Int. J. Nonlinear Anal. Appl. 14 (2023) 5, 121–131
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.27875.3742

On the properties of r-circulant matrices involving Mersenne and Fermat numbers

Munesh Kumari[a], Kalika Prasad[a,*], Jagmohan Tanti[b], Engin Özkan[c]

[a] Department of Mathematics, Central University of Jharkhand, Ranchi, India

[b] Department of Mathematics, Babasaheb Bhimrao Ambedkar University, India

[c] Department of Mathematics, Erzincan Binali Yıldırım University, Turkey

(Communicated by Choonkil Park)

Abstract

The aim of this study is to investigate r-circulant matrices containing Mersenne and Fermat numbers with arithmetic indices. We obtain the eigenvalues and determinants of these matrices implicitly. In addition, limits for matrix norms and spectral norms of these matrices are obtained. Thus, the results for right and skew-right circulant matrices appear immediately.

Keywords: Mersenne numbers, Fermat numbers, Circulant matrices, Eigenvalues, Determinants, Spectral norm, Euclidean norm
2020 MSC: Primary 11B37, 15B36; Secondary 65Q30

1 Introduction

Circulant matrices are one of the special types of matrices that can be identified by their first row. In recent years, special matrices had wide applications in coding theory, cryptography, signal processing, linear forecast, theory of statistical designs, engineering simulations, etc. (for instance see [5, 7, 8, 9]). Due to its special structure and wide application in various fields, it is becoming a more interesting subject among researchers.

Recently, many authors have investigated the algebraic properties of circulant and r-circulant matrices involving a special integer sequence such as Fibonacci, Lucas, Pell integer sequences etc. In their studies, they obtained the formula for the eigenvalues, the determinants, norms and bounds for the spectral norm of these matrices. For instance, Solak[15] obtained the norms for circulant matrices containing the Fibonacci and Lucas numbers. Zheng et al.[19] found the exact inverse of circulant matrices with Fermat and Mersenne numbers. For r-circulant matrices S.-Q., Shen et al.[14] obtained the bounds for the norms with Fibonacci and Lucas numbers and in [13], they obtained the spectral norms with k-Fibonacci and k-Lucas numbers. Some recent works in this direction can be seen in [11, 17, 18].

Our aim is to investigate the r-circulant matrices containing two special number sequences, Mersenne and Fermat sequences with arithmetic indices. Mersenne and Fermat sequences are Fibonacci-like sequences and can be obtained

*Corresponding author
Email addresses: muneshnasir94@gmail.com (Munesh Kumari), klkaprsd@gmail.com (Kalika Prasad), jagmohan.t@gmail.com (Jagmohan Tanti), eozkan@erzincan.edu.tr (Engin Özkan)

Received: July 2022 *Accepted:* March 2023

directly with the formulas $2^n - 1$ and $2^n + 1$, respectively. In [2, 3, 6, 10, 12, 16] some studies on recent developments in Mersenne and Mersenne-like sequences and their applications can be seen.

Throughout the paper, M_n (R_n respectively) denote the famous Mersenne (Fermat) numbers, and for $n \geq 0$ defined by the recursion $M_n = 3M_{n-1} - 2M_{n-2}$ ($R_n = 3R_{n-1} - 2R_{n-2},$) with initial values $M_0 = 0,\ M_1 = 1$ ($R_0 = 2,\ \ R_1 = 3$). The first few terms of these sequences are

n	**0**	**1**	2	3	4	5	6	7	8	...
M_n	**0**	**1**	3	7	15	31	63	127	255	...
R_n	**2**	**3**	5	9	17	33	65	129	257	...

The closed form formulas known as the Binet formula for Mersenne and Fermat numbers are given by, respectively,

$$M_n = 2^n - 1 \quad \text{and} \quad R_n = 2^n + 1. \tag{1.1}$$

The characteristic equation corresponding to the above recursion is given by $\alpha^2 - 3\alpha + 2 = 0$ and it has two roots, say, α_1 and α_2, which have the following properties:

$$\alpha_1 + \alpha_2 = 3, \quad \alpha_1 \alpha_2 = 2. \tag{1.2}$$

Different norms for a square matrix are given in the following lemma.

Lemma 1.1. Let $H = [h_{ij}]_{n \times n}$ be any square matrix, then we have

$$\begin{aligned} ||H||_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |h_{ij}|, \qquad ||H||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |h_{ij}|, \\ ||H||_F &= \sqrt{\sum_{1=1}^{n} \sum_{j=1}^{n} |h_{ij}|^2} \qquad \textit{and} \qquad ||H||_2 = \sqrt{\max_{1 \leq i \leq n} \mu_i(H^* H)}, \end{aligned}$$

where $\mu_i(H^*H)$ denote the eigenvalues of H^*H and H^* is the conjugate transpose of H. And for matrix H, these norms are related as

$$\frac{1}{\sqrt{n}} ||H||_F \leq ||H||_2 \leq ||H||_F. \tag{1.3}$$

Lemma 1.2. Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$, $B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ and if C is the Hadamard product of A and B, then we have

$$||C||_2 \leq u(A)\nu(B), \tag{1.4}$$

where $u(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}$ and $\nu(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}$.

Definition 1.3. [4] For $r \in \mathbb{C} - \{0\}$, a matrix C_r is said to be r-circulant matrix if it is of the form

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}_{n \times n}$$

and it is denoted by $C_r = Circ(r; \vec{c})$, where $\vec{c} = (c_0, c_1, ..., c_{n-1})$ is the first row vector. For $r = 1$ and $r = -1$, we get the right circulant and skew-right circulant matrices, respectively.

Some results used in our work are shown in the following lemmas.

Lemma 1.4. [4] Let C_r be r-circulant matrices then its eigenvalues μ_i are given by

$$\mu_i = \sum_{j=0}^{n-1} c_j(\rho\omega^{-i})^j, \quad i = 0, 1, 2, ..., n-1,$$

where ω is the nth root of unity and ρ is the nth root of r.

Lemma 1.5. The Euclidean norm of r-circulant matrix C_r is given by

$$||C_r||_E = \sqrt{\sum_{j=0}^{n-1} |C_j|^2[n - j(1 - |r|^2)]}. \tag{1.5}$$

Lemma 1.6. [1] For any a and b, we have

$$\prod_{i=0}^{n-1} (a - b\rho_i\omega_{-i}) = a^n - rb^n, \tag{1.6}$$

where ρ_i are the n^{th} roots of r.

Now, we obtain the eigenvalues, the determinant, Euclidean norms and bounds for the spectral norm of r-circulant matrices containing Mersenne and Fermat numbers with arithmetic indices. As consequences, we obtain many new identities for Mersenne and Fermat numbers.

2 Main results

Let s and t be non negative integers and $r \in \mathbb{C} - \{0\}$. The r-circulant matrices with Mersenne and Fermat numbers are denoted by M_r and R_r, respectively, and defined as follows.

Definition 2.1 (Mersenne r-circulant matrix). The Mersenne r-circulant matrix is defined as $M_r = Circ(r; \vec{c})$ where first row vector is $\vec{c} = (M_s, M_{s+t}, M_{s+2t}..., M_{s+(n-1)t})$, i.e., matrix of the form

$$M_r = \begin{bmatrix} M_s & M_{s+t} & M_{s+2t} & \cdots & M_{s+(n-2)t} & M_{s+(n-1)t} \\ rM_{s+(n-1)t} & M_s & M_{s+t} & \cdots & M_{s+(n-3)t} & M_{s+(n-2)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_{s+2t} & rM_{s+3t} & rM_{s+4t} & \cdots & M_s & M_{s+t} \\ rM_{s+t} & rM_{s+2t} & rM_{s+3t} & \cdots & rM_{s+(n-1)t} & M_s \end{bmatrix}. \tag{2.1}$$

Definition 2.2 (Fermat r-circulant matrix). The Fermat r-circulant matrix is defined as $R_r = Circ(r; \vec{c})$ where first row vector is $\vec{c} = (R_s, R_{s+t}, R_{s+2t}..., R_{s+(n-1)t})$, i.e., matrix of the form

$$R_r = \begin{bmatrix} R_s & R_{s+t} & R_{s+2t} & \cdots & R_{s+(n-2)t} & R_{s+(n-1)t} \\ rR_{s+(n-1)t} & R_s & R_{s+t} & \cdots & R_{s+(n-3)t} & R_{s+(n-2)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{s+2t} & rR_{s+3t} & rR_{s+4t} & \cdots & R_s & R_{s+t} \\ rR_{s+t} & rR_{s+2t} & rR_{s+3t} & \cdots & rR_{s+(n-1)t} & R_s \end{bmatrix}. \tag{2.2}$$

In the following theorems, we give the formula for the eigenvalues of the matrices M_r and R_r and as special cases for $r = 1, -1$, the eigenvalues for right circulant and skew-right circulant matrices are obtained.

Theorem 2.3. The eigenvalues of Mersenne r-circulant matrices M_r are

$$\mu_i(M_r) = \begin{cases} \dfrac{M_s - rM_{s+nt} - \rho\omega^{-i}[2^t M_{s-t} - rM_{s+(n-1)t}]}{(1 - \alpha_1^t\rho\omega^{-i})(1 - \alpha_2^t\rho\omega^{-i})} & : s > t, \\ \dfrac{M_s - rM_{(n+1)s} + \rho\omega^{-i} rM_{ns}}{(1 - \alpha_1^s\rho\omega^{-i})(1 - \alpha_2^s\rho\omega^{-i})} & : s = t, \\ \dfrac{M_s - rM_{s+nt} - \rho\omega^{-i}[2^s M_{t-s} - rM_{s+(n-1)t}]}{(1 - \alpha_1^t\rho\omega^{-i})(1 - \alpha_2^t\rho\omega^{-i})} & : s < t, \end{cases} \tag{2.3}$$

where $i = 0, 1, 2, ..., n-1$.

Proof . We have

$$
\begin{aligned}
\mu_i(M_r) &= \sum_{j=0}^{n-1} M_{s+jt}(\rho\omega^{-i})^j, \quad i = 0, 1, 2, ..., n-1 \\
&= \sum_{j=0}^{n-1} (\alpha_1^{s+jt} - \alpha_2^{s+jt})(\rho\omega^{-i})^j \\
&= \alpha_1^s \sum_{j=0}^{n-1} (\alpha_1^t \rho\omega^{-i})^j - \alpha_2^s \sum_{j=0}^{n-1} (\alpha_2^t \rho\omega^{-i})^j \\
&= \alpha_1^s \left[\frac{1-(r\alpha_1^{nt})}{1-\alpha_1^t\rho\omega^{-i}} \right] - \alpha_2^s \left[\frac{1-r(\alpha_2^{nt}}{1-\alpha_2^t\rho\omega^{-i}} \right] \\
&= \frac{(\alpha_1^s - \alpha_2^s) - r(\alpha_1^{s+nt} - \alpha_2^{s+nt}) - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r(\alpha_1^{s+nt}\alpha_2^t - \alpha_2^{s+nt}\alpha_1^t)]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} \\
&= \frac{M_s - rM_{s+nt} - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})}, \quad \text{(using Eq. (1.2))}.
\end{aligned}
\tag{2.4}
$$

And by using Eq. (1.2), we have

$$
\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t = \begin{cases} 2^t M_{s-t} & : s > t, \\ & : s = t, \\ 2^s M_{t-s} & : s < t. \end{cases}
$$

This completes the proof. □

Corollary 2.4. For $r = 1$ and $r = -1$, we get the eigenvalues for the Mersenne right circulant and Mersenne skew-right circulant matrices, respectively, given as

$$
\begin{aligned}
\mu_i(M_1) &= \frac{M_s - M_{s+nt} - \omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - 2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})}, \\
\mu_i(M_{-1}) &= \frac{M_s + M_{s+nt} - \zeta\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) + 2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\zeta\omega^{-i})(1-\alpha_2^t\zeta\omega^{-i})},
\end{aligned}
$$

where ζ is the nth root of -1 and

$$
\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t = \begin{cases} 2^t M_{s-t} & : s > t, \\ 0 & : s = t, \\ 2^s M_{t-s} & : s < t. \end{cases}
$$

Theorem 2.5. The eigenvalues of Fermat r-circulant matrices are

$$
\mu_i(R_r) = \begin{cases} \dfrac{R_s - rR_{s+nt} + \rho\omega^{-i}[r2^t R_{s+(n-1)t} - 2^t R_{s-t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} & : s > t, \\ \dfrac{R_s - rR_{(n+1)s} + 2^s\rho\omega^{-i}[rR_{ns} - 2]}{(1-\alpha_1^s\rho\omega^{-i})(1-\alpha_2^s\rho\omega^{-i})} & : s = t, \\ \dfrac{R_s - rR_{s+nt} + \rho\omega^{-i}[r2^t R_{s+(n-1)t} - 2^s R_{t-s}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} & : s < t. \end{cases}
$$

Proof . The argument is the same as Theorem 2.3. □

Corollary 2.6. For $r = 1$ and $r = -1$ in Theorem 2.5, we have

$$\mu_i(R_1) = \begin{cases} \dfrac{R_s - R_{s+nt} + \rho\omega^{-i}[2^t R_{s+(n-1)t} - 2^t R_{s-t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} & : s > t, \\ \dfrac{R_s - R_{(n+1)s} + 2^s\rho\omega^{-i}[R_{ns} - 2]}{(1-\alpha_1^s\rho\omega^{-i})(1-\alpha_2^s\rho\omega^{-i})} & : s = t, \\ \dfrac{R_s - R_{s+nt} + \rho\omega^{-i}[2^t R_{s+(n-1)t} - 2^s R_{t-s}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} & : s < t. \end{cases}$$

$$\mu_i(R_{-1}) = \begin{cases} \dfrac{R_s + R_{s+nt} - \zeta\omega^{-i}[2^t R_{s+(n-1)t} + 2^t R_{s-t}]}{(1-\alpha_1^t\zeta\omega^{-i})(1-\alpha_2^t\zeta\omega^{-i})} & : s > t, \\ \dfrac{R_s + R_{(n+1)s} - 2^s\zeta\omega^{-i}[R_{ns} + 2]}{(1-\alpha_1^s\zeta\omega^{-i})(1-\alpha_2^s\zeta\omega^{-i})} & : s = t, \\ \dfrac{R_s + R_{s+nt} - \zeta\omega^{-i}[2^t R_{s+(n-1)t} + 2^s R_{t-s}]}{(1-\alpha_1^t\zeta\omega^{-i})(1-\alpha_2^t\zeta\omega^{-i})} & : s < t. \end{cases}$$

As a consequence of the above results, we have the following sum identity for Mersenne and Fermat numbers.

Theorem 2.7. For a positive integer n, we have

$$\sum_{i=0}^{n-1} \frac{M_s - rM_{s+nt} - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} = n(2^s - 1).$$

Proof . To obtain the result, we use the fact that the trace of a square matrix is the sum of its eigenvalues. Hence for Mersenne r-circulant matrices M_r, we have

$$\sum_{i=0}^{n-1} \mu_i(M_r) = \sum_{i=0}^{n-1} \frac{M_s - rM_{s+nt} - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})}.$$

Since r-circulant matrices M_r are diagonal constant and the diagonal entries are M_s, the sum of diagonal entries of M_r is nM_s. Therefore,

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{M_s - rM_{s+nt} - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^t M_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} &= nM_s \\ &= n(2^s - 1). \end{aligned}$$

This completes the proof. □

If $s = 0$, then we have the following identity

$$\sum_{i=0}^{n-1} \frac{\rho\omega^{-i}[M_t + r2^t M_{(n-1)t}] - rM_{nt}}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} = 0.$$

Note that the sum identity proposed in the above theorem is independent of r.

Theorem 2.8. For $n \in \mathbb{N}$ and Fermat numbers R_s, we have

$$\sum_{i=0}^{n-1} \frac{R_s - rR_{s+nt} + \rho\omega^{-i}[r2^t R_{s+(n-1)t} - (\alpha_1^s\alpha_2^t + \alpha_2^s\alpha_1^t)]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} = n(2^s + 1).$$

and

$$\sum_{i=0}^{n-1} \frac{2 - rR_{nt} + \rho\omega^{-i}[r2^t R_{(n-1)t} - M_t]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} = 2n.$$

Proof . The argument is the same as in the proof of Theorem 2.7. □

In the next results, we give the determinants of r-circulant matrices involving Mersenne and Fermat numbers and as consequence, we get the determinant for the right and skew-right circulant matrices.

Theorem 2.9. Determinant of M_r is given by

$$\det(M_r) = \frac{(M_s - rM_{s+nt})^n - r[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^tM_{s+(n-1)t}]^n}{1 - rR_{nt} + 2^{nt}r^2}.$$

Proof . With eigenvalues μ_i of matrix M_r, the determinant is given by $\det(M_r) = \prod_{i=0}^{n-1}\mu_i$. Hence,

$$\begin{aligned}
\det(M_r) &= \prod_{i=0}^{n-1} \frac{(M_s - rM_{s+nt}) - \rho\omega^{-i}[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^tM_{s+(n-1)t}]}{(1-\alpha_1^t\rho\omega^{-i})(1-\alpha_2^t\rho\omega^{-i})} \\
&= \frac{(M_s - rM_{s+nt})^n - r[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^tM_{s+(n-1)t}]^n}{(1-r\alpha_1^{nt})(1-\alpha_2^{nt})} \qquad \text{(using Eq. (1.6))} \\
&= \frac{(M_s - rM_{s+nt})^n - r[(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - r2^tM_{s+(n-1)t}]^n}{1 - rR_{nt} + 2^{nt}r^2}.
\end{aligned}$$

This completes the proof. □

Corollary 2.10. The determinants of the Mersenne right circulant and skew-right circulant matrices are given as

$$\det(M_1) = \frac{(M_s - M_{s+nt})^n - [(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) - 2^tM_{s+(n-1)t}]^n}{1 - R_{nt} + 2^{nt}},$$
$$\det(M_{-1}) = \frac{(M_s + M_{s+nt})^n + [(\alpha_1^s\alpha_2^t - \alpha_2^s\alpha_1^t) + 2^tM_{s+(n-1)t}]^n}{1 + R_{nt} + 2^{nt}}.$$

Theorem 2.11. The determinant of the Fermat r-circulant matrix is given as

$$\det(R_r) = \frac{(R_s - rR_{s+nt})^n - r[(\alpha_1^s\alpha_2^t + \alpha_2^s\alpha_1^t) - r2^tR_{s+(n-1)t}]^n}{1 - rR_{nt} + 2^{nt}r^2}.$$

Proof . The argument is the same as Theorem 2.9. □

Corollary 2.12. The determinants of Fermat right and skew-right circulant matrices are given as

$$\begin{aligned}
\det(R_1) &= \frac{(R_s - R_{s+nt})^n - [(\alpha_1^s\alpha_2^t + \alpha_2^s\alpha_1^t) - 2^tR_{s+(n-1)t}]^n}{1 - R_{nt} + 2^{nt}}, \\
\det(R_{-1}) &= \frac{(R_s + R_{s+nt})^n + [(\alpha_1^s\alpha_2^t + \alpha_2^s\alpha_1^t) + 2^tR_{s+(n-1)t}]^n}{1 + R_{nt} + 2^{nt}}.
\end{aligned}$$

The sum identities

On setting $i = 0$, $r = 1$ and $\rho = 1$ in equations (2.3) and (2.5), the following sum identities are verified for the Mersenne and Fermat numbers.

$$\sum_{j=0}^{n-1} M_{s+jt} = \begin{cases} \dfrac{M_s - M_{s+nt} - 2^tM_{s-t} + M_{s+(n-1)t}}{1 - R_t + 2^t} & : s > t, \\ \dfrac{M_s - M_{(n+1)s} + M_{ns}}{1 - R_s + 2^s}, & : s = t, \\ \dfrac{M_s - M_{s+nt} - 2^sM_{t-s} + M_{s+(n-1)t}}{1 - R_t + 2^t,} & : s < t, \end{cases}$$

and

$$\sum_{j=0}^{n-1} R_{s+jt} = \begin{cases} \dfrac{R_s - R_{s+nt} - 2^tR_{s+(n-1)t} + 2^tR_{s-t}}{1 - R_t + 2^t} & : s > t, \\ \dfrac{R_s - R_{(n+1)s} + 2^s(R_{ns} + 2)}{1 - R_s + 2^s} & : s = t, \\ \dfrac{R_s - R_{s+nt} - 2^tR_{s+(n-1)t} + 2^sR_{t-s}}{1 - R_t + 2^t} & : s < t. \end{cases}$$

3 Norm of Mersenne and Fermat r-circulant matrices

Consider the following matrices (special case when $s = 0$ and $t = 1$),

$$M'_r = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-2} & M_{n-1} \\ rM_{n-1} & M_0 & M_1 & \cdots & M_{n-3} & M_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_2 & rM_3 & rM_4 & \cdots & M_0 & M_1 \\ rM_1 & rM_2 & rM_3 & \cdots & rM_{n-1} & M_0 \end{bmatrix} \tag{3.1}$$

and

$$R'_r = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-2} & R_{n-1} \\ rR_{n-1} & R_0 & R_1 & \cdots & R_{n-3} & R_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_2 & rR_3 & rR_4 & \cdots & R_0 & R_1 \\ rR_1 & rR_2 & rR_3 & \cdots & rR_{n-1} & R_0 \end{bmatrix}. \tag{3.2}$$

To obtain the different matrix norms, we need to prove some results on the sum of squares of the Mersenne and Fermat numbers which we show in the following lemmas.

Lemma 3.1. The finite sum of squares of the Mersenne numbers is given by

$$\sum_{j=0}^{n-1} M_j^2 = \frac{M_{2n} - 6M_n + 3n}{3}. \tag{3.3}$$

Proof . By using Eq. (1.1), we get

$$\begin{aligned} \sum_{j=0}^{n-1} M_j^2 &= \sum_{j=0}^{n-1} (2^j - 1)^2 \\ &= \sum_{j=0}^{n-1} (2^{2j} - 2^{j+1} + 1) \\ &= \frac{4^n - 1}{3} - 2(2^n - 1) + n \\ &= \frac{M_{2n} - 6M_n + 3n}{3}. \end{aligned}$$

This completes the proof. □

By a similar argument, the next lemma can be proved.

Lemma 3.2. The finite sum of squares of the Fermat numbers is given by

$$\sum_{j=0}^{n-1} R_j^2 = \frac{R_{2n} + 6R_n + 3n - 14}{3}.$$

Theorem 3.3. The maximum absolute column sum and maximum absolute row sum matrix norm for the matrix M'_r are given as

$$||M'_r||_1 = ||M'_r||_\infty = |r|(M_n - n).$$

Proof . Since, for r-circulant matrices we have $||.||_1 = ||.||_\infty$, from Lemma 1.1, we have

$$\begin{aligned} ||M'_r||_1 = ||M'_r||_\infty &= \max_{1 \le j \le n} \sum_{i=1}^{n} |m_{ij}| \\ &= M_0 + |r| \sum_{k=1}^{n-1} |M_k| \\ &= |r|(M_n - n) \qquad (\text{Since, } M_0 = 0). \end{aligned}$$

This completes the proof. □

Theorem 3.4. The Euclidean norm for the Mersenne r-circulant matrices is given by

$$||M'_r||_E = \sqrt{n\sum_{j=0}^{n-1}(R_{2j}-4)[n-j(1-|r|^2)]}.$$

Proof . By Eq. (1.5), we have

$$\begin{aligned}||M'_r||_E^2 &= \sum_{j=0}^{n-1}|M_j|^2[n-j(1-|r|^2)]\\ &= \sum_{j=0}^{n-1}(\alpha_1^j-\alpha_2^j)^2[n-j(1-|r|^2)]\\ &= \sum_{j=0}^{n-1}(\alpha_1^{2j}+\alpha_2^{2j}-2\alpha_1\alpha_2)[n-j(1-|r|^2)]\\ &= \sum_{j=0}^{n-1}(R_{2j}-4)[n-j(1-|r|^2)].\end{aligned}$$

This completes the proof. □

Theorem 3.5. The Euclidean norm for the Fermat r-circulant matrices is given by

$$||R'_r||_E = \sqrt{\sum_{j=0}^{n-1}(R_{2j}+4)[n-j(1-|r|^2)]}.$$

Proof . The argument is the same as the above theorem. □

Theorem 3.6. The bound for the spectral norm of the Mersenne r-circulant matrices is,

$$\begin{cases}\sqrt{\dfrac{M_{2n}-6M_n+3n}{3}} \le ||M'_r||_2 \le \sqrt{|r|^2\dfrac{M_{2n}-6M_n+3n}{3}}\sqrt{1+\dfrac{M_{2n}-6M_n+3n}{3}} & : |r|\ge 1,\\ |r|\sqrt{\dfrac{M_{2n}-6M_n+3n}{3}} \le ||M'_r||_2 \le \sqrt{n\dfrac{M_{2n}-6M_n+3n}{3}} & : |r|\le 1.\end{cases}$$

Proof . By Eq. (1.5), the Euclidean norm is given as,

$$||M'_r||_E^2 = \sum_{j=0}^{n-1}|M_j|^2[n-j(1-|r|^2)].$$

Case-1: If $|r|\ge 1$, then from Lemma 3.1, we get

$$\begin{aligned}||M'_r||_E^2 &= \sum_{j=0}^{n-1}(n-j)|M_j|^2+|r|^2\sum_{j=0}^{n-1}j|M_j|^2 \ge \sum_{j=0}^{n-1}(n-j)|M_j|^2+\sum_{j=0}^{n-1}j|M_j|^2\\ &\ge \sum_{j=0}^{n-1}n|M_j|^2\\ &\ge n\left(\frac{M_{2n}-6M_n+3n}{3}\right),\\ \text{which implies}\quad \frac{||M'_r||_E}{\sqrt{n}} &\ge \sqrt{\frac{M_{2n}-6M_n+3n}{3}}.\end{aligned}$$

And from Eq. (1.3), we get

$$||M'_r||_2 \geq \sqrt{\frac{M_{2n} - 6M_n + 3n}{3}}. \tag{3.4}$$

Now, to obtain the upper bound for the spectral norm, we write M'_r in the form of the Hadamard product of two matrices. Let, $X = \begin{bmatrix} M_0 & 1 & 1 & \cdots & 1 \\ rM_{(n-1)} & M_0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rM_2 & rM_3 & rM_4 & \cdots & 1 \\ rM_1 & rM_2 & rM_3 & \cdots & M_0 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & M_1 & M_2 & \cdots & M_{n-1} \\ 1 & 1 & M_1 & \cdots & M_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & M_1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$. Then clearly, $M'_r = X \circ Y$, where $\circ$ denotes the Hadamard product. Now,

$$\begin{aligned} u(X) &= \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |x_{ij}|^2} \\ &= \sqrt{M_0^2 + |r|^2 \sum_{j=1}^{n-1} M_j^2} \\ &= \sqrt{|r|^2 \frac{M_{2n} - 6M_n + 3n}{3}} \end{aligned}$$

and

$$\begin{aligned} \nu(Y) &= \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |y_{ij}|^2} \\ &= \sqrt{1 + \sum_{i=1}^{n-1} M_i^2} \\ &= \sqrt{1 + \frac{M_{2n} - 6M_n + 3n}{3}}. \end{aligned}$$

Thus, by Lemma 1.2, we write

$$||M'_r||_2 \leq u(X)\nu(Y) = \sqrt{|r|^2 \frac{M_{2n} - 6M_n + 3n}{3}} \sqrt{1 + \frac{M_{2n} - 6M_n + 3n}{3}}.$$

Hence, we have

$$\sqrt{\frac{M_{2n} - 6M_n + 3n}{3}} \leq ||M'_r||_2 \leq \sqrt{|r|^2 \frac{M_{2n} - 6M_n + 3n}{3}} \sqrt{1 + \frac{M_{2n} - 6M_n + 3n}{3}}.$$

Case-2: If $|r| < 1$, then from Eq. (1.5) and Lemma 3.1, we get

$$\begin{aligned} ||M'_r||_E^2 &\geq \sum_{j=0}^{n-1} (n-j)|r|^2 |M_j|^2 + \sum_{j=0}^{n-1} j|r|^2 |M_j|^2 \\ &\geq n|r|^2 \left(\frac{M_{2n} - 6M_n + 3n}{3} \right), \\ \text{which implies} \quad \frac{||M'_r||_E}{\sqrt{n}} &\geq |r| \sqrt{\frac{M_{2n} - 6M_n + 3n}{3}}. \end{aligned}$$

And from Eq. (1.3), we get

$$||M'_r||_2 \geq |r| \sqrt{\frac{M_{2n} - 6M_n + 3n}{3}}.$$

Next, we calculate the upper bound for the spectral norm of M'_r.
Let

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & 1 \\ r & r & r & \cdots & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-1} \\ M_{n-1} & M_0 & M_1 & \cdots & M_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_2 & M_3 & M_4 & \cdots & M_1 \\ M_1 & M_2 & M_3 & \cdots & M_0 \end{bmatrix}.$$

Then clearly, $M'_r = P \circ Q$, where $\circ$ denotes the Hadamard product. So,

$$u_1(P) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |p_{ij}|^2} = \sqrt{n}$$

and

$$\begin{aligned} \nu_1(Q) &= \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |q_{ij}|^2} \\ &= \sqrt{\sum_{j=0}^{n-1} M_j^2} \\ &= \sqrt{\frac{M_{2n} - 6M_n + 3n}{3}}. \end{aligned}$$

Hence, by Lemma 1.2, we have

$$||M'_r||_2 \le u_1(P)\nu_1(Q) = \sqrt{n\frac{M_{2n} - 6M_n + 3n}{3}}.$$

Thus,

$$|r|\sqrt{\frac{M_{2n} - 6M_n + 3n}{3}} \le ||M'_r||_2 \le \sqrt{n\frac{M_{2n} - 6M_n + 3n}{3}}.$$

This completes the proof. □

Theorem 3.7. Lower and upper bounds for the spectral norm of the Fermat r-circulant matrices are

$$\begin{cases} \sqrt{\dfrac{R_{2n} + 6R_n + 3n - 14}{3}} \le ||R'_r||_2 \le \sqrt{|r|^2\dfrac{R_{2n} + 6R_n + 3n - 14}{3}}\sqrt{1 + \dfrac{R_{2n} + 6R_n + 3n - 14}{3}} & : |r| \ge 1, \\ |r|\sqrt{\dfrac{R_{2n} + 6R_n + 3n - 14}{3}} \le ||R'_r||_2 \le \sqrt{n\dfrac{R_{2n} + 6R_n + 3n - 14}{3}} & : |r| < 1. \end{cases}$$

Proof . Using Lemma 3.2 and proceeding as the above theorem, we get the required result. □

Conclusion

In this study, we defined r-circulant matrices M_r and R_r involving Mersenne and Fermat numbers, respectively, having arithmetic indices. We obtained eigenvalues, determinant, Euclidean norm and lower and upper bounds for the spectral norm of these matrices in closed form. We also obtained all the algebraic properties discussed as a special case of the main results for $r = 1$ and $r = -1$ for right-circulant and skew-right circulant matrices. And, in conclusion, some interesting results and the sum identities have been obtained.

Acknowledgment

The authors are thankful to the anonymous referees for their insightful comments and suggestions. The first and second authors would like to thank the University Grant Commission, India for the financial support as Senior Research Fellowship.

References

[1] A.C.F. Bueno, *On the eigenvalues and the determinant of the right circulant matrices with Pell and Pell–Lucas numbers*, Int. J. Math. Sci. Comput. **4** (2014), no. 1, 19–20.

[2] P. Catarino, H. Campos, and P. Vasco, *On the Mersenne sequence*, Ann. Math. Inf. **46** (2016), 37–53.

[3] M. Chelgham and A. Boussayoud, *On the k-Mersenne–Lucas numbers*, Notes Number Theory Discrete Math. **1** (2021), no. 27, 7–13.

[4] R.E. Cline, R.J. Plemmons, and G. Worm, *Generalized inverses of certain Toeplitz matrices*, Linear Algebra Appl. **8** (1974), no. 1, 25–33.

[5] B. Fischer and J. Modersitzki, *Fast inversion of matrices arising in image processing*, Number Algorithms **22** (1999), no. 1, 1–11.

[6] R. Frontczak and T. Goy, *Mersenne-horadam identities using generating functions*, Carpathian Math. Publ. **12** (2020), no. 1, 34–45.

[7] S. Georgiou and C. Kravvaritis, *New good quasi-cyclic codes over GF(3)*, Int. J. Algebra **1** (2007), no. 1, 11–24.

[8] S. Kounias, C. Koukouvinos, N. Nikolaou, and A. Kakos, *The non-equivalent circulant d-optimal designs for* $n \equiv 2$ *mod 4*, $n \leq 54, n = 66$, J. Comb. Theory Ser. A. **65** (1994), no. 1, 26–38.

[9] I. Kra and S.R. Simanca, *On circulant matrices*, Notices AMS **59** (2012), no. 3, 368–377.

[10] M. Kumari, J. Tanti, and K. Prasad, *On some new families of k-Mersenne and generalized k-gaussian Mersenne numbers and their polynomials*, Contrib. Discrete Math. (2023), (to appear).

[11] E. Özkan, A. Göçer, and I. Altun, *A new sequence realizing Lucas numbers, and the Lucas bound*, Electronic J. Math. Anal. Appl. **5** (2017), no. 1.

[12] E. Özkan and M. Uysal, *Mersenne-Lucas hybrid number*, Math. Montisnigri **52** (2021), 17–29.

[13] S.-Q. Shen and J.-M. Cen, *On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers*, Int. J. Contemp. Math. Sci. **5** (2010), no. 12, 569–578.

[14] S.-Q. Shen and J.-M. Cen, *On the bounds for the norms of r-circulant matrices with the Fibonacci and Lucas numbers*, Appl. Math. Comput. **216** (2010), no. 10, 2891–2897.

[15] S. Solak, *On the norms of circulant matrices with the Fibonacci and Lucas numbers*, Appl. Math. Comput. **160** (2005), no. 1, 125–132.

[16] Y. Soykan, *A study on generalized Mersenne numbers*, J. Prog. Res. Math. **18** (2021), no. 3, 90–108.

[17] R. Türkmen and H. Gökbaş, *On the spectral norm of r-circulant matrices with the Pell and Pell-Lucas numbers*, J. Inequal. Appl. **2016** (2016), Article no. 65.

[18] Y. Yu and Z. Jiang, *On the norms and spreads of Fermat, Mersenne and Gaussian Fibonacci RFMLR-circulant matrices*, WSEAS Trans. Math. **15** (2016), Art. #4, 34–43.

[19] Y. Zheng and S. Shon, *Exact inverse matrices of Fermat and Mersenne circulant matrix*, Abstr. Appl. Anal. **2015** (2015), Article ID 760823.