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Starlikeness of an integral operator associated with Mittag-Leffler functions

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Abstract

In the present paper, we introduce a new integral operator involving with Mittag-Leffler function and the Salagean operator. Further, we obtain some sufficient conditions for this integral operator belonging to certain classes of starlike functions.

Keywords: Analytic function, Univalent function, Mittag-Leffler function, Starlike function, Salagean derivative, Integral operator
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1 Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we represent $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) which are also univalent in Δ.

In 1936, Robertson [22] (see also [24]) introduced two most important and widely used classes of univalent functions as follows:
A function $f(z) \in \mathcal{A}$ is said to be starlike of order ϵ if it satisfies the following analytic criteria

$$\Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \epsilon, \quad z \in \Delta, \quad \text{for some } \ \epsilon(0 \le \epsilon < 1).$$

Also, a function $f(z) \in \mathcal{A}$ is said to be convex of order ϵ if it satisfies the following analytic criteria

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \epsilon, \quad z \in \Delta, \text{for some } \quad \epsilon(0 \le \epsilon < 1).$$

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The classes of all starlike functions and convex functions of order ϵ are denoted by $\mathcal{S}^*(\epsilon)$ and $\mathcal{C}(\epsilon)$, respectively. For $\epsilon = 0$, these classes reduce to the classes $\mathcal{S}^*$ and $\mathcal{C}$, respectively. In 1983, Salagean [23] introduced an interesting derivative operator D^p known as Salagean derivative operator. Using this operator he generalized and unified the classes of starlike and convex functions, by investigating a new class $\mathcal{S}(p,\epsilon)$ consisting of functions f of the form (1.1) and satisfying the following analytic criteria

$$\Re\left\{\frac{D^{p+1}f(z)}{D^p f(z)}\right\} > \epsilon, \quad z \in \Delta, \quad p \in \mathbb{N}\cup\{0\}, \quad \text{for some } \epsilon(0 \le \epsilon < 1).$$

It is worthy to note that for $p = 0$ and $p = 1$, the class $\mathcal{S}(p,\epsilon)$ reduce to the classes $\mathcal{S}^*(\epsilon)$ and $\mathcal{C}(\epsilon)$, respectively. The class $\mathcal{S}(p,\epsilon)$ was further studied by Kadioğlu [9]. By using Salagean operator several researchers introduced various subclasses of analytic and harmonic univalent functions. Recent work on Salagean operator may be find in [7, 8, 10]. Analogues to the class $\mathcal{S}(p,\epsilon)$, Porwal and Kumar [19] introduced a new class $\mathcal{N}(p,\omega)$ consisting of functions f of the form (1.1) and satisfying the following analytic criteria

$$\Re\left\{\frac{D^{p+1}f(z)}{D^p f(z)}\right\} < \omega, \quad z \in \Delta,, \quad p \in \mathbb{N}\cup\{0\}, \quad \text{for some } \omega\left(1 < \omega \le \frac{2^p+1}{2^{p-1}+1}\right).$$

The applications of various special functions on integral operator is a current and interesting topic of research in geometric function theory. From time-to-time various integral operators associated with several special functions like Bessel functions, Mittag-Leffler functions, Dini functions, Struve function and Lommel functions are introduced and extensively studied by several researchers. Noteworthy, contribution in this direction may be found in [1, 5, 6, 11, 12, 15, 17, 18, 19, 20, 21, 26].

Now, we recall the definition of Mittag-Leffler function $E_\alpha(z)$ which was introduced by Mittag-Leffler[13] and defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad (z \in \mathbb{C}, \;\; \Re(\alpha) > 0).$$

In 1905, Wiman[27, 28] generalized the Mittag-Leffler function in $E_{\alpha,\,\beta}(z)$ by the relation

$$E_{\alpha,\,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{1.2}$$

where $z, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$. It should be easy to see that the function $E_{\alpha,\,\beta}(z)$ defined by (1.2) is not in class $\mathcal{A}$. Thus, first we normalize the Mittag-Leffler function as follows

$$\begin{aligned} \mathbb{E}_{\alpha,\,\beta}(z) &= \Gamma(\beta) z\, E_{\alpha,\,\beta}(z) \\ \mathbb{E}_{\alpha,\,\beta}(z) &= z + \sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\, z^n, \end{aligned} \tag{1.3}$$

where $z, \alpha, \beta \in \mathbb{C}$, $\beta \ne 0, -1, -2, \cdots, \Re(\alpha) > 0$. In the present work, we shall restrict our attention to the case for real-valued α, β and $z \in \Delta$. For specific values of α and β, the function $\mathbb{E}_{\alpha,\,\beta}(z)$ reduces to many well-known functions

$$\begin{aligned} \mathbb{E}_{2,\,1}(z) &= z\cosh\sqrt{z} \\ \mathbb{E}_{2,\,2}(z) &= \sqrt{z}\sinh\sqrt{z} \\ \mathbb{E}_{2,\,3}(z) &= 2[\cosh\sqrt{z}-1] \quad \text{and} \\ \mathbb{E}_{2,\,4}(z) &= \frac{6[\sinh\sqrt{z}-\sqrt{z}]}{\sqrt{z}}. \end{aligned}$$

For further study of Mittag-Leffler function and generalized Mittag-Leffler function, interesting reader may refer to [2, 3].

Motivated with the above mentioned work, Srivastava *et al.* [26] investigated a new integral operator associated with Mittag-Leffler functions and obtain various interesting results.

In the present work, we introduce a new integral operator involving Mittag-Leffler function in the following way

$$F_{\alpha_i,\,\beta_i,\,\gamma_i,\,\lambda_j}(p,z)=\int\limits_0^z\prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(t)}{t}\right)^{\gamma_i}\prod_{j=1}^{m}\left(\frac{D^p f_j(t)}{t}\right)^{\lambda_j}dt \tag{1.4}$$

where the functions $\mathbb{E}_{\alpha_i,\,\beta_i}(z)$ is normalized Mittag-Leffler functions defined by (1.3), parameters γ_i,λ_j are positive real numbers such that the integral in (1.4) exists. The integral operator defined by (1.4) reduces to various integral operators for specific values of parameters $\alpha_i,\beta_i,\gamma_i,\lambda_j,p$, studied earlier by various researchers.

1. For $\gamma_i=0(i=1,2,\dots,n)$, the integral operator studied by Porwal [16].
2. For $\gamma_i=0(i=1,2,\dots,n)$, $p=0,1$ the integral operator studied by Breaz [4].
3. For $\gamma_i=0(i=1,2,\dots,n)$, $p=1,m=1$ the integral operator studied by Passai and Pescar [14].
4. For $\lambda_j=0(j=1,2,\dots,m)$ the integral operator studied by Srivastava *et al.* [26].

In the present paper, we obtain some sufficient conditions for the integral operator defined by (1.4) is in the class $\mathcal{S}^*$.
For simplicity we can write

$$\mathbb{F}(p,z)=F_{\alpha_i,\,\beta_i,\,\gamma_i,\,\lambda_j}(p,z).$$

2 Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. ([26]) Let $\alpha\geq 1,\beta\geq 1$. Then

$$\left|\frac{z\,\mathbb{E}'_{\alpha,\,\beta}(z)}{\mathbb{E}_{\,\alpha,\,\beta}(z)}-1\right|\leq\frac{2\beta+1}{\beta^2-\beta-1},\qquad z\in\Delta.$$

Lemma 2.2. ([25]) If $f\in\mathcal{A}$ satisfies

$$\Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}<\frac{\delta+1}{2(\delta-1)},\qquad z\in\Delta,\ \text{for some}\quad 2\leq\delta<3,$$
$$\text{or}\quad\Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}<\frac{5\delta-1}{2(\delta+1)},\qquad z\in\Delta,\ \text{for some}\quad 1<\delta\leq 2,\ \ then\ \ f\in\mathcal{S}^*.$$

Lemma 2.3. ([25]) If $f\in\mathcal{A}$ satisfies

$$\Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>-\frac{\delta+1}{2\delta(\delta-1)},\qquad z\in\Delta,\ \text{for some}\quad \delta\leq -1,$$
$$\text{or}\quad\Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>\frac{3\delta+1}{2\delta(\delta+1)},\qquad z\in\Delta,\ \text{for some}\quad \delta>1,\ \ then\ \ f\in\mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right).$$

3 Main Results

Theorem 3.1. Let n,m be natural numbers and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n\geq 1,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n\geq\frac{1}{2}(1+\sqrt{5})$ and suppose that $\beta=\min\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ and suppose that $\gamma_i(i=1,2,\dots,n)$, $\lambda_j\ (j=1,2,\dots,m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $N(p,\omega_j)$ for $(j=1,2,\dots,m)$, also let $\omega=\max\{\omega_1,\omega_2,\dots,\omega_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\ \sum_{i=1}^{n}\gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j\leq\frac{3-\delta}{2(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}^*$ for some $2\leq\delta<3$.

Proof . Differentiating equation (1.4) we have

$$\mathbb{F}'(p,z)=\prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}{z}\right)^{\gamma_i}\prod_{j=1}^{m}\left(\frac{D^p f_j(z)}{z}\right)^{\lambda_j}.$$

Taking logarithmic differentiation, we have

$$\frac{\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}=\sum_{i=1}^{n}\gamma_i\left(\frac{\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-\frac{1}{z}\right)+\sum_{j=1}^{m}\lambda_j\left(\frac{(D^p f_j(z))'}{D^p f_j(z)}-\frac{1}{z}\right)$$

or equivalently

$$1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}=\sum_{i=1}^{n}\gamma_i\left(\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right)+\sum_{j=1}^{m}\lambda_j\left(\frac{D^{p+1}f_j(z)}{D^p f_j(z)}-1\right)+1. \tag{3.1}$$

Taking the real part of both side of (3.1) we have

$$\begin{aligned}\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}&=\sum_{i=1}^{n}\gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)-1}\right\}+\sum_{j=1}^{m}\lambda_j\Re\left(\frac{D^{p+1}f_j(z)}{D^p f_j(z)}-1\right)+1\\&\le 1+\sum_{i=1}^{n}\gamma_i\left|\frac{z\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right|+\sum_{j=1}^{m}\lambda_j\left(\omega_j-1\right) &(3.2)\\&\le 1+\sum_{i=1}^{n}\gamma_i\left(\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right)+\sum_{j=1}^{m}\lambda_j\left(\omega_j-1\right). &(3.3)\end{aligned}$$

For all $z\in\Delta$ and $(\beta_1,\ \beta_2,\ \cdots,\ \beta_n)\ge\frac{1}{2}(1+\sqrt{5})$. Since the function $\phi:\left(\frac{1}{2}(1+\sqrt{5}),\infty\right)\to\mathbb{R}$, defined by $\phi(x)=\frac{2x+1}{x^2-x-1}$ is decreasing. Therefore, for all $i\in\{1,\ 2,\ \cdots,\ n\}$, we obtain

$$\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\le\frac{2\beta+1}{\beta^2-\beta-1}.$$

Using this result, inequality (3.2) can be written as

$$\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\}\le 1+\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j.$$

Since

$$\begin{aligned}1+\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j&<\frac{\delta+1}{2(\delta-1)}\\ \frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i+(\omega-1)\sum_{j=1}^{m}\lambda_j&<\frac{\delta+1}{2(\delta-1)}-1\\&=\frac{3-\delta}{2(\delta-1)}.\end{aligned}$$

Therefore, from Lemma 2.2, $\mathbb{F}(p,z)\in\mathcal{S}^*$ for some $2\le\delta<3$. Thus, the proof of Theorem 3.1 is established. □

Theorem 3.2. Let n,m be natural numbers and $\alpha_i\ge 1$, $\beta_i\ge\frac{1}{2}(1+\sqrt{5})$ for $i=1,2,\ldots,n$; $\beta=\min\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ and suppose that $\gamma_i(i=1,2,\ldots,n)$, $\lambda_j\ (j=1,2,\ldots,m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $N(p,\omega_j)$ for $(j = 1, 2, \ldots, m)$, $p \in N_0, 1 < \omega_j \leq \dfrac{2^p+1}{2^{p-1}+1}$, also let $\omega = \max\{\omega_1, \omega_2, \ldots, \omega_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i + (\omega-1)\sum_{j=1}^{m}\lambda_j \leq \frac{3\,(\delta-1)}{2\delta+1}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}^*$ for some $1 < \delta \leq 2$.

Proof . The proof of above theorem is much similar to that Theorem 3.1. Therefore, we omit the detail. □

Theorem 3.3. Let n, m be natural numbers and $\alpha_i \geq 1$, $\beta_i \geq \frac{1}{2}(1+\sqrt{5})$ for $i = 1, 2, \ldots, n$; $\beta = \max\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ and suppose that $\gamma_i (i = 1, 2, \ldots, n)$, λ_j $(j = 1, 2, \ldots, m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $\mathcal{S}(p,\epsilon_j)$ for $(j = 1, 2, \ldots, m)$, $p \in N_0, 0 \leq \epsilon_j < 1$, also let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i + (1-\epsilon)\sum_{j=1}^{m}\lambda_j \leq \frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}\left(\dfrac{\delta+1}{2\delta}\right)$ for some $\delta \leq -1$.

Proof . The equation (3.1) can be re-written as

$$\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\} = \sum_{i=1}^{n}\ \gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} + \sum_{j=1}^{m}\lambda_j\Re\left(\frac{D^{p+1}f_j(z)}{D^p f_j(z)}\right) + 1 - \sum_{i=1}^{n}\gamma_i - \sum_{j=1}^{m}\lambda_j. \tag{3.4}$$

From Lemma 2.1, we have

$$\left|\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)} - 1\right| \leq \frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using the identity $\Re\{z\} \leq |z|$, we have

$$\Re\left\{1-\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} \leq \frac{2\beta_i+1}{\beta_i^2-\beta_i-1},$$

or

$$\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} \geq 1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using the above result in (3.4), we have

$$\begin{aligned}\Re\left\{1+\frac{z\,\mathbb{F}''(p,z)}{\mathbb{F}'(p,z)}\right\} &\geq \sum_{i=1}^{n}\ \gamma_i\left(1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right) + \sum_{j=1}^{m}\lambda_j\epsilon_j + 1 - \sum_{i=1}^{n}\gamma_i - \sum_{j=1}^{m}\lambda_j \\ &\geq 1 - \frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\ \gamma_i - (1-\epsilon)\sum_{j=1}^{m}\lambda_j \\ &\geq -\frac{\delta+1}{2\delta(\delta-1)}, \qquad \text{(by the given hypothesis).}\end{aligned}$$

□

Theorem 3.4. Let n, m be natural numbers and $\alpha_i \geq 1$, $\beta_i \geq \frac{1}{2}(1+\sqrt{5})$ for $i = 1, 2, \ldots, n$; $\beta = \max\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ and suppose that $\gamma_i (i = 1, 2, \ldots, n)$, λ_j $(j = 1, 2, \ldots, m)$ are positive real numbers. Further, we let $f_j(z)$ be of the form (1.1) in the class $\mathcal{S}(p,\epsilon_j)$ for $(j=1,2,\dots,m)$, $p \in N_0, 0 \leq \epsilon_j < 1$, also let $\epsilon = \min \epsilon_1, \epsilon_2, \dots, \epsilon_m$. Moreover, suppose that these numbers satisfy the following inequality

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i + (1-\epsilon)\sum_{j=1}^{m}\lambda_j \leq \frac{\delta+1-2\delta^2}{2\delta(\delta-1)}$$

is satisfied. Then the function $\mathbb{F}(p,z)$ defined by (1.4) is in the class $\mathcal{S}\left(\frac{\delta+1}{2\delta}\right)$ for some $\delta > 1$.

Proof . The proof of above theorem runs parallel to that of Theorem 3.3. Therefore, we omit the details involved. ☐

Remark 3.5.
1. If we put $\gamma_i = 0$ $(i=1,2,\dots,n)$ in Theorem 3.1–3.4 then we obtain the corresponding results for the integral operator introduced by Porwal [16].
2. If we put $\lambda_j = 0$ $(j=1,2,\dots,m$ in Theorem 3.1–3.4 then we obtain the corresponding results for the integral operator studied by Srivastava et al. [26]

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