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Independence fractals of fractal graphs

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Abstract

For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of $V(G)$ and a vertex $v \in V$, the metric representation of v with respect to W is a k-vector, which is defined as $r(v/W) = \{d(v, w_1), d(v, w_2), ..., d(v, w_k)\}$. The set W is called a resolving set for G if $r(u/W) = r(v/W)$ implies that $u = v$ for all $u, v \in V(G)$. The minimum cardinality of a resolving set of G is called the metric dimension of G. For two graphs G and H, the lexicographic product $G \wr H$ of H by G is obtained from G by replacing each vertex of G with a copy of H. A graph G is considered fractal if a graph Γ exists, with at least two vertices, such as $G \simeq \Gamma \wr G$. This paper intends to discuss the fractal graph of some graphs and corresponding independence fractals. Also, compare the independent fractals of the fractal graph G, fractal factor Γ and $\Gamma \wr G$.

Keywords: Fractal graph, Egamorphism, Metric dimension, Metric basis, Resolving set, Independence Fractals
2020 MSC: 28A80, 47H10, 54H25, 05C12, 05C63, 05C75, 05C76, 05E30

1 Introduction

The concepts of metric dimension of a graph and its related properties such as basis were introduced by P.J.Slater [12] and independently by Harary and Melter [6]. Slater introduced metric dimension by motivated from the robot navigation problem. The motivation of this paper came from the notion of fractal graphs which was introduced by Pierre Ille and Robert Woodrow[11]. The definition of fractal graphs was made with respect to the idempotency under the lexicographic product of graphs [10]. Since the definition requires a graph with at least two vertices, we start with the lexicographic products which contain six vertices which is obtained from the graphs with two and three vertices. An attempt to study the fractal properties of these graphs using this definition has been made in this paper which will help us to extend it to the advanced graphs, which is currently a less explored area of study.

2 Preliminaries

All the graphs considered in this paper are undirected, simple, finite and connected. We use standard terminology, the terms not defined here may found in [8], [7].

Definition 2.1. [9] A function $f : V(G) \to V(H)$ is an egamorphism from G to H if for $v, w \in V(G)$ such that $f(v) \neq f(w)$, we have $[v, w]_G = [f(v), f(w)]_H$.

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Definition 2.2. [9] For a graph G, G is a fractal if and only if there exists a graph Γ satisfying the properties, $V(\Gamma) = 2$ and $G \simeq \Gamma \wr G$. The graph Γ satisfying these properties is called the fractal factor of G. Here we use the weak notion of isomorphism of graphs which is given in Definition 2.1.

Definition 2.3. A partition P of $V(G)$ is a modular partition of G if each block of P is a module of G. A subset M of $V(G)$ is a module of G if for any $x, y \in M$ and $v \in V(G)\backslash M$, we have $[x, v]_G = [y, v]_G$.

Definition 2.4. Let $G = (V, E)$ be a connected, undirected graph and $v_1, v_2, v_3 \in V$. A vertex v is said to resolve the vertices v_1 and v_3 if the distance of v_1 from v_2 is different from distance of v_3 from v_2.

Definition 2.5. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of $V(G)$ and for any vertex $v \in V$, the (metric)representation of v with respect to W is the k-vector which is denoted and defined as $r(v/W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is called a resolving set for G if $r(v_1|W) = r(v_2|W)$ implies that $v_1 = v_2$ for all $v_1, v_2 \in V(G)$.

Definition 2.6. A resolving set of minimum cardinality for a graph G is called a minimum resolving set. A minimum resolving set is usually called a basis for G. The minimum cardinality of a resolving set of G is called the metric dimension of G and is denoted by $dim(G)$.

Definition 2.7. For two graphs G and H, the lexicographic product $G \wr H$ of H by G is obtained from G by replacing each vertex of G by a copy of H.

Theorem 2.8. [5] A connected graph G of order $n > 2$ has dimension $n - 1$ if and only if $G = K_n$.

Theorem 2.9. [5] A connected graph G of order n has dimension 1 if and only if $G = P_n$.

Theorem 2.10. [5] The metric dimension of C_n is $\dim C_n = 2$

3 Main Results

There are 112 graphs of order 6 exist. Among those graphs following are the only 3 graphs obtained as a lexicographic product of two graphs. That are $P_2 \wr P_3$, $P_3 \wr P_2$, $C_3 \wr P_2$ and $P_2 \wr C_3$. But $C_3 \wr P_2 \cong P_2 \wr C_3 \cong K_6$ as shown in Figure 1.

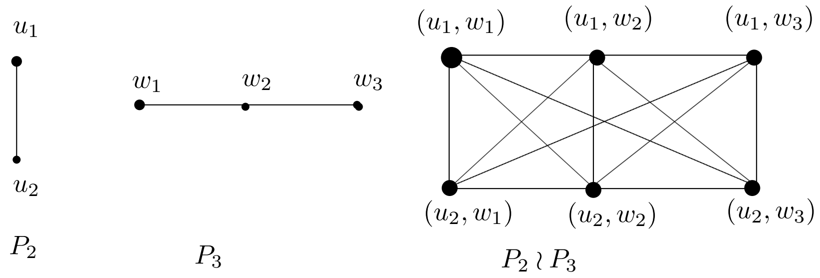

Figure 1(a)

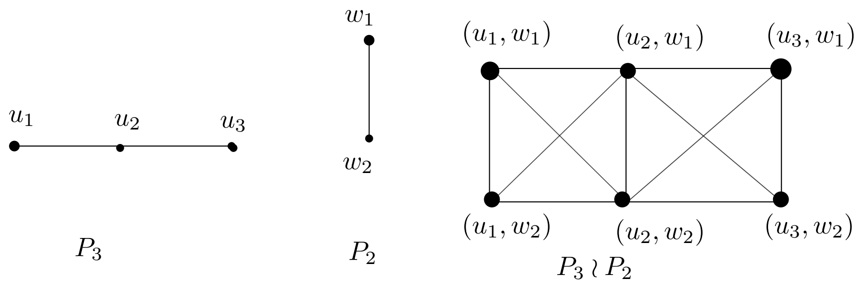

Figure 1(b)

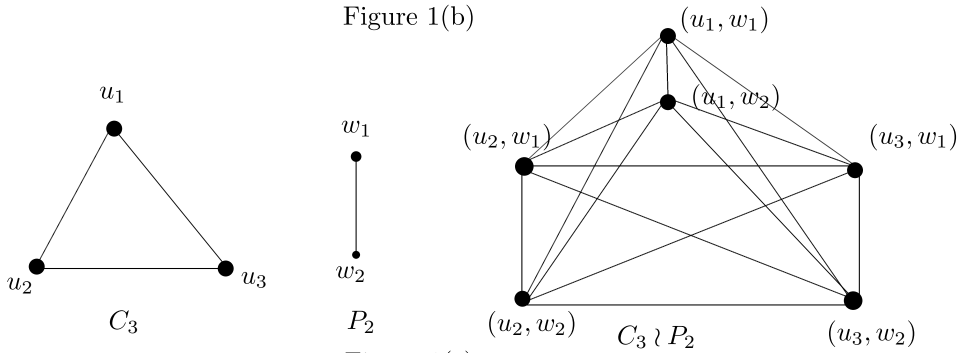

Figure 1(c)

Figure 1
Lexicographic products of order 6

In Figure 1(a), the set $W_1 = \{(u_1, w_1), (u_2, w_2), (u_2, w_1)\}$ form a basis. Since with respect W_1 all vertices of $V \setminus W_1$ have unique metric representation $\{(1, 1, 1), (2, 1, 1), (1, 1, 2)\}$. Therefore $dim(P_2 \wr P_3) = 3$. In $P_3 \wr P_2$, with respect to $W_2 = \{(u_2, w_1), (u_3, w_1), (u_1, w_2)\}$ every vertices in $V \backslash W_2$ have unique metric representation $\{(1, 2, 1), (1, 1, 1), (1, 1, 2)\}$. Hence $dim(G_2) = 3$. The lexicographic product $C_3 \wr P_2$ is same as $P_2 \wr C_3$ and is isomorphic to the complete graph K_6, therefore by Theorem 2.8, $dim(C_3 \wr P_2) = dim(P_2 \wr C_3) = 5$. In the above discussed three lexicographic products, it is clear that only connected bases exist. Considering the lexicographic product with six vertices, which is generated from a combination of graphs with two and three vertices as explained above, we consider the graphs $P_2 \wr P_3$, $P_3 \wr P_2$, $P_2 \wr C_3$, $C_3 \wr P_2$.

In the following discussions we consider the graph $P_2 \wr P_3$ as a case to discuss the fractal properties defined.

Definition 3.1. For the graphs P_2 and $P_2 \wr P_3$, define a function $f_1 : V(P_2 \wr P_3) \to V(P_2)$ as $f_1(v, w) = v$ and $f_1 : V(P_2 \wr P_3) \to V(P_3)$ as $g_1(v, w) = w$ are egamorphisms.

According to the definition, $f_1(u_1, w_3) = 1$, $f_1(u_1, w_4) = 1$, $f_1(u_1, w_5) = 1$ and $f_1(u_2, w_3) = 2$, $f_1(u_2, w_4) = 2$, $f_1(u_2, w_5) = 2$. For all the pairs for which $f_1(v) \neq f_1(w)$, we have $[v, w]_{P_2 \wr P_3} = [f_1(v), f_1(w)]_{P_2}$. Thus f_1 is an egamorphism from $P_2 \wr P_3$ to P_2. Similarly, $g_1(u_1, w_3) = 3$ and $g_1(u_2, w_3) = 3$, $g_1(u_1, w_4) = 4$ and $g_1(u_2, w_4) = 4$, $g_1(u_1, w_5) = 5$ and $g_1(u_2, w_5) = 5$. By the same argument the function g_1 from $P_2 \wr P_3$ to P_3 is an egamorphism.

Figure 1: $f^2_{P_3}$

Figure 2: zeros of $f^3_{P_3}$

Using the Similar steps as explained above, we can define a function $f_2 : V(P_3 \wr P_2) \to V(P_3)$ as $f_2(v,w) = v$ and $f_2 : V(P_3 \wr P_2) \to V(P_2)$ as $g_2(v,w) = w$ are egamorphisms.

Similarly we can define for the graphs C_3 and $C_3 \wr P_2$, a function f_3 as $f_3 : V(C_3 \wr P_2) \to V(P_2)$ as $f_3(v,w) = v$ and $g_3 : V(P_2 \wr C_3) \to V(C_3)$ as $g_3(v,w) = w$ are egamorphisms. Also $f_4 : V(C_3 \wr P_2) \to V(C_3)$ as $f_4(v,w) = v$ and $g_4 : V(C_3 \wr P_2) \to V(P_2)$ as $g_4(v,w) = w$ are egamorphisms.

Proposition 3.2. The graph $G = P_3$ is a fractal graph with the fractal factor P_2.

Proof . We try to give a characterisation of a fractal graph in terms of the lexicographic product for the graph $P_2 \wr P_3$. As per the definition we need a graph Γ such that $G \simeq \Gamma \wr G$. Let us consider the set $P = f(V(P_2 \wr P_3))$. We define the set $\pi(f) = \{f^{-1}(p) : p \in P\}$. Then $\pi(f) = \{f^{-1}(u_1), f^{-1}(u_2)\}$ is a modular partition of $P_2 \wr P_3$ and the function $f/\pi(f) : P_2 \wr P_3/\pi(f) \to P_2 \wr P_3$ is an isomorphism from $P_2 \wr P_3/\pi(f)$ onto $P_2 \wr P_3$.

To prove that a graph $G = P_3$ is a fractal graph, we have to find a $\Gamma = P_2$ such that $P_3 \simeq P_2 \wr P_3$. The egamorphism $f : V(P_2 \wr P_3) \to V(P_2)$ defined above induces an isomorphism $f/\pi(f)$ from $P_2 \wr P_3/\pi(f)$ onto $P_2 \wr P_3$. Thus the graph P_3 is a fractal graph and its fractal factor is P_2. □

In the similar way the results can be discussed for the other two graphs under consideration. That is The graph $G = P_2$ is a fractal graph with the fractal factor P_3. The graph $G = C_3$ is a fractal graph with the fractal factor P_2 and The graph $G = P_2$ is a fractal graph with the fractal factor C_3...

4 Independence Fractals of Fractal Graphs

From the proposition 3.2, P_3 is a fractal graph with the fractal factor P_2. Now our aim is to find independence fractal of P_2, P_3 and $P_2 \wr P_3$ and compare. The independence polynomial of P_2 is $1 + 2x$ and its independence fractal is $\{0\}$. Consider the graph P_3, path of 3 vertices. The reduced independence polynomial of P_3 is $x^2 + 3x$ and the roots are $\{0, -3\}$ Reduced independence polynomial of G^2 is

$$f^2_{P_3} = x^4 + 6x^3 + 12x^2 + 9x$$

zeros are {0,-3,-1.5+0.8660254037,-1.5-0.8660254037} 1. $f^3_{P_3} = x^8 + 12x^7 + 60x^6 + 162x^5 + 255x^4 + 234x^3 + 117x^2 + 27x$ 2

$f^4_{P_3} = x^{16} + 24x^{15} + 264x^{14} + 1764x^{13} + 7998x^{12} + 26028x^{11} + 62694x^{10} + 113562x^9 + 155532x^8 + 160524x^7 + 123354x^6 + 69012x^5 + 27090x^4 + 7020x^3 + 1080x^2 + 81x$ 3

$f^5_{P_3} = x^{32} + 48x^{31} + 1104x^{30} + 16200x^{29} + 170364x^{28} + 1367352x^{27} + 8709372x^{26} + 45196164x^{25} + 194659260x^{24} + 705275640x^{23} + 2171029500x^{22} + 5719669200x^{21} + 12964837320x^{20} + 25376373360x^{19} + 42985699164x^{18} + 63077397138x^{17}$

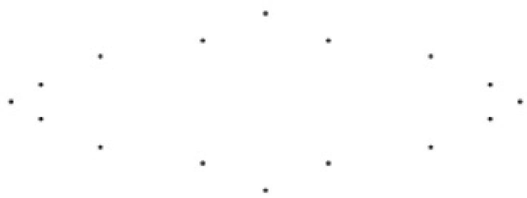

Figure 3: zeros of $f^4_{P_3}$

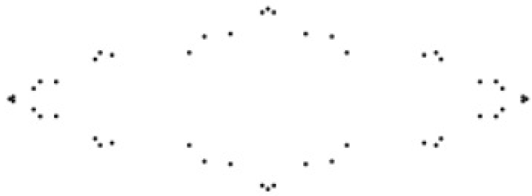

Figure 4: zeros of $f^5_{P_3}$

$+\ 80167328355x^{16} + 88129510128x^{15} + 83592981000x^{14} + 68160238128x^{13} + 47533167702x^{12} + 28162200528x^{11} + 14054267070x^{10} + 5843229030x^9 + 1995712704x^8 + 549874440x^7 + 119344806x^6 + 19758816x^5 + 2384910x^4 + 196020x^3 + 9801x^2 + 243x$ 4

$f^6_{P_3} = x^{64}+96x^{63}+4512x^{62}+138384x^{61}+3114744x^{60}+54859248x^{59}+787288248x^{58}+9465398856x^{57}+97284540936x^{56}+867969735408x^{55}+6803390274840x^{54}+47301219820800x^{53}+293996733240960x^{52}+1644168788246016x^{51}+8318329456894344x^{50}+...+4299232733079024600x^{15}+1088670520304593740x^{14}+244071483555679020x^{13}+48072209536084050x^{12}+8242879270340016x^{11}+1217238760267974x^{10}+152800018311474x^9+16046664935904x^8+1381943626668x^7+95133665214x^6 + 5060726748x^5 + 198480051x^4 + 5351346x^3 + 88452x^2 + 729x$ 5

$f^7_{P_3} = x^{128}+192x^{127}+18240x^{126}+1143072x^{125}+53157360x^{124}+1956526560x^{123}+59365133424x^{122}+1527201462672x^{121}+34001329348176x^{120}+665471551534560x^{119}+11591811962314416x^{118}+...+81230895019296631818x^{10}+3295418716560322710x^9+112450270038475329x^8+3162384373387500x^7+71412759548046x^6+1251240607386x^5+16221458925x^4+145017054x^3 + 796797x^2 + 2187x$ 6

Next consider the graph $P_2 \wr P_3$, path of 3 vertices. The reduced independence polynomial of $P_2 \wr P_3$ is $2x^2 + 6x$ and the roots are $\{0, -3\}$ Reduced independence polynomial of $(P_2 \wr P_3)^2$ is

$$f^2_{P_2 \wr P_3} = 8x^4 + 48x^3 + 84x^2 + 36x$$

zeros are {0,-3,-1.5+0.8660254037,-1.5-0.8660254037} 7
$f^3_{P_2 \wr P_3} = 128x^8 + 1536x^7 + 7296x^6 + 17280x^5 + 21072x^4 + 12384x^3 + 3096x^2 + 216x$ 8

$f^4_{P_2 \wr P_3} = 32768x^{16} + 786432x^{15} + 8454144x^{14} + 53673984x^{13} + 223420416x^{12} + 640106496x^{11} + 1289834496x^{10} + 1837043712x^9 + 1835721984x^8 + 1264131072x^7 + 582656256x^6 + 171673344x^5 + 29996640x^4 + 2749248x^3 + 111888x^2 + 1296x$ 9

Figure 5: zeros of $f^6_{P_3}$

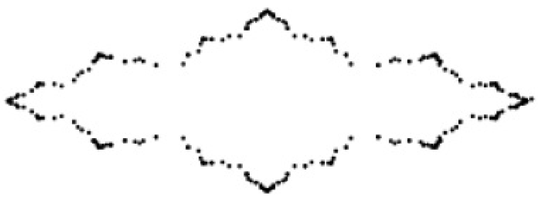

Figure 6: zeros of $f^7_{P_3}$

Figure 7: $f^2_{P_2 \wr P_3}$

$$f^5_{P_2 \wr P_3} = 2147483648x^{32} + 103079215104x^{31} + 2345052143616x^{30} + 33629593927680x^{29} + 341073016651776x^{28} + 2601790256185344x^{27} + 15499768708988928x^{26} + 73911918787559424x^{25} + 286898865724981248x^{24} + 917036565421621248x^{23} + 2432710724749885440x^{22} + 5382984053727166464x^{21} + 9963162798051557376x^{20} + 15438884485315166208x^{19} + 20015873232029614080x^{18} + 21663088583601291264x^{17} + 19501885484129648640x^{16} + 14528838842766065664x^{15} + 8897822396572631040x^{14} + 4441686131291848704x^{13} + 1788091567492055040x^{12} + 572801213100883968x^{11} + 143587145204932608x^{10} + 27581251970801664x^{9} + 3954820455654912x^{8} + 409733633378304x^{7} + 29435235909120x^{6} + 1386964062720x^{5} + 39469930560x^{4} + 596522880x^{3} + 4030560x^{2} + 7776x \quad 10$$

$$f^6_{P_2 \wr P_3} = 9223372036854775808x^{64} + 885443715538058477568x^{63} + 41394493701404233826304x^{62} + 1255780549561851435810816x^{61} + 27794382602016871818461184x^{60} + 478431968837375800570281984x^{59} + 6667139200089219482039353344x^{58} + 77311637987997737525698363392x^{57} + 760980840717088074268435021824x^{56} + 6454085254168595219931941830656x^{55} + ... + 23787679445219765178040320 0x^{11} + 95368467533441846898032 64x^{10} + 2959376679674078046289 92x^{9} + 6912504047062167487488x^{8} + 117457846074269282304x^{7} + 1391339527620940800x^{6} + 10853279541365760x^{5} + 51281895070080x^{4} + 128945675520x^{3} + 145115712x^{2} + 46656x \quad 11$$

$$f^7_{P_2 \wr P_3} = 170141183460469231731687303715884105728x^{128} + 32667107224410092492483962313449748299776x^{127} + 3095208409512856263662855429199363651403776x^{126} + 192940102044172108783733402413812575895552000x^{125} + 8900131925501413680454057339699115722876649472x^{124} + 324022069977248890691423948487280463230210670592x^{123} + 9696579501064061946575326207136595557780828454912x^{122} + 245298178738242656713799374698776237375206176325632x^{121} + 5354104711575856642198763390125764574148953934659584x^{120} + ... + 30134075621268019060292752834 56x^{9} + 116871822050604784073682186 24x^{8} + 330106020548293722926039 04x^{7} + 650550398121276201676 80x^{6} + 84483726112429885440x^{5} + 6648918885719884 8x^{4} + 27855748689408x^{3} + 5224258944x^{2} + 279936x \quad 12$$

To check the connectedness of independence fractal of $P_2 \wr P_3$, the critical point of $f_{P_2 \wr P_3}$ is -3/2 and its forward orbit is {-1.5, -4.5, 13.5, 445.5, 399613.5,...} which is unbounded. Hence the independence fractal of $P_2 \wr P_3$ is totally disconnected.

Proposition 4.1. Fractal graph P_3, its fractal factor P_2 and $P_2 \wr P_3$ have totally disconnected independence fractals.

Figure 8: zeros of $f^3_{P_2 \wr P_3}$

Figure 9: zeros of $f^4_{P_2 \wr P_3}$

Figure 10: zeros of $f^5_{P_2 \wr P_3}$

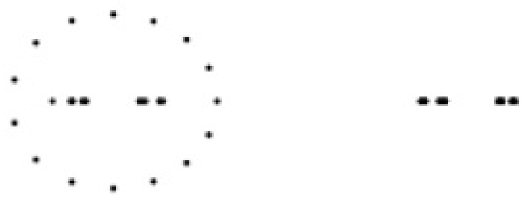

Figure 11: zeros of $f^6_{P_2 \wr P_3}$

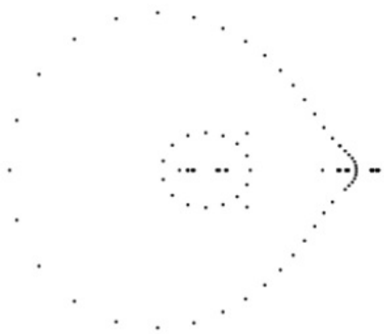

Figure 12: zeros of $f^7_{P_2 \wr P_3}$

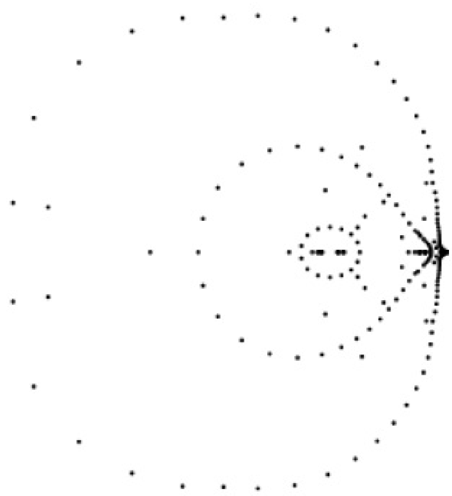

Figure 13: zeros of $f^8_{P_2 \wr P_3}$

Conclusion

Here we have tried to analyse the fractal properties of the lexicographic products of graphs with six vertices. Also determined the metric dimension of the fractal factor of the lexicographic products with six vertices which are detected using the egamorphisms.

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