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Coefficient bounds for a generalized subclass of bounded turning functions associated with Sigmoid function

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Abstract

In this paper, we introduce a subclass of analytic functions associated with the Sigmoid function and determine the upper bounds for various coefficient functionals such as Fekete-Szegő functional, second Hankel determinant, Zalcman functional and third Hankel determinant. Also, the concept is extended to two-fold and three-fold symmetric functions. The results proved earlier, follow as special cases of the results of this paper.

Keywords: Bounded turning functions, Subordination, Sigmoid function, Hankel determinant, Coefficient bounds
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1 Introduction

Special functions play an important role in the field of Science and Engineering. One of the most useful example of special functions is the activation function. Activation functions are further of three types and the most popular of these is the Sigmoid function whose working is analogous to the human brain. The Sigmoid function is of the form $\frac{1}{1+e^{-z}}$, and has the following properties:
(i) Its output ranges between 0 and 1;
(ii) It maps sufficiently large input domains onto a small output range;
(iii) It is a one-one function, so it never loses information.

The above properties make it clear that the Sigmoid function is quite useful in Geometric function theory. A variety of subclasses of analytic functions associated with Sigmoid function have been studied by various authors including Khan et al. [13, 14], Joseph et al. [11], Goel and Kumar [8], Ramachandran and Dhanalakshmi. [28] and Singh et al. [33]. Before defining our main class, firstly let's have an overview of the preliminary classes. The class of functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, is denoted by $\mathcal{A}$. Further, the subclass of $\mathcal{A}$ which consists of univalent functions, is denoted by $\mathcal{S}$. The notion of subordination plays an important role in the theory of univalent functions and it owes its origin to Lindelöf [18]. This concept is stated as, for two analytic functions f and g in E, f is said to be subordinate to g (denoted as $f \prec g$) if there exists

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a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

The classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, respectively and defined as

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

The class $\mathcal{CS}^*$ of close-to-star functions was introduced by Reade [29] and it consists of functions $f \in \mathcal{A}$ such that $Re\left(\frac{f(z)}{g(z)}\right) > 0,\ g \in \mathcal{S}^*$. For $g(z) = z$, MacGregor [22] studied the class $\mathcal{R}' = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E \right\}$, which is indeed a subclass of close-to-star functions. Also the class $\mathcal{R}$ defined as $\mathcal{R} = \{ f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E\}$, is the class of bounded turning functions which was introduced and studied by MacGregor [21]. Later on, Murugusundramurthi and Magesh [24] studied the class $\mathcal{R}(\alpha)$, which is a unification of the classes $\mathcal{R}'$ and $\mathcal{R}$. The class $\mathcal{R}(\alpha)$ is given by

$$\mathcal{R}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, z \in E \right\}.$$

For $\alpha = 0$ and $\alpha = 1$, the class $\mathcal{R}(\alpha)$ reduces to the classes $\mathcal{R}'$ and $\mathcal{R}$, respectively. Inspired by the above work on different subclasses of analytic functions, we define the following subclass of $\mathcal{A}$, associated with the Sigmoid function $\dfrac{2}{1+4e^{-z}}$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_\alpha(\Phi)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{2}{1+4e^{-z}}.$$

We have the following observations:
(i) $\mathcal{R}_0(\Phi) \equiv \mathcal{R}'(\Phi)$.
(ii) $\mathcal{R}_1(\Phi) \equiv \mathcal{R}(\Phi)$.

For $q \geq 1$ and $n \geq 1$, Noonan and Thomas [25] stated the q^{th} Hankel determinant as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q = 2, n = 1$ and $a_1 = 1$, the Hankel determinant reduces to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegő functional. Fekete and Szegő [7] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \mathcal{S}$. Also for $q = 2, n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is Hankel determinant of order 2. Further, for $q = 3, n = 1$, the Hankel determinant yields

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

which is the third order Hankel determinant. For $f \in \mathcal{S}$ and $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.1}$$

Ma [19] introduced a useful functional $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which is known as generalized Zalcman functional. For $n = 2$ and $m = 3$, this functional reduces to $J_{2,3}(f) = a_2 a_3 - a_4$. Various authors established the upper bound for the functional $J_{2,3}(f)$ for different subclasses of $\mathcal{A}$ as it is plays an important role in establishing the bounds for the third Hankel determinant.

A rizeable amount of work has been done on the estimation of second Hankel determinant by various authors including Noor [26], Ehrenborg [6], Layman [15], Singh [31], Mehrok and Singh [23] and Janteng et al. [10]. It is little bit complicated to establish the upper bound for the third order Hankel determinant. Babalola [3], was the first to obtain the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Later on, a few researchers including Shanmugam et al. [30], Bucur et al. [4], Altinkaya and Yalcin [1], Singh and Singh [32] have worked in the direction of third Hankel determinant for various subclasses of analytic functions.

In the present paper, we establish the upper bounds for the initial coefficients, Fekete-Szegő inequality, Zalcman functional, second Hankel determinant and third hankel determinant for the class $\mathcal{R}_\alpha(\Phi)$. Also various known results follow as particular cases.

Let $\mathcal{P}$ denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E. The class $\mathcal{P}$ was established by Carathéodory [5] and the functions of this class are known as Carathéodory functions.

2 Preliminaries

The following lemmas are very useful in the derivation of our main results:

Lemma 2.1. If $p \in \mathcal{P}$, then

$$|p_k| \leq 2, k \in \mathbb{N}.$$

The above well known result is due to Carathéodory [5]. Further Hayami and Owa [9, page 2577, Corollary 2.5], established the following result:

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1.$$

Also Ma and Minda [20, Page 162, Lemma 1] proved that if ρ is any complex number, then

$$|p_2 - \rho p_1^2| \leq 2 \max\{1, |2\rho - 1|\}.$$

Lemma 2.2. It is mentioned in [2] (page 1617, Lemma 2.2) that, for $p \in \mathcal{P}$,

$$|J p_1^3 - K p_1 p_2 + L p_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|$$

and in particular, $|p_1^3 - 2p_1 p_2 + p_3| \leq 2$.

Lemma 2.3. [16, 17] If $p \in \mathcal{P}$, then $2p_2 = p_1^2 + (4 - p_1^2)x$,

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

3 Bounds of $|H_3(1)|$ for the class $\mathcal{R}_\alpha(\Phi)$

This section is concerned with the estimation of upper bounds of various coefficient functionals, which lead to the bound of the third Hankel determinant.

Theorem 3.1. If $\mathcal{R}_\alpha(\Phi)$, then

$$|a_2| \leq \frac{1}{2(1+\alpha)}, \tag{3.1}$$

$$|a_3| \leq \frac{1}{2(1+2\alpha)}, \tag{3.2}$$

$$|a_4| \leq \frac{1}{2(1+3\alpha)}, \tag{3.3}$$

and

$$|a_5| \leq \frac{1}{2(1+4\alpha)}. \tag{3.4}$$

The estimates are sharp.

Proof . Since $f \in \mathcal{R}_\alpha(\Phi)$, using the principle of subordination, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{2}{1+4e^{-w(z)}}, \tag{3.5}$$

where the function w satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$. Let us define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$. On expanding (3.5), we get

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ... \tag{3.6}$$

Moreover,

$$\frac{2}{1+4e^{-w(z)}} = 1 + \tfrac{1}{4}p_1 z + \left(\tfrac{1}{4}p_2 - \tfrac{p_1^2}{8}\right) z^2 + \left(\frac{1}{4}p_3 - \frac{1}{4}p_1 p_2 + \frac{11}{192}p_1^3\right) z^3 + \left(\frac{1}{4}p_4 - \frac{3}{128}p_1^4 - \frac{1}{8}p_2^2 - \frac{1}{4}p_1 p_3 + \frac{11}{64}p_1^2 p_2\right) z^4 + ... \tag{3.7}$$

Using (3.6) and (3.7), (3.5) yields

$$\begin{aligned} &1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + \cdots \\ =&1 + \frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{p_1^2}{8}\right) z^2 + \left(\frac{1}{4}p_3 - \frac{1}{4}p_1 p_2 + \frac{11}{192}p_1^3\right) z^3 \\ &+ \left(\frac{1}{4}p_4 - \frac{3}{128}p_1^4 - \frac{1}{8}p_2^2 - \frac{1}{4}p_1 p_3 + \frac{11}{64}p_1^2 p_2\right) z^4 + \cdots . \end{aligned} \tag{3.8}$$

On equating the coefficients of z, z^2, z^3 and z^4 in (3.8) and on simplifying, we obtain

$$a_2 = \frac{1}{4(1+\alpha)}p_1, \tag{3.9}$$

$$a_3 = \frac{1}{1+2\alpha}\left[\frac{1}{4}p_2 - \frac{p_1^2}{8}\right], \tag{3.10}$$

$$a_4 = \frac{1}{1+3\alpha}\left[\frac{1}{4}p_3 - \frac{1}{4}p_1 p_2 + \frac{11}{192}p_1^3\right], \tag{3.11}$$

and

$$a_5 = \frac{1}{(1+4\alpha)}\left[\frac{3}{128}p_1^4 + \frac{1}{8}p_2^2 + \frac{1}{4}p_1 p_3 - \frac{11}{64}p_1^2 p_2 - \frac{1}{4}p_4\right]. \tag{3.12}$$

Using first inequality of Lemma 2.1 in (3.9), the result (3.1) is obvious. From (3.10), we have

$$|a_3| = \frac{1}{4(1+2\alpha)}\left|p_2 - \frac{1}{2}p_1^2\right|. \tag{3.13}$$

The result (3.2) can be easily obtained on using third inequality of Lemma 2.1 in (3.13). (3.11) can be expressed as

$$|a_4| = \frac{1}{192(1+3\alpha)}\left|48p_3 - 48p_1p_2 + 11p_1^3\right|. \tag{3.14}$$

Applying inequality 2 of Lemma 2.1 in (3.14), the result (3.3) is obvious. Furthermore, (3.12) can be reframed as

$$|a_5| = \frac{1}{4(1+4\alpha)}\left|\frac{3}{32}p_1^4 + \frac{1}{2}p_2^2 + p_1p_3 - \frac{11}{16}p_1^2p_2 - p_4\right|. \tag{3.15}$$

Using second inequality of Lemma 2.1, the result (3.4) follows from (3.15). The results (3.1), (3.2), (3.3) and (3.4) are sharp for the function f given by

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{2}{1+4e^{-z}}.$$

□

On putting $\alpha = 0$, Theorem 3.1 yields the following result:
Corollary 3.1 If $f \in \mathcal{R}^{'}(\Phi)$, then

$$|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{2}, |a_4| \leq \frac{1}{2}, |a_5| \leq \frac{1}{2}.$$

For $\alpha = 1$, Theorem 3.1 gives the following result due to Khan et al. [13]:
Corollary 3.2 If $f \in \mathcal{R}(\Phi)$, then

$$|a_2| \leq \frac{1}{4}, |a_3| \leq \frac{1}{6}, |a_4| \leq \frac{1}{8}, |a_5| \leq \frac{1}{10}.$$

Theorem 3.2. If $f \in \mathcal{R}_\alpha(\Phi)$, then

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)}. \tag{3.16}$$

Proof . From (3.9) and (3.10), we have

$$|a_3 - a_2^2| = \frac{1}{4(1+2\alpha)}\left|p_2 - \frac{2\alpha^2 + 6\alpha + 3}{4(1+\alpha)^2}p_1^2\right|. \tag{3.17}$$

Using third inequality of Lemma 2.1, (3.17) takes the following form:

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)} max\left\{1, \frac{1+2\alpha}{2(1+\alpha)^2}\right\}. \tag{3.18}$$

But $\frac{1+2\alpha}{2(1+\alpha)^2} \leq 1$ for $0 \leq \alpha \leq 1$. Hence, the result (3.16) is obvious from (3.18). □

Substituting $\alpha = 0$, Theorem 3.2 yields the following result:
Corollary 3.3 If $f \in \mathcal{R}^{'}(\Phi)$, then

$$|a_3 - a_2^2| \leq \frac{1}{2}.$$

Putting $\alpha = 1$, Theorem 3.2 yields the following result due to Khan et al. [13]:
Corollary 3.4 If $f \in \mathcal{R}(\Phi)$, then

$$|a_3 - a_2^2| \leq \frac{1}{6}.$$

Theorem 3.3. If $f \in \mathcal{R}_\alpha(\Phi)$, then

$$|a_2a_3 - a_4| \leq \frac{1}{2(1+3\alpha)}. \tag{3.19}$$

Proof . Using (3.9), (3.10), (3.11) and simplifying, we have

$$|a_2a_3 - a_4| = \frac{1}{192(1+\alpha)(1+2\alpha)(1+3\alpha)} \left|(17+51\alpha+22\alpha^2)p_1^3 - (60+180\alpha+96\alpha^2)p_1p_2 + 48(1+\alpha)(1+2\alpha)p_3\right|. \tag{3.20}$$

On applying Lemma 2.2 in (3.20), it yields (3.19). □

For $\alpha = 0$, the following result is a consequence of Theorem 3.3:

Corollary 3.5 If $f \in \mathcal{R}'(\Phi)$, then

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

For $\alpha = 1$, we can obtain the following result due to Khan et al. [13], from Theorem 3.3, :

Corollary 3.6 If $f \in \mathcal{R}(\Phi)$, then

$$|a_2a_3 - a_4| \leq \frac{1}{8}.$$

Theorem 3.4. If $f \in \mathcal{R}_\alpha(\Phi)$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4(1+2\alpha)^2}. \tag{3.21}$$

Proof . Using (3.9), (3.10) and (3.11), we have

$$|a_2a_4 - a_3^2| = \frac{1}{768(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big|48(1+2\alpha)^2p_1p_3$$

$$+48[(1+\alpha)(1+3\alpha)-(1+2\alpha)^2]p_1^2p_2 + [11(1+2\alpha)^2+12(1+\alpha)(1+3\alpha)]p_1^4 - 48(1+\alpha)(1+3\alpha)p_2^2\Big|.$$

On substituting for p_2 and p_3 from Lemma 2.3 and letting $p_1 = p$, we get

$$|a_2a_4 - a_3^2| = \frac{1}{768(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big|(68\alpha^2+92\alpha+23)p^4$$

$$-12(1+2\alpha)^2p^2(4-p^2)x^2 - 12(1+\alpha)(1+3\alpha)(4-p^2)^2x^2 + 24(1+2\alpha)^2p(4-p^2)(1-|x|^2)z\Big|.$$

Since $|p| = |p_1| \leq 2$, it can be assumed that $p \in [0,2]$. Then by using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0,1]$, we obtain

$$|a_2a_4 - a_3^2| \leq \tfrac{1}{768(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(68\alpha^2+92\alpha+23)p^4 + 12(1+2\alpha)^2p^2(4-p^2)t^2$$

$$+12(1+\alpha)(1+3\alpha)(4-p^2)^2t^2 + 24(1+2\alpha)^2p(4-p^2) - 24(1+2\alpha)^2p(4-p^2)t^2\Big] = F(p,t).$$

$$\frac{\partial F}{\partial t} = \frac{t(4-p^2)}{32(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(1+2\alpha)^2p^2 + (1+\alpha)(1+3\alpha)(4-p^2) - 2(1+2\alpha)^2p\Big],$$

which can be expressed as

$$\frac{\partial F}{\partial t} = \frac{t(4-p^2)}{32(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left\{(p^2-8p+12)\left[\alpha^2 + \frac{8(2-p)}{p^2-8p+12}\alpha\right] + 2(2-p)\right\}.$$

But, as $p \leq 2$ and $p^2-8p+12 = (p-2)(p-6) \geq 0$, so it is obvious that $\dfrac{\partial F}{\partial t} \geq 0$. Therefore, $F(p,t)$ is an increasing function of t and so

$$\begin{aligned}\max\{F(p,t)\} =& F(p,1)\\ =& \frac{1}{768(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(68\alpha^2+92\alpha+23)p^4 + 12(1+2\alpha)^2p^2(4-p^2) + 12(1+\alpha)(1+3\alpha)(4-p^2)^2\Big]\\ =& H(p),\end{aligned}$$

$$H'(p)=\frac{1}{768(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left[4(68\alpha^2+92\alpha+23)p^3+48(1+2\alpha)^2p(2-p^2)-48(1+\alpha)(1+3\alpha)(4-p^2)p\right].$$

$H'(p)=0$ gives $p=0$. Also $H''(p)<0$ for $p=0$. This implies $\max\{H(p)\}=H(0)=\dfrac{1}{4(1+2\alpha)^2}$, which proves (3.21). □

Putting $\alpha=0$, Theorem 3.4 gives the following result:
Corollary 3.7 If $f\in\mathcal{R}^{'}(\Phi)$, then

$$|a_2a_4-a_3^2|\leq\frac{1}{4}.$$

Substituting for $\alpha=1$, the following result due to Khan et al. [13], is obvious from Theorem 3.4:
Corollary 3.8 If $f\in\mathcal{R}(\Phi)$, then

$$|a_2a_4-a_3^2|\leq\frac{1}{36}.$$

Theorem 3.5. If $f\in\mathcal{R}_\alpha(\Phi)$, then

$$|H_3(1)|\leq\frac{5+50\alpha+179\alpha^2+268\alpha^3+136\alpha^4}{8(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}. \tag{3.22}$$

Proof . Using (3.2), (3.3), (3.4), (3.16), (3.19) and (3.21) in (1.1), the result (3.22) can be easily obtained. □

For $\alpha=0$, Theorem 3.5 yields the following result:
Corollary 3.9 If $f\in\mathcal{R}^{'}(\Phi)$, then

$$|H_3(1)|\leq\frac{5}{8}.$$

For $\alpha=1$, Theorem 3.5 yields the following result due to Khan et al. [13]:
Corollary 3.10 If $f\in\mathcal{R}(\Phi)$, then

$$|H_3(1)|\leq\frac{319}{8640}.$$

4 Bounds of $|H_3(1)|$ for two-fold and three-fold symmetric functions

In this section, we establish the bounds of third Hankel determinant for the subclasses $\mathcal{R}_\alpha^{(2)}(\Phi)$ and $\mathcal{R}_\alpha^{(3)}(\Phi)$ of two-fold and three-fold symmetric functions, respectively. A function f is said to be n-fold symmetric if is satisfies the following condition:

$$f(\xi z)=\xi f(z)$$

where $\xi=e^{\frac{2\pi i}{n}}$, $n\in\mathbb{N}$ and $z\in E$. By $\mathcal{S}^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class $\mathcal{S}$. The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z)=z+\sum_{k=1}^{\infty}a_{nk+1}z^{nk+1}. \tag{4.1}$$

An analytic function f of the form (4.1) belongs to the family $\mathcal{R}_\alpha^{(n)}(\Phi)$ if and only if

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)=\frac{2}{1+4e^{-\left(\frac{p(z)-1}{p(z)+1}\right)}},p\in\mathcal{P}^{(n)},$$

where

$$\mathcal{P}^{(n)}=\left\{p\in\mathcal{P}:p(z)=1+\sum_{k=1}^{\infty}p_{nk}z^{nk},z\in E\right\}. \tag{4.2}$$

Theorem 4.1. If $f\in\mathcal{R}_\alpha^{(2)}(\Phi)$, then

$$|H_3(1)|\leq\frac{1}{4(1+2\alpha)(1+4\alpha)}. \tag{4.3}$$

Proof . If $f \in \mathcal{R}_{\alpha}^{(2)}(\Phi)$, so by definition there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{2}{1+4e^{-\left(\frac{p(z)-1}{p(z)+1}\right)}}. \tag{4.4}$$

Using (4.1) and (4.2) for $n = 2$, (4.4) yields

$$a_3 = \frac{1}{4(1+2\alpha)}p_2, \tag{4.5}$$

$$a_5 = \frac{1}{4(1+4\alpha)}\left(p_4 - \frac{1}{2}p_2^2\right). \tag{4.6}$$

Also

$$H_3(1) = a_3a_5 - a_3^3. \tag{4.7}$$

Using (4.5) and (4.6) in (4.7), it can be expressed as

$$|H_3(1)| = \frac{1}{16(1+2\alpha)(1+4\alpha)}p_2\left|p_4 - \frac{2(1+2\alpha)^2-(1+4\alpha)}{4(1+2\alpha)^2}p_2^2\right|. \tag{4.8}$$

On using second inequality of Lemma 2.1 in the above expression, we can easily get the result (4.3). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 4.1:

Corollary 4.1 If $f \in \mathcal{R}'^{(2)}(\Phi)$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 4.1 agrees with the following result:

Corollary 4.2 If $f \in \mathcal{R}^{(2)}(\Phi)$, then

$$|a_3 - a_2^2| \leq \frac{1}{60}.$$

Theorem 4.2. If $f \in \mathcal{R}_{\alpha}^{(3)}(\Phi)$, then

$$|H_3(1)| \leq \frac{1}{4(1+3\alpha)^2}. \tag{4.9}$$

Proof . If $f \in \mathcal{R}_{\alpha}^{(3)}(\Phi)$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{2}{1+4e^{-\left(\frac{p(z)-1}{p(z)+1}\right)}}. \tag{4.10}$$

Using (4.1) and (4.2) for $n = 3$, (4.10) gives

$$a_4 = \frac{1}{4(1+3\alpha)}p_3. \tag{4.11}$$

Also

$$H_3(1) = -a_4^2. \tag{4.12}$$

Using (4.11) in (4.12), it yields

$$H_3(1) = -\frac{1}{16(1+3\alpha)^2}p_3^2. \tag{4.13}$$

Applying triangle inequality and using first inequality of Lemma 2.1, (4.9) can be easily obtained.
□

For $\alpha = 0$, Theorem 4.2 yields the following result:

Corollary 4.3 If $f \in \mathcal{R}'^{(3)}(\Phi)$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 4.2 yields the following result:
Corollary 4.4 If $f \in \mathcal{R}^{(3)}(\Phi)$, then

$$|H_3(1)| \leq \frac{1}{64}.$$

5 Conclusion

In this paper, we have introduced a generalized subclass of bounded turning functions associated with the sigmoid function. Various prperties of this class such as coefficient estimates, the bounds for Fekete-Szegő inequality, Zalcman functional, second Hankel determinant and third Hankel determinant, have been established. Some earlier known results follow as special cases of the results obtained in this paper. It will pave the way for other researchers to study some more subclasses of analytic functions.

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