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Study of Langmuir waves for Zakharov equation using Sardar sub-equation method

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Abstract

The Zakharov equation is a nonlinear plasma fluid model, used for ion-acoustic waves in a magnetized plasma. In the present study, Langmuir waves of the dimensionless Zakharov equation are investigated by using the Sardar-subequation method. The obtained solutions lead to a variety of exact solutions in the form of dark, bright, periodic singular, singular and combined dark-bright type solutions. These acquired solutions are depicted graphically by the 2D, contour and 3D plots which show the physical behaviour of obtained solutions. All the graphs confirm the validity of the obtained solutions. These types of solutions have a large range of applications in mathematical and applied sciences..

Keywords: Sardar-subequation method (SSM), Dimensionless Zakharov equation, Traveling wave solution
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1 Introduction

Non-Linear equations play imperative role in the domain of engineering and sciences such as solid state physics, cell recognition, fluid mechanics, plasma physics, fiber optics, biology, heat flow occurrence, quantum mechanics, electricity and condensed matter physics [5, 6, 7, 11, 14, 15, 21, 22, 23, 30]. The applications of soliton solutions of NLEs play important role in different fields like neutral physics and diffusion process. Traveling wave solution is a wave that proceed in a specific direction with addition of retaining a fixed shape and plays an important role in physical models. In this regard, many efficient methods such as the $(\frac{G'}{G})$-expansion technique [4], the sine-cosine technique [34], the tanh-function technique [20], the first integral technique [13], extended tanh [35, 36], Hirota's direct [38, 39], functional variable [8, 12], Exp$[-\varphi(\xi)]$-Expansion [31, 33], Jacobi elliptic ansatz [2, 18], sub equation [3, 19] and [9, 22, 29, 32, 41] have been established .

In this paper, propagation of Langmuir waves for Zakharov equation by using through traveling wave solutions is investigated. The Langmuir waves is actually the illustration of the electron plasma waves and is used to generate instabilities during fluctuations of electrons [10]. Zakharov system is introduced by Zakharov [37, 40] and is generalized

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form of Korteweg-de-Vries equation [16]. This equation explains the relation between low frequency acoustic waves and high frequency Langmuir waves which has many applications in high-energy physical processes, environmental science, fluid mechanics and electronics [17]. The Zakharov equation is read as [1]

$$\begin{aligned} \iota\chi_t + \chi_{xx} + \mu\Pi(|\chi|^2)\chi &= \chi\psi, \\ \psi_{tt} - \psi_{xx} &= \left(|\chi|^{2c}\right)_{xx}, \end{aligned} \tag{1.1}$$

where μ is arbitrary constant, $\chi = \chi(x,t)$ represents the envelope of the high-frequency electric field and $\psi = \psi(x,t)$ represents the plasma density. Using following values

$$\Pi(|\chi|^2) = |\chi|^2,\ c = 1,$$

in equation (1), it becomes

$$\begin{aligned} \iota\chi_t + \chi_{xx} + \mu|\chi|^2\chi &= \chi\psi, \\ \psi_{tt} - \psi_{xx} &= (|\chi|^2)_{xx}. \end{aligned} \tag{1.2}$$

In this study, we implement the SSM [28] to provide the brief classification of traveling wave solutions for Zakharov equation. This method has been used to get the solutions in the form of hyperbolic and trigonometric solutions [25, 27].

This paper is organized as follows, in section 2, the Sardar-subequation method is discussed. In section 3, traveling wave solutions of dimensionless Zakharov Equation are presented. Graphical representation is explained in section 4 and the conclusion is given in section 5. The appendix gives the details of calculations of some lengthy solutions.

2 The Sardar-subequation Method

The Sardar-subequation method [28] is regarded as one of general from which, under specific circumstances, different techniques can be generated such as the first integral technique and the functional variable method. Consider the following NLEs for $\phi(x,t)$

$$\Re(\phi,\ \phi_t,\ \phi_x,\ \phi_{tt},\ \phi_{xx}, ...) = 0, \tag{2.1}$$

where $\Re$ is a polynomial in ϕ and its partial derivatives. Using the following traveling wave transformation

$$\phi(x,t) = \phi(\xi), \quad \xi = x - \beta_1 t, \tag{2.2}$$

in equation (2.1), where $\beta_1 \neq 0$ is constant, following ODE w.r.t ξ is generated

$$Q(\phi,\ \phi^{'},\ \phi^{''},\ \phi^{'''}, ...) = 0, \tag{2.3}$$

where $\phi = \phi(\xi), \phi^{'} = \frac{d\phi}{d\xi}, \phi^{''} = \frac{d^2\phi}{d\xi^2}, \ldots$. Solution of equation (2.3) has following form

$$\phi(\xi) = \sum_{i=0}^{n} \varpi_i \Psi^i(\xi), \tag{2.4}$$

where coefficients ϖ_i, $(i = 0, 1, \cdots, n)$ to be find out with $(\varpi_n \neq 0)$ and $\Psi(\xi)$ is the solution of the equation

$$(\Psi'(\xi))^2 = \rho + \alpha\Psi^2(\xi) + \Psi^4(\xi). \tag{2.5}$$

where ρ and α are the real constants. The solutions of equation (2.5) are given as:

Case I: If $\rho = 0$ and $\alpha > 0$, then

$$\Psi_1^{\pm}(\xi) = \pm\sqrt{-pq\alpha}\ sech_{pq}(\sqrt{\alpha}\xi), \qquad \text{and} \qquad \Psi_2^{\pm}(\xi) = \pm\sqrt{pq\alpha}\ csch_{pq}(\sqrt{\alpha}\xi),$$

where $\sec h_{pq}(\xi) = \frac{2}{pe^{\xi}+qe^{-\xi}}$, $\csc h_{pq}(\xi) = \frac{2}{pe^{\xi}-qe^{-\xi}}$.

Case II: If $\rho = 0$ and $\alpha < 0$, then

$$\Psi_3^{\pm}(\xi) = \pm\sqrt{-pq\alpha}\ \sec_{pq}(\sqrt{-\alpha}\xi), \qquad \text{and} \qquad \Psi_4^{\pm}(\xi) = \pm\sqrt{-pq\alpha}\ \csc_{pq}(\sqrt{-\alpha}\xi),$$

where $\sec_{pq}(\xi) = \frac{2}{pe^{\iota\xi}+qe^{-\iota\xi}}$, $\csc_{pq}(\xi) = \frac{2\iota}{pe^{\iota\xi}-qe^{-\iota\xi}}$.

Case III: If $\rho = \frac{\alpha^2}{4}$ and $\alpha < 0$, then

$$\Psi_5^{\pm}(\xi) = \pm\sqrt{\frac{-\alpha}{2}}\tanh_{pq}(\sqrt{\frac{-\alpha}{2}}\xi), \qquad \Psi_6^{\pm}(\xi) = \pm\sqrt{\frac{-\alpha}{2}}\coth_{pq}(\sqrt{\frac{-\alpha}{2}}\xi),$$

$$\Psi_7^{\pm}(\xi) = \pm\sqrt{\frac{-\alpha}{2}}(\tanh_{pq}(\sqrt{-2\alpha}\xi) \pm \sqrt{-pq}\ \sec h_{pq}(\sqrt{-2\alpha}\xi)),$$

$$\Psi_8^{\pm}(\xi) = \pm\sqrt{\frac{-\alpha}{2}}(\coth_{pq}(\sqrt{-2\alpha}\xi) \pm \sqrt{pq}\ \csc h_{pq}(\sqrt{-2\alpha}\xi)),$$

$$\Psi_9^{\pm}(\xi) = \pm\sqrt{\frac{-\alpha}{8}}(\tanh_{pq}(\sqrt{\frac{-\alpha}{8}}\xi) + \coth_{pq}(\sqrt{\frac{-\alpha}{8}}\xi)),$$

where $\tanh_{pq}(\xi) = \frac{pe^{\xi}-qe^{-\xi}}{pe^{\xi}+qe^{-\xi}}$, $\coth_{pq}(\xi) = \frac{pe^{\xi}+qe^{-\xi}}{pe^{\xi}-qe^{-\xi}}$.

Case IV: If $\rho = \frac{\alpha^2}{4}$ and $\alpha > 0$, then

$$\Psi_{10}^{\pm}(\xi) = \pm\sqrt{\frac{\alpha}{2}}\tan_{pq}(\sqrt{\frac{\alpha}{2}}\xi),$$

$$\Psi_{11}^{\pm}(\xi) = \pm\sqrt{\frac{\alpha}{2}}\cot_{pq}(\sqrt{\frac{\alpha}{2}}\xi),$$

$$\Psi_{12}^{\pm}(\xi) = \pm\sqrt{\frac{\alpha}{2}}(\tan_{pq}(\sqrt{2\alpha}\xi) \pm \sqrt{pq}\ \sec_{pq}(\sqrt{2\alpha}\xi)),$$

$$\Psi_{13}^{\pm}(\xi) = \pm\sqrt{\frac{\alpha}{2}}(\cot_{pq}(\sqrt{2\alpha}\xi) \pm \sqrt{pq}\ \csc_{pq}(\sqrt{2\alpha}\xi)),$$

$$\Psi_{14}^{\pm}(\xi) = \pm\sqrt{\frac{\alpha}{8}}(\tan_{pq}(\sqrt{\frac{\alpha}{8}}\xi) + \cot_{pq}(\sqrt{\frac{\alpha}{8}}\xi)),$$

where $tan_{pq}(\xi) = -\iota\frac{pe^{\iota\xi}-qe^{-\iota\xi}}{pe^{\iota\xi}+qe^{-\iota\xi}}$, $cot_{pq}(\xi) = \iota\frac{pe^{\iota\xi}+qe^{-\iota\xi}}{pe^{\iota\xi}-qe^{-\iota\xi}}$. This method begins by finding out n with the help of the balance principle. When n is obtained, the solution which is predicted and it's necessary derivatives together with equation (2.5) are placed into equation (2.3). Then by taking all the coefficient of power of $\Psi(\xi)$ equal to zero, a system of algebraic equations is obtained and solved for α and $\varpi_i s$. When $\varpi_i s$ and α are find out the solutions are generated by using these parameters.

3 Traveling Wave Solutions of the Dimensionless Zakharov Equation

In this section, solutions are derived by applying SSM. The obtained solutions are very convenient and by giving the different conditions of parameters, we get different types of solutions.

Using these transformation $\chi = \phi e^{\iota\theta}$, $\phi = \phi(\xi)$, $\xi = x - \beta_1 t$, $\theta = -\beta_2 x + \beta_3 t + \beta_4$ in equation (1.2) where β_i $(i = 1, 2, 3, 4)$ are fixed arbitrary constants, it becomes

$$\begin{aligned}(\iota\beta_1\phi' - \beta_3\phi + \phi'' - 2\iota\beta_2\phi' - \beta_2^2\phi + \mu\phi^3 - \phi\psi)e^{\iota\theta} &= 0,\\ (\beta_1^2 - 1)\psi'' - (\phi^2)'' &= 0.\end{aligned} \tag{3.1}$$

For $\beta_1 + 2\beta_2 = 0$, integrating the second equation in equation (3.1) twice, assuming constant of integration zero then utilizing the result in first equation of the same system, the following equation is generated

$$\phi'' - (\beta_3 + \beta_2^2)\phi + (\mu - \frac{1}{\beta_1^2 - 1})\phi^3 = 0. \tag{3.2}$$

Equating the highest nonlinear order with the highest order derivative $[\phi'' : \phi^3]$ in equation (3.2), we obtain $n = 1$ and the equation (6) will become

$$\phi(\xi) = \varpi_0 + \varpi_1 \Psi^1(\xi). \tag{3.3}$$

Substituting equation (3.3) into equation (3.2) with equation (2.5) and equating all the coefficient of $\Psi(\xi)$ to zero, the following system of algebraic equations is obtained

$$-\tfrac{\varpi_0^3}{\beta_1^2-1} - \beta_2^2\varpi_0 - \beta_3\varpi_0 + \mu\varpi_0^3 = 0, \quad \alpha\varpi_1 - \beta_2^2\varpi_1 - \beta_3\varpi_1 - \tfrac{3\varpi_0^2\varpi_1}{\beta_1^2-1} + 3\mu\varpi_0^2\varpi_1 = 0,$$
$$3\mu\varpi_0\varpi_1^2 - \tfrac{3\varpi_0\varpi_1^2}{\beta_1^2-1} = 0, \quad -\tfrac{\varpi_1^3}{\beta_1^2-1} + \mu\varpi_1^3 + 2\varpi_1 = 0.$$

Solving this system of algebraic equations, we get

$$\varpi_0 = 0, \ \ \varpi_1 = \pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}, \ \ \alpha = \beta_2^2 + \beta_3. \tag{3.4}$$

Some of the traveling wave solution are given as

Case I: If $\beta_2^2 + \beta_3 > 0$ and $\rho = 0$ then

$$\chi_1(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\sqrt{(\beta_2^2+\beta_3)(-p)q}\right)\operatorname{sech}_{pq}\left(\sqrt{\beta_2^2+\beta_3}\xi\right), \tag{3.5}$$

$$\chi_2(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\sqrt{(\beta_2^2+\beta_3)\,pq}\right)\operatorname{csch}_{pq}\left(\sqrt{\beta_2^2+\beta_3}\xi\right). \tag{3.6}$$

Case II: If $\beta_2^2 + \beta_3 < 0$ and $\rho = 0$ then

$$\chi_3(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\sqrt{(\beta_2^2+\beta_3)(-p)q}\right)\sec_{pq}\left(\sqrt{-\beta_2^2-\beta_3}\xi\right), \tag{3.7}$$

$$\chi_4(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\sqrt{(\beta_2^2+\beta_3)(-p)q}\right)\csc_{pq}\left(\sqrt{-\beta_2^2-\beta_3}\xi\right). \tag{3.8}$$

Case III: If $\beta_2^2 + \beta_3 < 0$ and $\rho = \frac{(\beta_2^2+\beta_3)^2}{4}$ then

$$\chi_5(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\tanh_{pq}\left(\frac{\sqrt{-\beta_2^2-\beta_3}\xi}{\sqrt{2}}\right), \tag{3.9}$$

$$\chi_6(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\coth_{pq}\left(\frac{\sqrt{-\beta_2^2-\beta_3}\xi}{\sqrt{2}}\right). \tag{3.10}$$

Case IV: If $\beta_2^2 + \beta_3 > 0$ and $\rho = \frac{(\beta_2^2+\beta_3)^2}{4}$ then

$$\chi_{10}(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\tan_{pq}\left(\frac{\sqrt{\beta_2^2+\beta_3}\xi}{\sqrt{2}}\right), \tag{3.11}$$

$$\chi_{11}(x,t) = e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu - \mu - 1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\cot_{pq}\left(\frac{\sqrt{\beta_2^2+\beta_3}\xi}{\sqrt{2}}\right). \tag{3.12}$$

4 Graphical Representation

This section contains some graphs of solutions obtained by SSM using Maple 18. The 3d, contour and 2d plot of traveling wave solutions $\chi_1(x,t), \chi_3(x,t)$, $\chi_5(x,t)$, $\chi_6(x,t)$, $\chi_7(x,t)$, $\chi_{14}(x,t)$ are shown in figure 1-5. It is noted that the SSM method has ability to develop different types of soliton solutions such as bright, periodic, dark, singular, combined dark-bright, combined dark-singular soliton solutions. The physical interpretation of sketched solutions are as follow:

• The figure 1 presents the bright soliton solutions of Eq.(3.5) in 3d, contour and 2d plots with the parameters $\beta_1 = 2, \beta_2 = 0.5, \beta_3 = 1, \beta_4 = 0.75, \mu = 0.05, p = 0.8, q = 0.98$.

• Figure 2 represents the periodic wave solutions of Eq. (3.7) under the parameters $\beta_1 = 2, \beta_2 = 0.5, \beta_3 = -1, \beta_4 = 0.75, \mu = 0.05, p = 0.8, q = 0.98$. in 3d, contour and 2d plots.

• The dark soliton solution for Eq. (3.9) is depicted in figure 3 along parameters $\beta_1 = 2, \beta_2 = 0.5, \beta_3 = -1, \beta_4 = 0.75, \mu = 0.05, p = 0.8, q = 0.98$.

• The 3d, contour and 2d view of Eq. (3.10) is given in figure 4 which exhbits the singular soliton for the values of $\beta_1 = 2, \beta_2 = 0.5, \beta_3 = -1, \beta_4 = 0.75, \mu = 0.05, p = 0.8, q = 0.98$.

• The graph of Eq. (3.12) shows combined dark-bright soliton solutions for the values of $\beta_1 = 2, \beta_2 = 0.5, \beta_3 = -1, \beta_4 = 0.75, \mu = 0.05, p = 0.8, q = 0.98$ as shown in 3d, contour and 2d plot of figure 5.

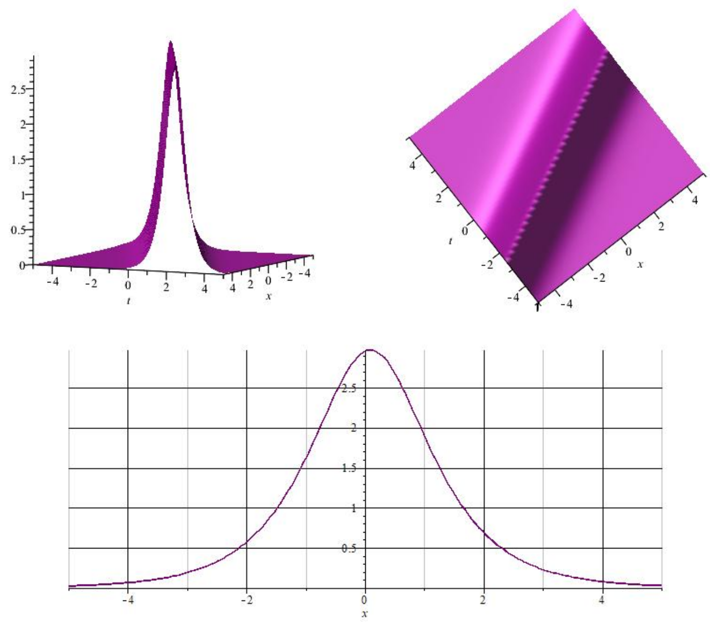

Figure 1: Dynamical behaviors of solution (3.5) (a) 3D graph with $t \leq 5, -5 \leq x$.(b) Contour graph with $t \leq 5, -5 \leq x$.(c) 2D graph with $t = 0$ and $-5 \leq x \leq 5$.

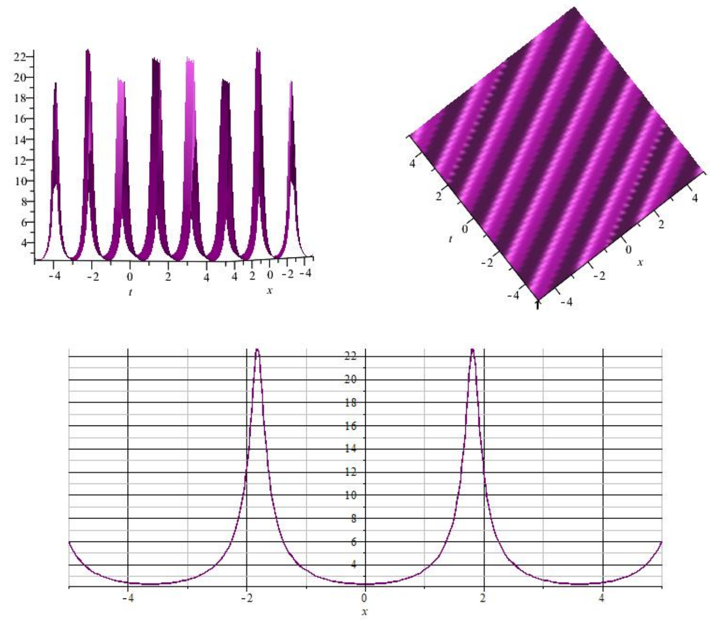

Figure 2: Dynamical behaviors of solution (3.7) (a) 3D graph with $t \leq 5, -5 \leq x$.(b) Contour graph with $t \leq 5, -5 \leq x$.(c) 2D graph with $t = 0$ and $-5 \leq x \leq 5$.

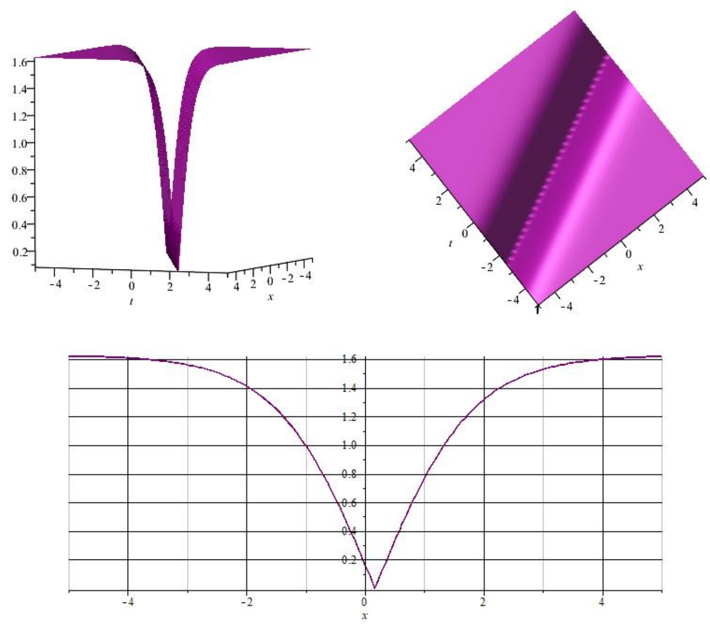

Figure 3: Dynamical behaviors of solution (3.9) (a) 3D graph with $t \leq 5, -5 \leq x$.(b) Contour graph with $t \leq 5, -5 \leq x$.(c) 2D graph with $t = 0$ and $-5 \leq x \leq 5$.

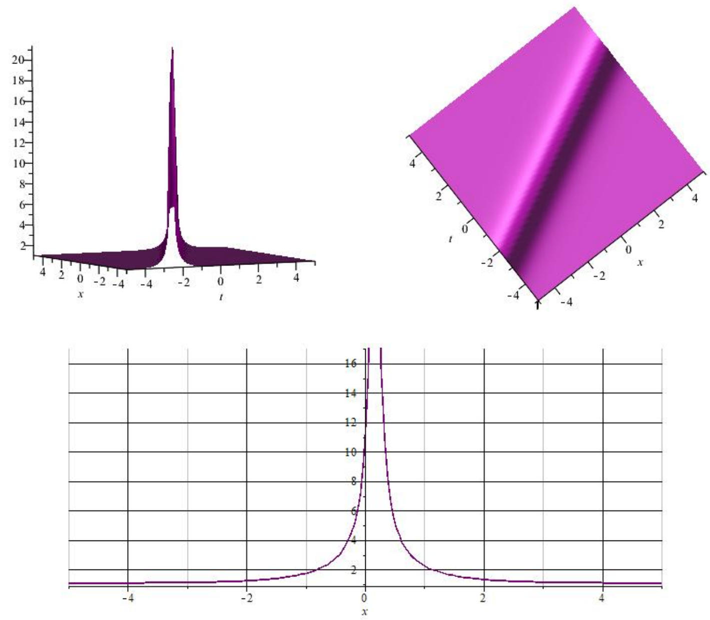

Figure 4: Dynamical behaviors of solution (3.10) (a) 3D graph with $t \leq 5, -5 \leq x$.(b) Contour graph with $t \leq 5, -5 \leq x$.(c) 2D graph with $t = 0$ and $-5 \leq x \leq 5$.

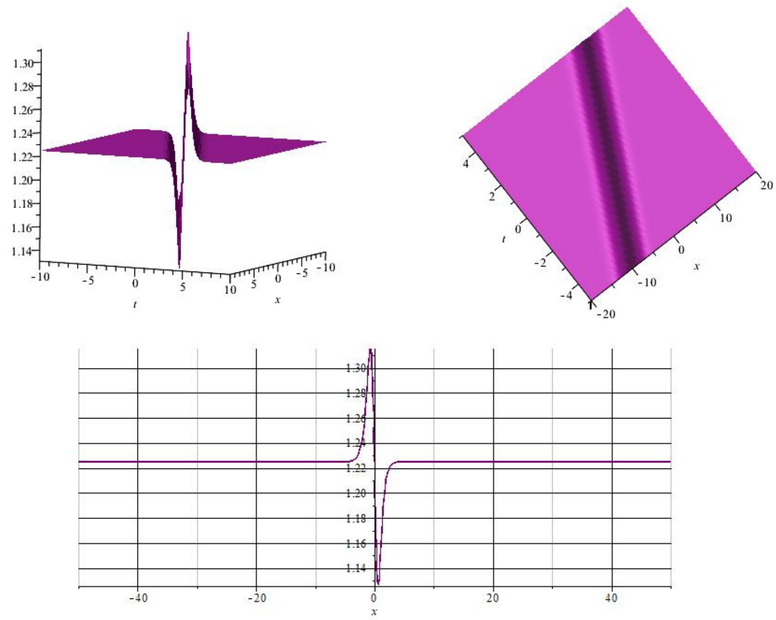

Figure 5: Dynamical behaviors of solution (3.11) (a) 3D graph with $-10 \leq x \leq 10, -10 \leq t \leq 10$.(b) Contour graph with $-10 \leq x \leq 10, -10 \leq t \leq 1-$.(c) 2D graph with $t = 0$ and $-40 \leq x \leq 40$.

5 Conclusion

Sardar subequation method (SSM) which is one of the powerful and effective technique is used to analyze the relation between (low and high) frequency and Langmuir waves of Zakharov equation in plasma. In dimensionless Zakharov equation, dark, bright, periodic singular, singular, combined dark-bright and combined dark-singular solutions are derived. To add more physical meaning of these solutions some 2D, 3D and contour graphs are presented. It has been observed that the method is powerful, easy and effective in finding the solutions of nonlinear PDEs. The conclusions of the present work provide much support to future work.

Appendix

Case III: If $\beta_2^2+\beta_3<0$ and $\rho=\frac{(\beta_2^2+\beta_3)^2}{4}$ then

$$\chi_7(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\left(\tanh_{pq}\left(\sqrt{2}\sqrt{-\beta_2^2-\beta_3}\xi\right)\right)$$
$$\pm e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\left(\sqrt{-pq}\,\mathrm{sech}_{pq}\left(\sqrt{2}\sqrt{-\beta_2^2-\beta_3}\xi\right)\right), \tag{5.1}$$

$$\chi_8(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\left(\coth_{pq}\left(\sqrt{2}\sqrt{-\beta_2^2-\beta_3}\xi\right)\right)$$
$$\pm e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{\sqrt{2}}\right)\left(\sqrt{-pq}\,\mathrm{csch}_{pq}\left(\sqrt{2}\sqrt{-\beta_2^2-\beta_3}\xi\right)\right), \tag{5.2}$$

$$\chi_9(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{-\beta_2^2-\beta_3}}{2\sqrt{2}}\right)\left(\coth_{pq}\left(\frac{\sqrt{-\beta_2^2-\beta_3}\xi}{2\sqrt{2}}\right)+\tanh_{pq}\left(\frac{\sqrt{-\beta_2^2-\beta_3}\xi}{2\sqrt{2}}\right)\right). \tag{5.3}$$

Case IV: If $\beta_2^2+\beta_3>0$ and $\rho=\frac{(\beta_2^2+\beta_3)^2}{4}$ then

$$\chi_{11}(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\cot_{pq}\left(\frac{\sqrt{\beta_2^2+\beta_3}\xi}{\sqrt{2}}\right), \tag{5.4}$$

$$\chi_{12}(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\left(\tan_{pq}\left(\sqrt{2}\sqrt{\beta_2^2+\beta_3}\xi\right)\right)$$
$$\pm e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\left(\sqrt{pq}\sec_{pq}\left(\sqrt{2}\sqrt{\beta_2^2+\beta_3}\xi\right)\right), \tag{5.5}$$

$$\chi_{13}(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\left(\cot_{pq}\left(\sqrt{2}\sqrt{\beta_2^2+\beta_3}\xi\right)\right)$$
$$\pm e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{\sqrt{2}}\right)\left(\sqrt{pq}\csc_{pq}\left(\sqrt{2}\sqrt{\beta_2^2+\beta_3}\xi\right)\right), \tag{5.6}$$

$$\chi_{14}(x,t)=e^{\iota\theta}\left(\pm\frac{\sqrt{2}\sqrt{1-\beta_1^2}}{\sqrt{\beta_1^2\mu-\mu-1}}\right)\left(\frac{\sqrt{\beta_2^2+\beta_3}}{2\sqrt{2}}\right)\left(\cot_{pq}\left(\frac{\sqrt{\beta_2^2+\beta_3}\xi}{2\sqrt{2}}\right)+\tan_{pq}\left(\frac{\sqrt{\beta_2^2+\beta_3}\xi}{2\sqrt{2}}\right)\right). \tag{5.7}$$

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