz}{z-\zeta} + \frac{zq'(z)}{q(z)}.$$

A simple calculation shows that for $z\in\Delta$

$$\lim_{z\to\zeta}(z-\zeta)\frac{zp'(z)}{p(z)} = \begin{cases} \lim_{z\to\zeta} mz + \lim_{z\to\zeta}(z-\zeta)\frac{zq'(z)}{q(z)} & \text{for}\quad \zeta\neq 0\\ m+\lim_{z\to\zeta}\frac{z^2q'(z)}{q(z)} & \text{for}\quad \zeta=0\end{cases}$$
$$= \begin{cases} m\zeta & \text{for}\quad \zeta\neq 0,\\ m & \text{for}\quad \zeta=0.\end{cases}$$

Then zp'/p has a simple pole at ζ, which contradicts (2.11). We conclude that $p(z)\neq 0$, as required. Let $p(z)\not\prec \mathfrak{q}_\lambda(z)$ on Δ. Then by Lemma 1.3 there exist $z_0\in\Delta$ and $\zeta_0\in\partial\Delta$ with $\zeta_0\neq -1$ such that

$$p(z_0)=\mathfrak{q}_\lambda(\zeta_0), \qquad z_0p'(z_0)=m\zeta_0\mathfrak{q}'_\lambda(\zeta_0) \qquad m\ge 1.$$

Thus

$$\Re\left\{\frac{z_0p'(z_0)}{p(z_0)}\right\} = \Re\left\{\frac{m\zeta_0\mathfrak{q}'_\lambda(\zeta_0)}{\mathfrak{q}_\lambda(\zeta_0)}\right\} = m\lambda\Re\left\{\frac{\zeta_0}{1+\zeta_0}\right\} = \frac{m\lambda}{2}\ge\frac{\lambda}{2}.$$

But this contradicts our assumption (2.11) and therefore $p\prec\mathfrak{q}_\lambda$ on Δ. □

Taking into account $p(z)=f'(z)$ in Theorem 2.6, the norm of pre-Schwarzian derivatives and univalency of functions on class $\mathcal{G}(\lambda)$ are investigated.

Lemma 2.7. If a function f belongs to the class $\mathcal{G}(\lambda)$, then $f' \in \mathcal{P}(\mathfrak{q}_\lambda)$. Also, f is univalent function in Δ and

$$z\exp\left(\int_0^z \frac{f'(t)-1}{t}\,\mathrm{d}t\right) \in \mathcal{S}_L^*(\lambda) \qquad \text{and} \qquad \int_0^z \exp\left(\int_0^w \frac{f'(t)-1}{t}\,\mathrm{d}t\right)\mathrm{d}w \in \mathcal{CV}_L(\lambda).$$

Lemma 2.8. Let f be a function in $\mathcal{G}(\lambda)$. Then $\|T_f\| \leq 2\lambda$. Moreover, equality holds for f given by $f(z) = \overline{\mu}\Phi(\mu z)$, where μ is an unimodular constant and

$$\Phi(z) = \frac{(1+z)^{1+\lambda}-1}{1+\lambda} \qquad (z \in \Delta)\,. \tag{2.12}$$

Proof . Suppose that $f \in \mathcal{G}(\lambda)$. Making use of Lemma 2.7 there exits $\omega \in \mathcal{B}$ such that $f'(z) = (1+\omega(z))^\lambda$ and

$$\left|\frac{f''(z)}{f'(z)}\right| = \frac{\lambda\,|\omega'(z)|}{|1+\omega(z)|} \qquad (z \in \Delta)\,.$$

By the Schwarz-Pick Lemma,

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2} \qquad (z \in \Delta)\,, \tag{2.13}$$

we conclude

$$\left|\frac{f''(z)}{f'(z)}\right| = \frac{\lambda\,|\omega'(z)|}{|1+\omega(z)|} \leq \frac{\lambda(1-|\omega(z)|^2)}{(1-|z|^2)\,(1-|\omega(z)|)} \leq \frac{\lambda(1+|z|)}{1-|z|^2}$$

and

$$\|T_f\| \leq \sup_{z\in\Delta} \lambda(1+|z|) \leq 2\lambda.$$

We have equality in the Schwarz-Pick lemma inequality (2.13), if and only if $\omega(z) = \mu z$ with $|\mu| = 1$ and μ is complex number. Thus for function

$$f'(z) = (1+\omega(z))^\lambda = (1+\mu z)^\lambda \qquad \text{or} \qquad f(z) = \overline{\mu}\Phi(\mu z),$$

where Φ given by (2.12), it follows that $\|T_f\| = 2\lambda$. □

For $p(z) = f(z)/z$ or $p(z) = z/f(z)$ in Theorem 2.6 and taking into account relation 2.9a, we get the following results.

Corollary 2.9. 1. Let $f \in \mathcal{A}$. If $f \in \mathcal{N}(\lambda)$, then

$$\frac{f(z)}{z} \prec \mathfrak{q}_\lambda(z) \quad \text{and} \quad z\exp\left(\int_0^z \frac{f(t)-t}{t^2}\,\mathrm{d}t\right) \in \mathcal{S}_L^*(\lambda).$$

2. Let $f \in \mathcal{A}$. If $f \in f \in \mathcal{ST}(1-\lambda/2)$, then

$$\frac{z}{f(z)} \prec \mathfrak{q}_\lambda(z) \quad \text{and} \quad z\exp\left(\int_0^z \frac{t-f(t)}{tf(t)}\,\mathrm{d}t\right) \in \mathcal{S}_L^*(\lambda).$$

Taking into account $p(z) = zf'(z)/f(z)$ in Theorem 2.6, we have the following corollary is a starlikeness condition for analytic functions of the unit disk.

Corollary 2.10. If a function $f \in \mathcal{H}$ satisfy the condition

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} < \frac{\lambda}{2} \qquad (z \in \Delta)\,,$$

then $f \in \mathcal{S}_L^*(\lambda) \subset \mathcal{ST}$.

Example 2.11. The Corollary 2.9 provides many examples of functions in class $\mathcal{S}_L^*(\lambda)$. Let

$$f_1(z) = z + A_1 z^n, (n \geq 2)\,, \qquad f_2(z) = \frac{z}{1-A_2 z}, \qquad f_3(z) = \frac{z}{(1-A_3 z)^2}.$$

For

$$0 < |A_1| \leq \frac{\lambda}{2n-2-\lambda}, \qquad 0 < |A_2| \leq \frac{\lambda}{2+\lambda}, \qquad 0 < |A_3| \leq \frac{\lambda}{4+\lambda},$$

the functions f_i, $i = 1, 2, 3$ belong in class $\mathcal{N}(\lambda)$. Then the appropriate functions

$$g_1(z) = z \exp\left(\frac{A_1 z^{n-1}}{n-1}\right), \quad g_2(z) = \frac{z}{1-A_2 z}, \quad g_3(z) = \frac{z}{1-A_3 z} \exp\left(\frac{A_3 z}{1-A_3 z}\right),$$

belong to the class $\mathcal{S}^*_L(\lambda)$.

Example 2.12. For $0 < |A| \leq \lambda/(2-\lambda)$

$$f(z) = \frac{1}{A}\left(\mathrm{e}^{Az} - 1\right) \in \mathcal{CV}_L(\lambda)$$

and for $0 < |A| \leq \lambda/(2+\lambda)$

$$f(z) = -\frac{1}{A} \ln(1 - Az) \in \mathcal{CV}_L(\lambda).$$

From the results in [8], function (2.9d), and Lemma 2.1, we have the following sharp estimates for function $f \in \mathcal{S}^*_L(\lambda)$ ($f \in \mathcal{CV}_L(\lambda)$ resp.).

Theorem 2.13. If $f \in \mathcal{S}^*_L(\lambda)$ and $|z| = r < 1$, then

1. Growth Theorem: $-F_\lambda(-r) \leq |f(z)| \leq F_\lambda(r)$,
2. Distortion Theorem: $F'_\lambda(-r) \leq |f'(z)| \leq F'_\lambda(r)$,
3. Rotation Theorem: $|\operatorname{Arg}\{f(z)/z\}| \leq \max_{|z|=r} \operatorname{Arg}\{F_\lambda(z)/z\}$. Equality holds for some $z \neq 0$ if and only if f is a rotation of F_λ given by (2.9d).
4. Covering Theorem: If $f \in \mathcal{S}^*_L(\lambda)$, then either f is a rotation of F_λ or

$$\{w \in \mathbb{C}: \quad |w| \leq -F_\lambda(-1)\} \subset f(\Delta).$$

Here $-F_\lambda(-1) = \lim_{r \to 1^-} -F_\lambda(-r)$.

Theorem 2.14. If $f \in \mathcal{CV}_L(\lambda)$ and $|z| = r < 1$, then

1. Growth Theorem: $-K_\lambda(-r) \leq |f(z)| \leq K_\lambda(r)$,
2. Distortion Theorem: $K'_\lambda(-r) \leq |f'(z)| \leq K'_\lambda(r)$,
3. Rotation Theorem: $|\operatorname{Arg}\{f'(z)\}| \leq \max_{|z|=r} \operatorname{Arg}\{K'_\lambda(z)\}$. Equality holds for some $z \neq 0$ if and only if f is a rotation of K_λ given by (2.10b).
4. Covering Theorem: If $f \in \mathcal{S}^*_L(\lambda)$, then either f is a rotation of F_λ or

$$\{w \in \mathbb{C}: \quad |w| \leq -K_\lambda(-1)\} \subset f(\Delta).$$

Here $-K_\lambda(-1) = \lim_{r \to 1^-} -K_\lambda(-r)$.

For the special case $\lambda = 1/2$, results for functions belonging to the class

$$\begin{aligned} \mathcal{S}^*_L := \mathcal{S}^*_L(1/2) &= \left\{ f \in \mathcal{SS}^*\left(\frac{1}{2}\right): \quad \left| \left[\frac{zf'(z)}{f(z)}\right]^2 - 1 \right| < 1,\, z \in \Delta \right\}, \\ &= \left\{ f \in \mathcal{ST}: \quad \Re\left\{ \left[\frac{zf'(z)}{f(z)}\right]^{-2} \right\} > \frac{1}{2},\, z \in \Delta \right\} \end{aligned}$$

and its generalizations can be found in [1, 2, 3, 13, 15, 16, 17, 18, 19]. The function $f \in \mathcal{S}^*_L$ if and only if quantity $zf'(z)/f(z)$ lies in the region bounded by the right loop of the *lemniscate of Bernoulli*

$$\begin{aligned} \mathbb{LB}\left(\frac{1}{2}\right) &= \left\{ \rho \mathrm{e}^{\mathrm{i}\varphi} : \rho = (2\cos 2\varphi)^{1/2}, \quad -\frac{\pi}{4} < \varphi \leq \frac{\pi}{4} \right\} \\ &= \left\{ w \in \mathbb{C}: \quad \Re\{w\} > 0, \qquad \Re\left\{\frac{1}{w^2}\right\} = \frac{1}{2} \right\} \cup \{0\}. \end{aligned}$$

Below, we get the sharp radius of convexity of the class $\mathcal{S}^*_L(\lambda)$.

Theorem 2.15. Let r_0 denote the positive root of the equation

$$(1-r)^{1+\lambda} = \lambda r \qquad r \in [0,1).$$

If $f \in \mathcal{S}^*_L(\lambda)$, then f is convex in the disk $|z| < r_0$. This result is sharp.

Proof . Let $f \in \mathcal{S}^*_L(\lambda)$. Then from Definition 2.4 we obtain

$$\frac{zf'(z)}{f(z)} = [1+w(z)]^{\lambda} \qquad (z \in \Delta)\,, \tag{2.14}$$

where $\omega \in \mathcal{B}$ with $|\omega(z)| \leq |z|$, $z \in \Delta$. Logarithmic differentiation of (2.14) yields that

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \Re\left\{[1+\omega(z)]^{\lambda} - \frac{\lambda z \omega'(z)}{[1+\omega(z)]}\right\}.$$

From Lemma 2.1 and inequality (2.13), it follows that

$$\begin{aligned}\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &\geq \Re\left\{[1+\omega(z)]^{\lambda}\right\} - \lambda|z|\,\frac{1-|\omega(z)|^2}{[1-|\omega(z)|\,]\,[1-|z|^2]} \\ &\geq (1-|z|)^{\lambda} - \frac{\lambda|z|}{(1-|z|)}.\end{aligned}$$

The function $g(x) = (1-r)^{\lambda} - \frac{\lambda r}{(1-r)}$ with $|z| = r \in [0,1)$ is decreasing in $[0,1)$ and $g(0) = 1$. The equation $g(r) = 0$ is equivalent to

$$(1-r)^{1+\lambda} = \lambda r \qquad r \in [0,1). \tag{2.15}$$

The only real positive root of (2.15) is equal to r_0. For a function F_λ given by (2.9d), we have

$$\Re\left\{1 + \frac{zF''_\lambda(z)}{F'_\lambda(z)}\right\} = \Re\left\{(1+z)^{\lambda} + \frac{\lambda z}{1+z}\right\} =: G(z)$$

and $G(-r_0) = 0$, this shows the sharpness of r_0. $\square$

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