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Approximation properties of bivariate generalized Baskakov-Kantorovich operators

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Abstract

The purpose of this paper is to study the bivariate extension of the generalized Baskakov-Kantorovich operators and obtain results on the degree of approximation, Voronovskaja type theorems and their first order derivatives in polynomial weighted spaces. Furthermore, we illustrate the convergence of the bivariate operators to a certain function through graphics using Matlab algorithm. We also discuss the comparison of the convergence of the bivariate generalized Baskakov Kantorovich operators and the bivariate Szász-Kantorovich operators to the function through illustrations using Matlab.

Keywords: Rate of convergence, simultaneous approximation, bivariate modulus of continuity
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1 Introduction

The approximation of functions by linear positive operators is an interesting area of research in approximation theory. Many new sequences and classes of linear positive operators have been considered in the literature and extensively studied by researchers. After the integral modifications of Bernstein polynomials by Kantorovich and Durrmeyer, the Kantorovich and Durrmeyer variants of several sequences of linear positive operators were defined and studied.

In [11], Erençin defined the Durrmeyer type modification of generalized Baskakov operators introduced by Mihesan [15], as

$$L_n^a(f;x)=\sum_{k=0}^{\infty}W_{n,k}^a(x)\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}f(t)dt,\ x\geq 0,$$

where
$W_{n,k}^a(x)=e^{\frac{-ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}, P_k(n,a)=\sum_{i=0}^{k}\binom{n}{k}(n)_i a^{k-i}$, and $(n)_0=1, (n)_i=n(n+1)...(n+i-1)$ for $i\geq 1$ and

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discussed some approximation properties. Very recently, Agrawal et al. [4] studied the simultaneous approximation and approximation of functions having derivatives of bounded variation by these operators.

Inspired by the above work, in [3], we proposed the Kantorovich modification of generalized Baskakov operators for the function f defined on $C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma)$ as $t \to \infty$, for some $\gamma > 0\}$ as follows :

$$K_n^a(f;x) = (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,\ a \geq 0. \tag{1.1}$$

In the present paper, our aim is to consider the bivariate extension of these operators. In this direction, Stancu [20] first introduced new linear positive operators in two and several dimensional variables. After that Barbosu [8] studied the bivariate extension of Stancu generalization of $q-$Bernstein operators. Doğru and Gupta [10] constructed a bivariate generalization of the Meyer-Konig and Zeller operators based on $q-$integers while in [2] Agratini presented two-dimensional extension of some univariate positive approximation processes expressed by series.

Many papers were published on approximation by modified Szász-Mirakyan and Baskakov operators for functions of one or two variables cf. [5, 6, 12, 13, 18, 19, 25] which deal with convergence, degree of approximation and Voronovskaja type theorems as well as convergence of partial derivatives of these operators. Wafi and Khatoon [21] defined the generalized Baskakov operators for functions of two variables in polynomial and exponential weighted spaces and discussed the rate of convergence and direct results. Later, in [22], the convergence of first order derivatives of these operators and a Voronovskaja type theorem were studied. For the recent background knowledge in this direction, readers can refer [1, 14, 16, 17, 23, 24].

2 Construction of the operators

Let $I = [0,\infty) \times [0,\infty)$ and $\omega_\gamma(x) = (1+x^\gamma)^{-1}$ for $\gamma \in \mathbb{N}^0$(set of all non-negative integers). Further, for fixed $\gamma_1, \gamma_2 \in \mathbb{N}^0$, let $\omega_{\gamma_1,\gamma_2}(x,y) = \omega_{\gamma_1}(x)\omega_{\gamma_2}(y)$.

Then, for $f \in C_{\gamma_1,\gamma_2}(I) := \{f \in C(I) : \omega_{\gamma_1,\gamma_2}(x,y)f(x,y)$ is bounded and uniformly continuous on I$\}$, we define a bivariate extension of the operators (1.1) as follows:
$K_{n_1,n_2}^a(f;x,y)$

$$= (n_1+1)(n_2+1)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} W_{n_1,n_2,k_1,k_2}^a(x,y)\int_{\frac{k_2}{n_2+1}}^{\frac{k_2+1}{n_2+1}}\int_{\frac{k_1}{n_1+1}}^{\frac{k_1+1}{n_1+1}} f(u,v)du\,dv, \tag{2.1}$$

where

$$W_{n_1,n_2,k_1,k_2}^a(x,y) = \frac{x^{k_1}y^{k_2}p_{k_1}(n_1,a)p_{k_2}(n_2,a)e^{\frac{-ax}{1+x}}e^{\frac{-ay}{1+y}}}{k_1!k_2!(1+x)^{n_1+k_1}(1+y)^{n_2+k_2}}.$$

If $f \in C_{\gamma_1,\gamma_2}(I)$ and if $f(x,y) = f_1(x)f_2(y)$ for all $(x,y) \in I$, then

$$K_{n_1,n_2}^a(f(u,v);x,y) = K_{n_1}^a(f_1(u);x)K_{n_2}^a(f_2(v);y), \tag{2.2}$$

for $(x,y) \in I$ and $n_1, n_2 \in \mathbb{N}$. The sup norm on $C_{\gamma_1,\gamma_2}(I)$ is given by

$$\| f \|_{\gamma_1,\gamma_2} = \sup_{(x,y)\in I} |f(x,y)|\omega_{\gamma_1,\gamma_2}(x,y), f \in C_{\gamma_1,\gamma_2}(I). \tag{2.3}$$

For $f \in C_{\gamma_1,\gamma_2}(I)$, we define the modulus of continuity

$$\omega(f, C_{\gamma_1,\gamma_2}; t, s) := \sup_{0<h<t, 0<\delta<s} \| \Delta_{h,\delta}f(.,.) \|_{\gamma_1,\gamma_2},\ \ t,s \geq 0, \tag{2.4}$$

where $\Delta_{h,\delta}f(x,y) := f(x+h, y+\delta) - f(x,y)$ for $(x,y) \in I$ and $h,\delta > 0$. Moreover, for fixed $m \in \mathbb{N}$, let $C_{\gamma_1,\gamma_2}^m(I)$ be the space of all functions $f \in C_{\gamma_1,\gamma_2}(I)$ having the partial derivatives $\dfrac{\partial^k f}{\partial x^s \partial y^{k-s}} \in C_{\gamma_1,\gamma_2}(I), s = 1,2,...,k;\ k = 1,2,...,m.$
The paper is organized as follows:

In Section 3 of this paper, we give some definitions and auxiliary results. In Section 4, we prove the main results of this paper wherein we study the degree of approximation, Voronovskaja type theorems and simultaneous approximation of first order derivatives for bivariate Baskakov Kantorovich operators $K^a_{n_1,n_2}$. The section 5 is devoted to the illustrations of the convergence of the operators $K^a_{n_1,n_2}$ to a certain function and the comparison of convergence with bivariate Szász Kantorovich operators to the function using Matlab.

3 Auxiliary Results

Lemma 3.1. In [3], for the rth order central moment of K^a_n, defined as

$$u^a_{n,r}(x) := K^a_n((t-x)^r; x), \text{ we have}$$

(i) $u^a_{n,0}(x) = 1, u^a_{n,1}(x) = \frac{1}{n+1}\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right)$
and $u^a_{n,2}(x) = \frac{1}{(n+1)^2}\left\{(n+1)x^2 + (n-1)x + \frac{a^2x^2}{(1+x)^2} + 2ax\left(\frac{1-x}{1+x}\right) + \frac{1}{3}\right\}$;

(ii) $u^a_{n,r}(x)$ is a rational function of x depending on the parameters a and r;

(iii) for each $x \in (0,\infty), u^a_{n,r}(x) = O\left(\frac{1}{n^{[\frac{r+1}{2}]}}\right)$, where $[\beta]$ denotes the integer part of β.

Lemma 3.2. Let $e_{ij} : I \to I, e_{ij} = x^i y^j, 0 \le i,j \le 2$ be two-dimensional test functions.
Then the bivariate operators defined in (2.1) satisfy the following results:

$$\begin{aligned}
(i)\, K^a_{n_1,n_2}(e_{00}; x, y) &= 1;\\
(ii)\, K^a_{n_1,n_2}(e_{10}; x, y) &= \frac{1}{n_1+1}\left(n_1 x + \frac{ax}{1+x} + \frac{1}{2}\right);\\
(iii)\, K^a_{n_1,n_2}(e_{01}; x, y) &= \frac{1}{n_2+1}\left(n_2 y + \frac{ay}{1+y} + \frac{1}{2}\right);\\
(iv)\, K^a_{n_1,n_2}(e_{20}; x, y) &= \frac{1}{(n_1+1)^2}\left(n_1^2x^2 + n_1x^2 + 2n_1x + \frac{2an_1x^2}{1+x} + \frac{a^2x^2}{(1+x)^2} + \frac{2ax}{1+x} + \frac{1}{3}\right);\\
(v)\, K^a_{n_1,n_2}(e_{02}; x, y) &= \frac{1}{(n_2+1)^2}\left(n_2^2y^2 + n_2y^2 + 2n_2y + \frac{2an_2y^2}{1+y} + \frac{a^2y^2}{(1+y)^2} + \frac{2ay}{1+y} + \frac{1}{3}\right);\\
(vi)\, K^a_{n_1,n_2}(e_{30}; x, y) &= \frac{1}{(n_1+1)^3}\left(n_1^3x^3 + \frac{3n_1^2x^2}{2}(3+2x) + \frac{3n_1x^2}{2} + \frac{n_1x}{2}(4x^2+6x+5) + \frac{3ax^3n_1^2}{1+x}\right.\\
&\left. + \frac{3an_1x^2}{1+x}\left\{(3+x) + \frac{ax}{1+x}\right\} + \frac{ax}{1+x}\left\{\frac{7}{2} + \frac{7}{2}\frac{ax}{(1+x)} + \frac{a^2x^2}{(1+x)^2}\right\} + \frac{1}{4}\right);\\
(vii)\, K^a_{n_1,n_2}(e_{03}; x, y) &= \frac{1}{(n_2+1)^3}\left(n_2^3y^3 + \frac{3n_2^2y^2}{2}(3+2y) + \frac{3n_2y^2}{2} + \frac{n_2y}{2}(4y^2+6y+5) + \frac{3ay^3n_2^2}{1+y}\right.\\
&\left. + \frac{3an_2y^2}{1+y}\left\{(3+y) + \frac{ay}{1+y}\right\} + \frac{ay}{1+y}\left\{\frac{7}{2} + \frac{7}{2}\frac{ay}{(1+y)} + \frac{a^2y^2}{(1+y)^2}\right\} + \frac{1}{4}\right).
\end{aligned}$$

Lemma 3.3. For $n_1, n_2 \in \mathbb{N}$, we have

$$\begin{aligned}
(i)\, K^a_{n_1,n_2}(u-x; x, y) &= \frac{1}{n_1+1}\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right);\\
(ii)\, K^a_{n_1,n_2}(v-y; x, y) &= \frac{1}{n_2+1}\left(-y + \frac{ay}{1+y} + \frac{1}{2}\right);\\
(iii)\, K^a_{n_1,n_2}((u-x)^2; x, y) &= \frac{1}{(n_1+1)^2}\left((n_1+1)x^2 + (n_1-1)x + \frac{a^2x^2}{(1+x)^2} + 2ax\left(\frac{1-x}{1+x}\right) + \frac{1}{3}\right);\\
(iv)\, K^a_{n_1,n_2}((v-y)^2; x, y) &= \frac{1}{(n_2+1)^2}\left((n_2+1)y^2 + (n_2-1)y + \frac{a^2y^2}{(1+y)^2} + 2ay\left(\frac{1-y}{1+y}\right) + \frac{1}{3}\right).
\end{aligned}$$

Remark 3.4. For every $x \in [0, \infty)$ and $n \in \mathbb{N}$, we have

$$K^a_{n_1,n_2}((u-x)^2; x, y) \leq \frac{\{\delta^a_{n_1}(x)\}^2}{n_1+1},$$

where $\{\delta^a_{n_1}(x)\}^2 = \phi^2(x) + \frac{(1+a)^2}{n_1+1}$ and $\phi(x) = \sqrt{x(1+x)}$.

Proof . From Lemma 3.3 (iii), we have

$$K^a_{n_1,n_2}((u-x)^2; x, y) \leq \frac{1}{n_1+1}\left(x(1+x) + \frac{(1+a)^2}{n_1+1}\right) = \frac{\{\delta^a_{n_1}(x)\}^2}{n_1+1}.$$

□

Lemma 3.5. For every $\gamma_1 \in \mathbb{N}^0$ there exist positive constants $M_k(\gamma_1), k = 1, 2$ such that

(i) $\omega_{\gamma_1}(x) K^a_n\left(\frac{1}{\omega_{\gamma_1}(t)}; x\right) \leq M_1(\gamma_1)$,

(ii) $\omega_{\gamma_1}(x) K^a_n\left(\frac{(t-x)^2}{\omega_{\gamma_1}(t)}; x\right) \leq M_2(\gamma_1) \frac{\{\delta^a_n(x)\}^2}{n+1}$,

for all $x \in \mathbb{R}^0 = \mathbb{R}_+ \cup \{0\}, \mathbb{R}_+ = (0, \infty)$ and $n \in \mathbb{N}$.

Lemma 3.6. For every $\gamma_1, \gamma_2 \in \mathbb{N}^0$ there exist positive constant $M_3(\gamma_1, \gamma_2)$, such that

$$\| K^a_{n_1,n_2}(f; ., .) \|_{\gamma_1,\gamma_2} \leq M_3(\gamma_1, \gamma_2) \| f \|_{\gamma_1,\gamma_2} \tag{3.1}$$

for every $f \in C_{\gamma_1,\gamma_2}(I)$ and for all $n_1, n_2 \in \mathbb{N}$.

Proof . From equation (2.2) and Lemma 3.5, we get

$$\begin{aligned} \omega_{\gamma_1,\gamma_2}(x,y) K^a_{n_1,n_2}\left(\frac{1}{\omega_{\gamma_1,\gamma_2}(u,v)}; x, y\right) &= \left(\omega_{\gamma_1}(x) K^a_{n_1}\left(\frac{1}{\omega_{\gamma_1}(u)}; x\right)\right)\left(\omega_{\gamma_2}(y) K^a_{n_2}\left(\frac{1}{\omega_{\gamma_2}(v)}; y\right)\right) \\ &\leq M_1(\gamma_1)\ M_2(\gamma_2), \text{ for all } (x,y) \in I \text{ and } n_1, n_2 \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Taking supremum on the left side of the inequality of (3.2) and using (2.3), we obtain

$$\left\| K^a_{n_1,n_2}\left(\frac{1}{\omega_{\gamma_1,\gamma_2}(u,v)}; ., .\right)\right\|_{\gamma_1,\gamma_2} \leq M_4(\gamma_1, \gamma_2). \tag{3.3}$$

$$\text{Now, } \| K^a_{n_1,n_2}(f; ., .) \|_{\gamma_1,\gamma_2} \leq \| f \|_{\gamma_1,\gamma_2} \left\| K^a_{n_1,n_2}\left(\frac{1}{\omega_{\gamma_1,\gamma_2}(u,v)}; ., .\right)\right\|_{\gamma_1,\gamma_2}.$$

From (3.3), we get the desired result. □

4 Main results

4.1 Local approximation

For $f \in C_B(I)$ (the space of all bounded and uniformly continuous functions on I, let $C^2_B(I) = \{f \in C_B(I) : f^{(p,q)} \in C_B(I), 0 \leq p+q \leq 2\}$, where $f^{(p,q)}$ is (p,q)th-order partial derivative with respect to x, y of f, equipped with the norm

$$||f||_{C^2_B(I)} = ||f||_{C_B(I)} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial x^i} \right\|_{C_B(I)} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C_B(I)}.$$

The Peetre's $K-$functional of the function $f \in C_B(I)$ is given by

$$\mathcal{K}(f;\delta) = \inf_{g\in C_B^2(I)}\{||f-g||_{C_B(I)} + \delta||g||_{C_B^2(I)}, \delta > 0\}.$$

It is also known that the following inequality

$$\mathcal{K}(f;\delta) \leq M_1\{\overline{\omega}_2(f;\sqrt{\delta}) + \min(1,\delta)||f||_{C_B(I)}\} \tag{4.1}$$

holds for all $\delta > 0$ ([9], page 192). The constant M_1 is independent of δ and f and $\overline{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity.
For $f \in C_B(I)$, the complete modulus of continuity for bivariate case is defined as follows:
$\omega(f;\delta) = \sup\left\{|f(u,v) - f(x,y)| : (u,v),(x,y) \in I \text{ and } \sqrt{(u-x)^2+(v-y)^2} \leq \delta\right\}.$
Further, the partial moduli of continuity with respect to x and y is defined as
$\omega_1(f;\delta) = \sup\left\{|f(x_1,y) - f(x_2,y)| : y \in \mathbb{R}^0 \ \ and \ \ |x_1 - x_2| \leq \delta\right\},$
$\omega_2(f;\delta) = \sup\left\{|f(x,y_1) - f(x,y_2)| : x \in \mathbb{R}^0 \ \ and \ \ |y_1 - y_2| \leq \delta\right\}.$
It is clear that they satisfy the properties of the usual modulus of continuity. The details of the modulus of continuity for the bivariate case can be found in [7].

Now, we find the order of approximation of the sequence $K^a_{n_1,n_2}(f;x,y)$ to the function $f(x,y) \in C_B(I)$ by Petree's $K-$functional.

Theorem 1. For the function $f \in C_B(I)$, the following inequality

$$\begin{aligned}
|K^a_{n_1,n_2}(f;x,y) - f(x,y)| &\leq 4\mathcal{K}(f;M_{n_1,n_2}(x,y)) \\
&\quad +\omega\left(f;\sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right) \\
&\leq M\left\{\overline{\omega}_2\left(f;\sqrt{M_{n_1,n_2}(x,y)}\right) + \min\{1, M_{n_1,n_2}(x,y)\}||f||_{C_B^2(I)}\right\} \\
&\quad +\omega\left(f;\sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right)
\end{aligned}$$

holds. The constant $M > 0$ is independent of f and $M_{n_1,n_2}(x,y)$, where

$$M_{n_1,n_2}(x,y) = \frac{\{\delta^a_{n_1}(x)\}^2}{n_1+1} + \frac{\{\delta^a_{n_2}(y)\}^2}{n_2+1}.$$

Proof . We define the auxiliary operators as follows:

$$\overline{K}^a_{n_1,n_2}(f;x,y) = K^a_{n_1,n_2}(f;x,y) - f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right), \frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right) + f(x,y). \tag{4.2}$$

Then, from Lemma 3.3, we have $\overline{K}^a_{n_1,n_2}((u-x);x,y) = 0$ and $\overline{K}^a_{n_1,n_2}((v-y);x,y) = 0$.
Let $g \in C_B^2(I)$ and $(u,v) \in I$. Using the Taylor's theorem, we have

$$g(u,v) - g(x,y) = \frac{\partial g(x,y)}{\partial x}(u-x) + \int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha + \frac{\partial g(x,y)}{\partial y}(v-y) + \int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta. \tag{4.3}$$

Operating $\overline{K}^a_{n_1,n_2}$ on both sides of (4.3), we get

$$\begin{aligned}
\overline{K}^a_{n_1,n_2}(g;x,y) - g(x,y) &= \overline{K}^a_{n_1,n_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha;x,y\right) + \overline{K}^a_{n_1,n_2}\left(\int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta;x,y\right) \\
&= K^a_{n_1,n_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha;x,y\right)
\end{aligned}$$

$$-\int_x^{\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)}\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-\alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha+K^a_{n_1,n_2}\left(\int_y^v(v-\beta)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta;x,y\right)$$
$$-\int_y^{\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)}\left(\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-\beta\right)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta.$$

Hence,

$$\begin{aligned}|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)| &\leq K^a_{n_1,n_2}\left(\left|\int_x^u|u-\alpha|\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right|d\alpha\right|;x,y\right)\\
&+\left|\int_x^{\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)}\left|\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-\alpha\right|\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right|d\alpha\right|\\
&+K^a_{n_1,n_2}\left(\left|\int_y^v|v-\beta|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right|d\beta\right|;x,y\right)\\
&+\left|\int_y^{\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)}\left|\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-\beta\right|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right|d\beta\right|\\
&\leq \left\{K^a_{n_1,n_2}((u-x)^2;x,y)+\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-x\right)^2\right\}||g||_{C^2_B(I)}\\
&+\left\{K^a_{n_1,n_2}((v-y)^2;x,y)+\left(\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-y\right)^2\right\}||g||_{C^2_B(I)}\\
&\leq \left\{\frac{1}{n_1+1}\{\delta^a_{n_1}(x)\}^2+\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2\right.\\
&\left.+\frac{1}{n_2+1}\{\delta^a_{n_2}(y)\}^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2\right\}||g||_{C^2_B(I)}.\end{aligned}$$

Thus, we get

$$|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)| \leq \left\{\frac{2}{n_1+1}\{\delta^a_{n_1}(x)\}^2+\frac{2}{n_2+1}\{\delta^a_{n_2}(y)\}^2\right\}||g||_{C^2_B(I)}.$$

Also,

$$\begin{aligned}|\overline{K}^a_{n_1,n_2}(f;x,y)| &\leq |K^a_{n_1,n_2}(f;x,y)|+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)\right|+|f(x,y)|\\
&\leq 3||f||_{C_B(I)}.\end{aligned}\tag{4.4}$$

Now, from equation (4.4), we get

$$\begin{aligned}|K^a_{n_1,n_2}(f;x,y)-f(x,y)| &\leq |\overline{K}^a_{n_1,n_2}(f-g;x,y)|+|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)|+|g(x,y)-f(x,y)|\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\\
&\leq 3||f-g||_{C_B(I)}+||f-g||_{C_B(I)}+|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)|\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\\
&\leq 4||f-g||_{C_B(I)}+\left\{\frac{2}{n_1+1}\{\delta^a_{n_1}(x)\}^2+\frac{2}{n_2+1}\{\delta^a_{n_1}(y)\}^2\right\}||g||_{C^2_B(I)}\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\end{aligned}$$

$$\leq \Big(4||f-g||_{C_B(I)}+2M_{n_1,n_2}(x,y)||g||_{C_B^2(I)}\Big)$$
$$+\omega\Bigg(f;\sqrt{\Big(\frac{1}{n_1+1}\Big(-x+\frac{ax}{1+x}+\frac{1}{2}\Big)\Big)^2+\Big(\frac{1}{n_2+1}\Big(-y+\frac{ay}{1+y}+\frac{1}{2}\Big)\Big)^2}\Bigg).$$

Taking infimum on the right hand side over all $g\in C_B^2(I)$ and using (4.1), we obtain

$$\begin{aligned}|K^a_{n_1,n_2}(f;x,y)-f(x,y)| &\leq 4\mathcal{K}(f;M_{n_1,n_2}(x,y))\\ &+\omega\Bigg(f;\sqrt{\Big(\frac{1}{n_1+1}\Big(-x+\frac{ax}{1+x}+\frac{1}{2}\Big)\Big)^2+\Big(\frac{1}{n_2+1}\Big(-y+\frac{ay}{1+y}+\frac{1}{2}\Big)\Big)^2}\Bigg)\\ &\leq M\Big\{\overline{\omega}_2\Big(f;\sqrt{M_{n_1,n_2}(x,y)}\Big)+\min\{1,M_{n_1,n_2}(x,y)\}||f||_{C_B^2(I)}\Big\}\\ &+\omega\Bigg(f;\sqrt{\Big(\frac{1}{n_1+1}\Big(-x+\frac{ax}{1+x}+\frac{1}{2}\Big)\Big)^2+\Big(\frac{1}{n_2+1}\Big(-y+\frac{ay}{1+y}+\frac{1}{2}\Big)\Big)^2}\Bigg).\end{aligned}$$

Hence, the proof is completed. □

4.2 Rate of convergence of bivariate operators

Theorem 2. Suppose that $f\in C^1_{\gamma_1,\gamma_2}(I)$ with $\gamma_1,\gamma_2\in\mathbb{N}^0$ then there exist a positive constant $M_5(\gamma_1,\gamma_2)$ such that for all $(x,y)\in I$ and $n_1,n_2\in\mathbb{N}$

$$\omega_{\gamma_1,\gamma_2}(x,y)|K^a_{n_1,n_2}(f;x,y)-f(x,y)|\leq M_5(\gamma_1,\gamma_2)\Big\{\parallel f_x\parallel_{\gamma_1,\gamma_2}\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}}+\parallel f_y\parallel_{\gamma_1,\gamma_2}\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}\Big\}.$$

Proof . Let $(x,y)\in I$ be a fixed point. Then, we have

$$\begin{aligned}f(t,z)-f(x,y) &= \int_x^t f_u(u,z)du+\int_y^z f_v(x,v)dv \quad \text{for } (t,z)\in I\\ K^a_{n_1,n_2}(f(t,z);x,y)-f(x,y) &= K^a_{n_1,n_2}\Big(\int_x^t f_u(u,z)du;x,y\Big)+K^a_{n_1,n_2}\Big(\int_y^z f_v(x,v)dv;x,y\Big). \qquad (4.5)\end{aligned}$$

Now, by using (2.3), we get

$$\Big|\int_x^t f_u(u,z)du\Big|\leq\parallel f_x\parallel_{\gamma_1,\gamma_2}\Big|\int_x^t\frac{du}{\omega_{\gamma_1,\gamma_2}(u,z)}\Big|\leq\parallel f_x\parallel_{\gamma_1,\gamma_2}\Big(\frac{1}{\omega_{\gamma_1,\gamma_2}(t,z)}+\frac{1}{\omega_{\gamma_1,\gamma_2}(x,z)}\Big)|t-x|,$$

and analogously

$$\Big|\int_y^z f_v(x,v)dv\Big|\leq\parallel f_y\parallel_{\gamma_1,\gamma_2}\Big(\frac{1}{\omega_{\gamma_1,\gamma_2}(x,z)}+\frac{1}{\omega_{\gamma_1,\gamma_2}(x,y)}\Big)|z-y|.$$

By using these inequalities and from (2.2), we obtain for $n_1,n_2\in\mathbb{N}$

$$\begin{aligned}&\omega_{\gamma_1,\gamma_2}(x,y)\Big|K^a_{n_1,n_2}\Big(\int_x^t f_u(u,z)du;x,y\Big)\Big|\leq\omega_{\gamma_1,\gamma_2}(x,y)K^a_{n_1,n_2}\Big(\Big|\int_x^t f_u(u,z)du\Big|;x,y\Big)\\ \leq\ &\parallel f_x\parallel_{\gamma_1,\gamma_2}\omega_{\gamma_1,\gamma_2}(x,y)\Big\{K^a_{n_1,n_2}\Big(\frac{|t-x|}{\omega_{\gamma_1,\gamma_2}(t,z)};x,y\Big)+K^a_{n_1,n_2}\Big(\frac{|t-x|}{\omega_{\gamma_1,\gamma_2}(x,z)};x,y\Big)\Big\}\\ =\ &\parallel f_x\parallel_{\gamma_1,\gamma_2}\omega_{\gamma_2}(y)K^a_{n_2}\Big(\frac{1}{\omega_{\gamma_2}(z)};y\Big)\times\Big\{\omega_{\gamma_1}(x)K^a_{n_1}\Big(\frac{|t-x|}{\omega_{\gamma_1}(t)};x\Big)+K^a_{n_1}(|t-x|;x)\Big\}, \qquad (4.6)\end{aligned}$$

and analogously

$$\omega_{\gamma_1,\gamma_2}(x,y)\left|K^a_{n_1,n_2}\left(\int_y^z f_v(x,v)dv;x,y\right)\right| \leq \parallel f_y \parallel_{\gamma_1,\gamma_2} \left\{\omega_{\gamma_2}(y)K^a_{n_2}\left(\frac{|z-y|}{\omega_{\gamma_2}(z)};y\right)+K^a_{n_2}(|z-y|;y)\right\}. \tag{4.7}$$

By the the Cauchy Schwarz inequality and Remark 3.4, we get for $n_1 \in \mathbb{N}$

$$K^a_{n_1}(|t-x|;x) \leq \left(K^a_{n_1}((t-x)^2;x)\right)^{1/2}\left(K^a_{n_1}(1;x)\right)^{1/2} \leq \frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}} \tag{4.8}$$

and

$$\omega_{\gamma_1}(x)K^a_{n_1}\left(\frac{|t-x|}{\omega_{\gamma_1}(t)};x\right) \leq \omega_{\gamma_1}(x)\left(K^a_{n_1}\left(\frac{(t-x)^2}{\omega_{\gamma_1}(t)};x\right)\right)^{1/2}\left(K^a_{n_1}\left(\frac{1}{\omega_{\gamma_1}(t)};x\right)\right)^{1/2} \leq M_6(\gamma_1)\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}},$$

in view of Lemma 3.5. Analogously for $n_2 \in \mathbb{N}$, we have

$$K^a_{n_2}(|z-y|;y) \leq \frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}, \tag{4.9}$$

and

$$\omega_{\gamma_2}(y)K^a_{n_2}\left(\frac{|z-y|}{\omega_{\gamma_2}(z)};y\right) \leq M_7(\gamma_2)\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}. \tag{4.10}$$

From equations (4.5)-(4.10), we obtain

$$\omega_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z);x,y)-f(x,y) \mid \quad \leq \quad M_8(\gamma_1,\gamma_2)\left\{\parallel f_x \parallel_{\gamma_1,\gamma_2}\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}}+\parallel f_y \parallel_{\gamma_1,\gamma_2}\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}\right\},$$

for all $n_1, n_2 \in \mathbb{N}$. Thus the proof is completed. □

Theorem 3. Suppose that $f \in C_{\gamma_1,\gamma_2}(I)$ with some $\gamma_1, \gamma_2 \in \mathbb{N}^0$. Then there exists a positive constant $M_9(\gamma_1,\gamma_2)$ such that

$$\omega_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z);x,y)-f(x,y) \mid \leq M_9(\gamma_1,\gamma_2)\omega\left(f;C_{\gamma_1,\gamma_2};\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}},\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}\right),$$

for all $(x,y) \in I$ and $n_1, n_2 \in \mathbb{N}$.

Proof . Let $f_{h,\delta}$ be the Steklov function of $f \in C_{\gamma_1,\gamma_2}(I)$, defined by the formula

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u,y+v)dv\,du, \tag{4.11}$$

$(x,y) \in I$ and $h, \delta \in \mathbb{R}_+$. From (4.11) it follows that

$$\begin{aligned}
f_{h,\delta}(x,y)-f(x,y) &= \frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,v}f(x,y)dv\,du,\\
\frac{\partial}{\partial x}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^\delta \Delta_{h,0}f(x,y+v)dv\\
&= \frac{1}{h\delta}\int_0^\delta (\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y))dv,\\
\frac{\partial}{\partial y}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^h \Delta_{0,\delta}f(x+u,y)du\\
&= \frac{1}{h\delta}\int_0^\delta (\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y))du.
\end{aligned}$$

Thus, from (2.3) and (2.4) we obtain

$$\| f_{h,\delta} - f \|_{\gamma_1,\gamma_2} \leq \omega(f, C_{\gamma_1,\gamma_2}; h, \delta), \tag{4.12}$$

$$\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{\gamma_1,\gamma_2} \leq 2h^{-1}\omega(f, C_{\gamma_1,\gamma_2}; h, \delta), \tag{4.13}$$

$$\left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{\gamma_1,\gamma_2} \leq 2\delta^{-1}\omega(f, C_{\gamma_1,\gamma_2}; h, \delta). \tag{4.14}$$

For $h, \delta \in \mathbb{R}_+$, we can write

$$\begin{aligned} \omega_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z); x, y) - f(x,y) \mid \leq\ & \omega_{\gamma_1,\gamma_2}(x,y)\{K^a_{n_1,n_2}(f(t,z) - f_{h,\delta}(t,z); x, y) \\ & + \mid K^a_{n_1,n_2}(f_{h,\delta}(t,z); x, y) - f_{h,\delta}(x,y) \mid \\ & + \mid f_{h,\delta}(x,y) - f(x,y) \mid\} := R_1 + R_2 + R_3. \end{aligned} \tag{4.15}$$

By (2.3), Lemma 3.6 and (4.12) it follows that

$$\begin{aligned} R_1 \leq \| K^a_{n_1,n_2}(f - f_{h,\delta;.,.}) \|_{\gamma_1,\gamma_2} &\leq M_{10}(\gamma_1,\gamma_2)\|f - f_{h,\delta}\|_{\gamma_1,\gamma_2} \\ &\leq M_{10}(\gamma_1,\gamma_2)\omega(f, C_{\gamma_1,\gamma_2}; h, \delta), \end{aligned}$$

and

$$R_3 \leq \omega(f, C_{\gamma_1,\gamma_2}; h, \delta).$$

By using Theorem 2 and (4.13) and (4.14), we get

$$\begin{aligned} R_2 &\leq M_{11}(\gamma_1,\gamma_2)\left\{ \left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{\gamma_1,\gamma_2} \frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}} + \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{\gamma_1,\gamma_2} \frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}} \right\} \\ &\leq 2M_{11}(\gamma_1,\gamma_2)\omega(f, C_{\gamma_1,\gamma_2}; h, \delta)\left\{ h^{-1}\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}} + \delta^{-1}\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}} \right\}. \end{aligned}$$

Consequently, we drive from (4.15)

$$\omega_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z); x, y) - f(x,y) \mid \leq M_{12}(\gamma_1,\gamma_2)\omega(f, C_{\gamma_1,\gamma_2}; h, \delta)\left\{ 1 + h^{-1}\frac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}} + \delta^{-1}\frac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}} \right\}$$

for all $(x,y) \in I, n_1, n_2 \in \mathbb{N}$ and $h, \delta \in \mathbb{R}_+$. On choosing $h = \dfrac{\delta^a_{n_1}(x)}{\sqrt{n_1+1}}$ and $\delta = \dfrac{\delta^a_{n_2}(y)}{\sqrt{n_2+1}}$, we immediately obtain the required result. □

As a consequence of Theorem 3, we have

Theorem 4. Let $f \in C_{\gamma_1,\gamma_2}(I)$ with some $\gamma_1, \gamma_2 \in \mathbb{N}^0$. Then for every $(x,y) \in I$,

$$\lim_{n_1,n_2\to\infty} K^a_{n_1,n_2}(f; x, y) = f(x,y).$$

Theorem 5. (Voronovskaja type theorem) Let $f \in C^2_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y) \in I$,

$$\begin{aligned} \lim_{n\to\infty} n\{K^a_{n,n}(f; x, y) - f(x,y)\} &= \left(-x + \frac{ax}{1+x} + \frac{1}{2} \right) f_x(x,y) + \left(-y + \frac{ay}{1+y} + \frac{1}{2} \right) f_y(x,y) \\ &+ \frac{x}{2}(1+x)f_{xx}(x,y) + \frac{y}{2}(1+y)f_{yy}(x,y). \end{aligned}$$

Proof . Let $(x,y)\in I$ be fixed. By Taylor formula, we have

$$\begin{aligned} f(u,v) &= f(x,y)+f_x(x,y)(u-x)+f_y(x,y)(v-y)\\ &+\frac{1}{2}\{f_{xx}(x,y)(u-x)^2+2f_{xy}(x,y)(u-x)(v-y)+f_{yy}(x,y)(v-y)^2\}\\ &+\psi(u,v;x,y)\sqrt{(u-x)^4+(v-y)^4}, \end{aligned}$$

where $\psi(.,.;x,y)\equiv\psi(.,.)\in C_{\gamma_1,\gamma_2}(I)$ and $\psi(x,y)=0$. Thus, we get

$$\begin{aligned} K^a_{n,n}(f(u,v);x,y) &= f(x,y)+f_x(x,y)K^a_n(u-x;x)+f_y(x,y)K^a_n(v-y;y)\\ &+\frac{1}{2}\{f_{xx}(x,y)K^a_n((u-x)^2;x)+2f_{xy}(x,y)K^a_n(u-x;x)K^a_n(v-y;y)\\ &+f_{yy}(x,y)K^a_n((v-y)^2;y)\}+K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y). \end{aligned}$$

Hence, using Lemma 3.1, we have

$$\begin{aligned} \lim_{n\to\infty} n\{K^a_{n,n}(f(u,v);x,y)-f(x,y)\} &= f_x(x,y)\Big(-x+\frac{ax}{1+x}+\frac{1}{2}\Big)+f_y(x,y)\Big(-y+\frac{ay}{1+y}+\frac{1}{2}\Big)\\ &+\frac{1}{2}\{x(1+x)f_{xx}(x,y)+y(1+y)f_{yy}(x,y)\}\\ &+\lim_{n\to\infty} nK^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y). \end{aligned} \tag{4.16}$$

Applying the Hölder's inequality, we have

$$\begin{aligned} |K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)| &\le \{K^a_{n,n}(\psi^2(u,v);x,y)\}^{1/2}\left\{K^a_{n,n}\left(\left((u-x)^4+(v-y)^4\right);x,y\right)\right\}^{1/2}\\ &\le \{K^a_{n,n}(\psi^2(u,v);x,y)\}^{1/2}\{K^a_n((u-x)^4;x)+K^a_n((v-y)^4;y)\}^{1/2}. \end{aligned}$$

By Theorem 4

$$\lim_{n\to\infty} K^a_{n,n}(\psi^2(u,v);x,y)=\psi^2(x,y)=0,$$

and from Lemma 3.1 (iii), for each $(x,y)\in I$, $K^a_n((u-x)^4;x)=O\left(\frac{1}{n^2}\right)$ and $K^a_n((v-y)^4;y)=O\left(\frac{1}{n^2}\right)$. Hence

$$\lim_{n\to\infty} nK^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)=0. \tag{4.17}$$

By combining (4.16) and (4.17), we obtain the desired result. □

4.3 Simultaneous approximation

Theorem 6. Let $f\in C^1_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y)\in\mathbb{R}^2_+=\mathbb{R}_+\times\mathbb{R}_+$,

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial\omega}K^a_{n,n}(f;\omega,y)\right)_{\omega=x} = \frac{\partial f}{\partial x}(x,y), \tag{4.18}$$

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial\nu}K^a_{n,n}(f;x,\nu)\right)_{\nu=y} = \frac{\partial f}{\partial y}(x,y). \tag{4.19}$$

Proof . We shall prove only (4.18) because the proof of (4.19) is similar. By the Taylor formula for $f\in C^1_{\gamma_1,\gamma_2}(I)$, we have

$$f(u,v)=f(x,y)+f_x(x,y)(u-x)+f_y(x,y)(v-y)+\psi(u,v;x,y)\sqrt{(u-x)^2+(v-y)^2}\ \text{ for } (u,v)\in I,$$

where $\psi(u,v;x,y) \equiv \psi(.,.) \in C_{\gamma_1,\gamma_2}(I)$ and $\psi(x,y)=0$. Operating $K^a_{n,n}(.;.,y)$ to the above inequality and then by using Lemma 3.3, we get

$$\begin{aligned}
\left(\frac{\partial}{\partial\omega}K^a_{n,n}(f(u,v);\omega,y)\right)_{\omega=x} &= f(x,y)\left(\frac{\partial}{\partial\omega}K^a_{n,n}(1;\omega,y)\right)_{\omega=x} + f_x(x,y)\left(\frac{\partial}{\partial\omega}K^a_{n,n}(u-x;\omega,y)\right)_{\omega=x}\\
&\quad + f_y(x,y)\left(\frac{\partial}{\partial\omega}K^a_{n,n}(v-y;\omega,y)\right)_{\omega=x}\\
&\quad + \left(\frac{\partial}{\partial\omega}K^a_{n,n}(\psi(u,v;x,y)\sqrt{(u-x)^2+(v-y)^2};\omega,y)\right)_{\omega=x}, \text{ for } (u,v)\in I\\
&= f_x(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{1}{n+1}\left(n\omega+\frac{a\omega}{1+\omega}+\frac{1}{2}\right)\right)\right\}_{\omega=x}\\
&\quad + f_y(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{1}{n+1}\left(ny+\frac{ay}{1+y}+\frac{1}{2}\right)\right)\right\}_{\omega=x} + E, \text{ say}\\
&= \frac{n}{n+1}f_x(x,y)+E.
\end{aligned}$$

It is sufficient to prove that $E\to 0$ as $n\to\infty$.

$$\begin{aligned}
E &= (n+1)^2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\frac{\partial}{\partial\omega}W^a_{n,n,k_1,k_2}(\omega,y)\right)_{\omega=x}\int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv\\
&= (n+1)^2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\{(k_1-nx)(1+x)-ax\}}{x(1+x)^2}W^a_{n,n,k_1,k_2}(x,y)\\
&\quad \times\int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv\\
&= \frac{n(n+1)^2}{x(1+x)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\frac{k_1}{n}-x\right)W^a_{n,n,k_1,k_2}(x,y)\int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv\\
&\quad -\frac{a}{(1+x)^2}K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y)\\
&= E_1+E_2, \text{ say.}
\end{aligned}$$

First, we estimate E_1 by using Schwarz inequality.

$$\begin{aligned}
E_1 &\le \frac{n}{x(1+x)}\left(\sum_{k_1=0}^{\infty}W^a_{n,k_1}(x)\left(\frac{k_1}{n}-x\right)^2\right)^{1/2}\\
&\quad \times\left((n+1)^2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}W^a_{n,n,k_1,k_2}(x,y)\int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\psi^2(u,v)((u-x)^2+(v-y)^2)\,du\,dv\right)^{1/2}\\
&\le \frac{n}{x(1+x)}\left(\sum_{k_1=0}^{\infty}W^a_{n,k_1}(x)\left(\frac{k_1}{n}-x\right)^2\right)^{1/2}\{K^a_{n,n}(\psi^4(u,v);x,y)(K^a_n((u-x)^4;x)\\
&\quad +2K^a_n((u-x)^2;x)(K^a_n((v-y)^2;y)+K^a_n((v-y)^4;y))\}^{1/4}
\end{aligned}$$

$$|E_1| \le M_{12}(x,y)\{K^a_{n,n}(\psi^4(u,v);x,y)\}^{1/4}, \text{ in view of Lemma 3.1.}$$

From Theorem 4, we obtain

$$\lim_{n\to\infty}K^a_{n,n}(\psi^4(u,v);x,y)=\psi^4(x,y)=0, \text{ for } (x,y)\in\mathbb{R}^2_+.$$

To estimate E_2, proceeding in a manner similar to the estimate of E_1, we get $E_2\to 0$ as $n\to\infty$. Combining the estimates of E_1 and E_2, it follows that $E\to 0$ as $n\to\infty$. This completes the proof. □

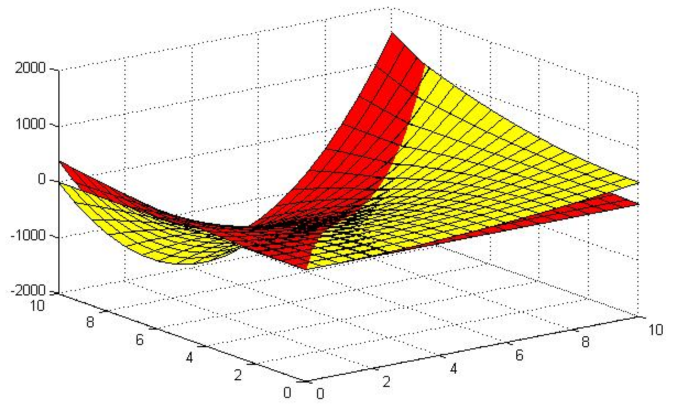

Figure 1: The Convergence of $K^{10}_{100,100}(f;x,y)$ (red) to $f(x,y)$ (yellow)

Similarly, we can prove the following theorem:

Theorem 7. Let $f \in C^3_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y) \in \mathbb{R}^2_+$, we have

$$\begin{aligned}\lim_{n\to\infty} n\bigg\{\bigg(\frac{\partial}{\partial\omega}K^a_{n,n}(f;\omega,y)\bigg)_{\omega=x} - \frac{\partial f}{\partial x}(x,y)\bigg\} &= \bigg(-1+\frac{a}{(1+x)^2}\bigg)f_x(x,y) + \bigg(1+\frac{ax}{1+x}\bigg)f_{xx}(x,y)\\ &+\bigg(-y+\frac{ay}{1+y}+\frac{1}{2}\bigg)f_{xy}(x,y)+\frac{y}{2}(1+y)f_{xyy}(x,y)\\ &+\frac{x}{2}(1+x)f_{xxx}(x,y)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n\bigg\{\bigg(\frac{\partial}{\partial\nu}K^a_{n,n}(f;x,\nu)\bigg)_{\nu=y} - \frac{\partial f}{\partial y}(x,y)\bigg\} &= \bigg(-1+\frac{a}{(1+y)^2}\bigg)f_y(x,y) + \bigg(1+\frac{ay}{1+y}\bigg)f_{yy}(x,y)\\ &+\bigg(-x+\frac{ax}{1+x}+\frac{1}{2}\bigg)f_{xy}(x,y)+\frac{x}{2}(1+x)f_{xxy}(x,y)\\ &+\frac{y}{2}(1+y)f_{yyy}(x,y).\end{aligned}$$

5 Numerical Examples

In the following, we give some numerical results regarding the approximation properties of bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ using Matlab algorithms for construction of operators.

Let us consider the function $f: I \to \mathbb{R}^0$, $f(x,y) = x^2y^2 - 9xy^2 + 4x^2$. The convergence of the bivariate generalized Baskakov-Kantorovich operators to the function f is illustrated in Example 1.

Example 1. For $n_1 = n_2 = 100$; $n_1 = n_2 = 500$ and $a = 10$, the convergence of the bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ (red) to the function $f(x,y) = x^2y^2 - 9xy^2 + 4x^2$ (yellow) is illustrated in figures 1 and 2 respectively. We observe that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operators becomes smaller.

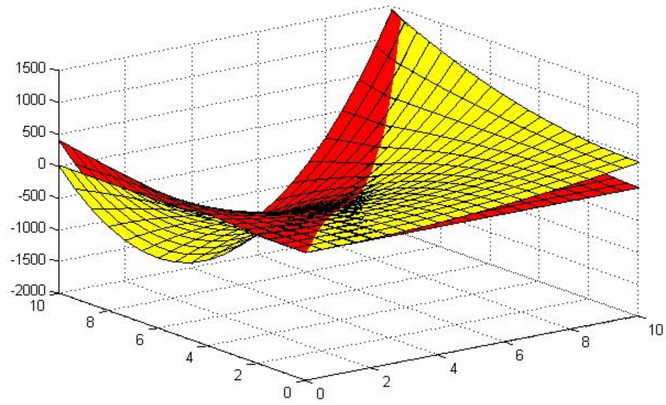

Figure 2: The Convergence of $K^{10}_{500,500}(f;x,y)$ (red) to $f(x,y)$ (yellow)

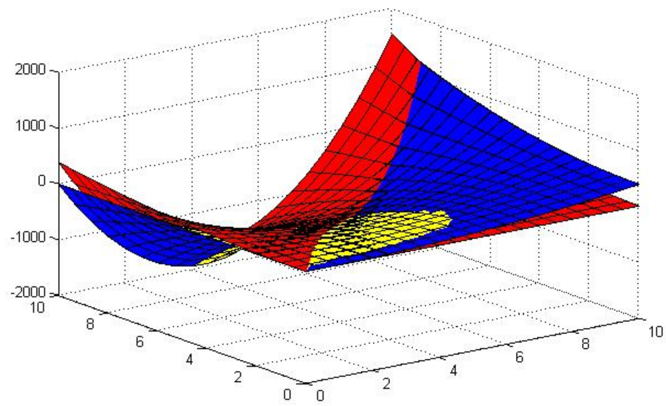

Figure 3. The Comparison of bivariate Szász-Kantorovich (blue) and
bivariate generalized Baskakov-Kantorovich $K^{10}_{100,100}(f;x,y)$ (red) to $f(x,y)$ (yellow)

Example 2. For $n_1 = n_2 = 100$; $n_1 = n_2 = 500$ and $a = 10$, the comparison of the bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ (red) and the bivariate Szász-Kantorovich operators (blue) to the function $f(x,y) = x^2y^2 - 9xy^2 + 4x^2$ (yellow) is illustrated in figures 5 and 5 respectively. It is observed that the error in the approximation of f by the bivariate Szász-Kantorovich operators is smaller than the bivariate generalized Baskakov-Kantorovich operators.

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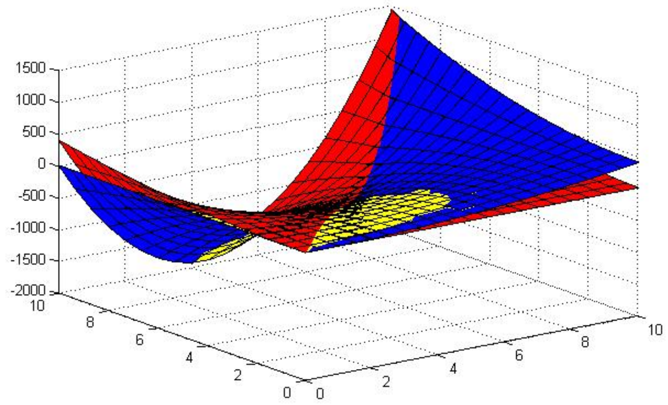

Figure 4.The Comparison of bivariate Szász-Kantorovich (blue) and
bivariate generalized Baskakov-Kantorovich $K^{10}_{500,500}(f;x,y)$ (red) to $f(x,y)$ (yellow)

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