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Existence, asymptotic stability and blow-up results for a variable-exponent viscoelastic double-Kirchhoff-type wave equation

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Abstract

The present study examined the behavior of solutions for a viscoelastic double-Kirchhoff-type wave equation with nonlocal degenerate damping term and variable exponent nonlinearities. Under appropriate conditions for the data and exponents, we prove the global existence, asymptotic stability, and blow up of solutions with arbitrary initial energy.

Keywords: Global existence, asymptotic stability, blow-up, Kirchhoff-type equation, nonlocal damping, variable exponent
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1 Introduction

In this work, we consider the following boundary value problem

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u + (g*\Delta u)(x,t) + \sigma(\|\nabla u\|^2)h(u_t) = \phi(u), \qquad (x,t)\in\Omega\times(0,\infty) \tag{1.1}$$

$$u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0, \qquad (x,t)\in\partial\Omega\times(0,\infty) \tag{1.2}$$

$$u(x,0) = u_0,\ u_t(x,0) = u_1(x), \qquad x\in\Omega. \tag{1.3}$$

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Here $a, b > 0$ and Ω is a bounded domain of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. Also, $\Delta_{p(x)}$ is called $p(x)-$Laplacian operator which defined as

$$\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2} \nabla u).$$

Moreover, suppose that r is a positive constant in which

$$M(s) = 1 + \sigma(s), \quad \sigma(s) = s^r,$$

and the viscoelastic term is defined as

$$(g * \Delta u)(x,t) = \int_0^t g(t-s)\Delta u(s) ds,$$

where the kernel of the memory satisfies the following condition:
(A1) g is a nonincreasing and nonnegative function satisfying

$$g(t) \geq 0, \quad 1 - \int_0^\infty g(t)dt = l > 0,$$
$$g'(t) \leq -\xi(t)g(t), \quad \int_0^{+\infty} \xi(t)dt = +\infty,$$

and ξ is a positive differentiable function.
The variable-exponent nonlinearity terms, i.e., $h(.)$ and $\phi(.)$ are defined as:

$$h(v) = |v|^{m(x)-2} u_t,$$
$$\phi(v) = |v|^{q(x)-2} u.$$

In this study, the following additional condition is considered on the variable exponents:
(A2) the exponents $p(\cdot)$, $m(\cdot)$ and $q(\cdot)$ are given measurable functions on $\overline{\Omega}$ such that:

$$2 \leq p^- \leq p(x) \leq p^+ < \infty,$$
$$2 \leq m^- \leq m(x) \leq m^+ < \infty,$$
$$2 \leq q^- \leq q(x) \leq q^+ < \infty,$$

with

$$p^- := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(x), \; p^+ := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} p(x),$$
$$m^- := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} m(x), \; m^+ := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} m(x),$$
$$q^- := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} q(x), \quad q^+ := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} q(x).$$

Before going any further, some previous works in the literature about the Kirchhoff-type models with nonlocal damping terms and constant-exponent nonlinearities are reviewed. Ikehata and Matsuyama [12] studied the associated mixed problem for the following Kirchhoff-type equation

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \delta|u_t|^{p-1}u_t = \mu|u_t|^{q-1}u.$$

Under some restrictions to the initial data, they proved the existence of global solutions and decay of the associated energy. In another study, Lazo [14], proved the existence of a global solution for the Kirchhoff-type equation of the form

$$u_{tt} + M(|A^{\frac{1}{2}}u|^2)Au + N(|A^\alpha u|^2)A^\alpha u_t = f,$$

where A is a positive self-adjoint operator which defined in a real Hilbert space H, and $0 < \alpha \leq 1$. Also, it is assumed that there exist constants $m_0, n_0 > 0$ such that the functions M and N satisfy the following conditions

$$M \in C^0((0,+\infty); R), \; M(s) \geq m_0 \text{ for all } s \geq 0.$$
$$N \in C^0((0,+\infty); R), \; N(s) \geq n_0 \text{ for all } s \geq 0.$$

In [6], Chueshov examined the following Kirchhoff wave model with nonlocal and nonlinear strong damping

$$u_{tt} - \Phi(\|\nabla u(t)\|_2^2)\Delta u - \sigma(\|\nabla u(t)\|_2^2)\Delta u_t + f(u) = h.$$

He demonstrated the existence and uniqueness of weak solutions and studied their properties for a wide class of nonlinearities which covers the case of possible degeneration (or even negativity) of the stiffness coefficient and the case of a supercritical source term. He proved that in the natural energy space endowed with a partially strong topology there exists a global finite-dimensional attractor. In a recent study, Narciso [19] investigated well-posedness and long-time behavior of the solutions for the following quasi-linear Kirchhoff-type wave model with nonlocal nonlinear damping

$$u_{tt} - \Phi(\|\nabla u(t)\|_2^2)\Delta u - \sigma(\|\nabla u(t)\|_2^2)g(u_t) + f(u) = h,$$

The existence of a local attractor was also proved under appropriate conditions on data.
Zhang et. al. [30] considered a wave equation with nonlocal nonlinear damping and source terms of the form

$$u_{tt} - \Delta u + M(\|\nabla u\|^2)g(u_t) = f(u),$$

in a bounded domain. They proved a general energy decay property for the solutions by constructing a stable set and using the multiplier technique. In this regard, see also Zhang et. al. [31]. Note that, numerous researchers have studied the different kinds of generalizations of the Kirchhoff-type wave models. Thus, due to the extensive literature on the topic, we selectively refer to [5, 7, 13, 18, 20, 28, 29] and the references therein.

Nonetheless, the stability and blow-up of solutions of viscoelastic Kirchhoff-type wave equations with variable-exponent nonlinearities is a less investigated area. For example, Pişkin [21], proved the blow-up of solutions for the following Kirchhoff-type equation with variable exponents nonlinearities:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u.$$

Afterward, Shahrouzi and Kargarfard [27] considered the following Kirchhoff-type equation with $m(x)-$Laplacian operator

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta_{m(x)}u + h(x,t,u,\nabla u) + \beta u_t = \phi_{p(x)}(u),$$

where $\phi_{p(x)}(u) = |u|^{p(x)}u$ and $h(x,t,u,\nabla u)$ is a function that satisfies

$$|h(x,t,u,\nabla u)| \leq M_1|u| + M_2|\nabla u|,$$

for some positive M_1, M_2. They proved the blow up of solutions with positive initial energy and suitable conditions on datum and variable exponents.
Recently, Antontsev et. al [2] looked into the following nonlinear Timoshenko equation with variable exponents:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u,$$

and demonstrated the local existence of the solution under suitable conditions. Moreover, the nonexistence of solutions was proved with negative initial energy.
Dai and Hao [8] studied the following equation

$$-M\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)div(|\nabla u|^{p(x)-2}\nabla u) = f(x,u).$$

Using a direct variational approach and the theory of the variable-exponent Sobolev spaces, they established conditions through which the existence and multiplicity of solutions for the problem were verified. In another study, Hamdani et. al. [11] investigated the following nonlocal $p(x)-$Kirchhoff-type equation

$$-(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)div(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u + g(x,u).$$

They obtained a nontrivial weak solution by using the Mountain Pass theorem. The relevant equations with variable-exponent nonlinearities have also been studied in [3, 10, 16, 17, 22, 23, 24, 25].

What distinguishes the current research from the other studies is the investigation of a new viscoelastic wave model with double-Kirchhoff-type terms as well as the inclusion of a product of $\sigma(\|\nabla u\|_2^2)$ and variable-exponent nonlinearity term $h(u_t)$. Our study extends and improves the results in the literature as obtained in [19, 26, 30, 31] to viscoelastic double-Kirchhoff-type equation with degenerate nonlocal damping and variable-exponent nonlinearities.

The rest of the paper is organized as follows. In section 2 we present some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, the Sobolev space, $W^{1,p(\cdot)}(\Omega)$, that we use for main results. Section 3 proves the global existence of solutions for the problem (1.1)-(1.3). In section 4 the asymptotic stability of solutions for appropriate initial data has been proved. Finally, the blow-up of solutions is proved with arbitrary initial energy and suitable conditions on datum, in section 5.

2 Preliminaries

In order to study problem (1.1)-(1.3), some theories about Lebesgue and Sobolev spaces with variable-exponents are required (for more details see [4, 9]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of R^n. we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \Big\{u|\ u \ is\ measurable\ in\ \Omega\ and\ \int_\Omega |\lambda u(x)|^{p(x)}dx < \infty\ for\ some\ \lambda > 0\Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(\cdot)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0\Big|\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\Big\}.$$

Lemma 2.1. [9] Let Ω be a bounded domain in R^n. The space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds

$$|\int_\Omega fg dx| \le (\frac{1}{p^-} + \frac{1}{q^-})\|f\|_{p(\cdot)}\|g\|_{q(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

The relation between the modular $\int_\Omega |f|^{p(x)}dx$ and the norm follows from

$$\min(\|f\|^{p^-}_{p(\cdot)}, \|f\|^{p^-}_{p(\cdot)}) \le \int_\Omega |f|^{p(x)}dx \le \max(\|f\|^{p^+}_{p(\cdot)}, \|f\|^{p^+}_{p(\cdot)}).$$

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u\ exists\ and\ |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Furthermore, let $W^{1,p(\cdot)}_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W^{1,p(\cdot)}_0(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

Let the variable exponent $p(\cdot)$ satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \le \frac{A}{log\frac{1}{|x-y|}},\ for\ all\ x, y \in \Omega\ with\ |x - y| < \delta,$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.2. (The Poincare inequality) Assume that Ω is a bounded domain of R^n and $p(\cdot)$ satisfies log-Hölder condition, then

$$\|u\|_{p(x)} \leq C_* \|\nabla u\|_{p(x)}, \ for\ all\ u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.1}$$

where $C_* = C_*(p^-, p^+, |\Omega|) > 0$.

Lemma 2.3. Let $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function that satisfy

$$ess\ inf_{x \in \overline{\Omega}}(p^*(x) - q(x)) > 0.$$

Then, the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & if\ p^- < n, \\ \infty, & if\ p^- \geq n. \end{cases}$$

If in addition $p(\cdot)$ satisfies log-Hölder condition, then

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & if\ p(x) < n, \\ \infty, & if\ p(x) \geq n. \end{cases}$$

We sometimes use the Young's inequality

$$ab \leq \delta a^{q(x)} + C_\delta(x) b^{q'(x)}, \quad a, b \geq 0, \quad \delta > 0, \quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{2.2}$$

where $C_\delta(x) = \frac{1}{q'(x)}(\delta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (2.2)

$$ab \leq \frac{a^{q(x)}}{q(x)} + \frac{b^{q'(x)}}{q'(x)}. \tag{2.3}$$

We set

$$V = \{v \in W^{1,p(x)} \cap H^1(\Omega)\} : v = 0\ on\ \partial\Omega\}.$$

The energy function is defined as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{1}{2}(g \odot \nabla u)(t) + \frac{1}{2(r+1)}\|\nabla u\|^{2(r+1)} \\ &+ a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx, \end{aligned} \tag{2.4}$$

where

$$(g \odot \nabla u)(t) = \int_0^t g(t-s) \int_\Omega (\nabla u(s) - \nabla u)^2 dx ds.$$

Now, we first state a well-known lemma that will be needed later to prove the asymptotic stability result for problem (1.1)-(1.3).

Lemma 2.4. [15] Let $E : R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $\gamma \geq 1$ and $A > 0$ such that

$$\int_0^{+\infty} E^{(\gamma+1)/2}(t)dt \leq AE(S), \quad 0 \leq S < +\infty,$$

then

$$\begin{aligned} E(t) &\leq cE(0)(1+t)^{2/(\gamma-1)}, \quad t \geq 0, \text{ if } \gamma > 1, \\ E(t) &\leq cE(0)e^{-\omega t}, \quad t \geq 0, \text{ if } \gamma = 1, \end{aligned}$$

where c and ω are positive constants independent of the initial energy $E(0)$.

For the sake of completeness, the local existence result for the problem (1.1)-(1.3) is stated without providing the proof. Indeed, by using the Faedo-Galerkin method and a combination of the works [1, 2, 19], one could prove this Theorem.

Theorem 2.5. (Local existence and monotonicity of energy) Let us assume that (A1) and (A2) hold. If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then problem (1.1)-(1.3) possesses a unique weak solution in the case

$$u \in L^\infty\big((0,T), H^2(\Omega) \cap V\big), \qquad u_t \in L^\infty\big((0,T), L^{m(\cdot)}(\Omega)\big),$$

$$u_{tt} \in L^\infty\big((0,T), L^2(\Omega) \cap W^{-1,m'(x)}(\Omega)\big),$$

where $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1$. Moreover, for any solution of problem (1.1)-(1.3) the energy functional along the solution satisfies

$$E'(t) \leq \frac{1}{2}(g' \odot \nabla u)(t) - \|\nabla u\|^{2r} \int_\Omega |u_t|^{m(x)} dx \leq 0. \tag{2.5}$$

3 Global existence

In this section, we are going to prove the global existence of solutions for the problem (1.1)-(1.3). For this goal, we define the following functionals to obtain the potential well.

$$I(u(t)) =(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + a\int_\Omega |\nabla u|^{p(x)}dx + b\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 - \int_\Omega |u|^{q(x)}dx. \tag{3.1}$$

$$\begin{aligned} J(u(t)) =&\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{1}{2}(g \odot \nabla u)(t) + \frac{1}{2(r+1)}\|\nabla u\|^{2(r+1)} \\ &+ a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx. \end{aligned} \tag{3.2}$$

From the definitions (3.1) and (3.2), we have

$$E(t) = J(u(t)) + \frac{1}{2}\|u_t\|^2. \tag{3.3}$$

Lemma 3.1. Under the assumptions of Theorem 2.5, we assume that $I(u(0)) > 0$, $q^- > \max\{2(r+1),\ 2(p^+)^2\}$ and moreover,

$$\gamma_0 := \max\{C_q^{q^-}\left(\frac{E(0)}{\beta l}\right)^{\frac{q^--2}{2}},\ C_q^{q^+}\left(\frac{E(0)}{\beta l}\right)^{\frac{q^+-2}{2}}\} < 1, \tag{3.4}$$

where C_q is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ and

$$\beta := \min\{\frac{q^- - 2}{2q^-},\ \frac{q^- - 2(r+1)}{2q^-(r+1)},\ \frac{a(q^- - p^+)}{p^+q^-},\ \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2q^-}\}.$$

Then we have

$$I(u(t)) > 0, \text{for all } t \in [0,T].$$

Proof . By using the continuity of $u(t)$ and since $I(u(0)) > 0$, thus there exists a time $T^* < T$ such that

$$I(u(t)) \geq 0, \text{for all } t \in [0,T^*].$$

Now, under (A1) and from the definition of $J(u(t))$ in (3.2), we deduce that

$$\begin{aligned} J(u(t)) \geq&\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{1}{2}(g \odot \nabla u)(t) + \frac{1}{2(r+1)}\|\nabla u\|^{2(r+1)} \\ &+ \frac{a}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{b}{2(p^+)^2}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 - \frac{1}{q^-}\int_\Omega |u|^{q(x)}dx. \end{aligned}$$

From definition of $I(u(t))$, we obtain

$$\begin{aligned}J(u(t)) \geq& \frac{q^- - 2}{2q^-}(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{q^- - 2}{2q^-}(g \odot \nabla u)(t) + \frac{q^- - 2(r+1)}{2q^-(r+1)}\|\nabla u\|^{2(r+1)} + \frac{a(q^- - p^+)}{p^+ q^-}\int_\Omega |\nabla u|^{p(x)}dx \\ &+ \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2 q^-}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 + \frac{1}{q^-}I(u(t)).\end{aligned}$$

Thanks to the assumptions of Lemma, since $q^- > \max\{2(r+1),\ 2(p^+)^2\}$, therefore we get

$$\begin{aligned}J(u(t)) \geq& \beta\Big((1 - \int_0^t g(s)ds)\|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_\Omega |\nabla u|^{p(x)}dx + \Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2\Big) + \frac{1}{q^-}I(u(t)) \\ \geq& \beta\Big(l\|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_\Omega |\nabla u|^{p(x)}dx + \Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2\Big), \end{aligned} \tag{3.5}$$

where (A2) has been used and

$$\beta := \min\{\frac{q^- - 2}{2q^-},\ \frac{q^- - 2(r+1)}{2q^-(r+1)},\ \frac{a(q^- - p^+)}{p^+ q^-},\ \frac{b(q^- - 2(p^+)^2)}{2(p^+)^2 q^-}\}.$$

Inequality (3.5) yields

$$l\|\nabla u\|^2 \leq \beta^{-1}J(u(t)) \leq \beta^{-1}E(t) \leq \beta^{-1}E(0),$$

and therefore we have

$$\|\nabla u\|^2 \leq \frac{E(0)}{\beta l}. \tag{3.6}$$

At this point, we shall prove that $I(u(t)) > 0,\ \forall t \in [0, T^*]$. For this purpose, suppose that C_q is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, we have

$$\begin{aligned}\int_\Omega |u|^{q(x)}dx \leq& \max\{\Big(\int_\Omega |u|^{q(x)}\Big)^{q^-},\ \Big(\int_\Omega |u|^{q(x)}\Big)^{q^+}\} \\ \leq& \max\{C_q^{q^-}\|\nabla u\|^{q^-},\ C_q^{q^+}\|\nabla u\|^{q^+}\} \\ =& \max\{C_q^{q^-}\|\nabla u\|^{q^- - 2},\ C_q^{q^+}\|\nabla u\|^{q^+ - 2}\}\|\nabla u\|^2 \\ \leq& \max\{C_q^{q^-}\left(\frac{E(0)}{\beta l}\right)^{\frac{q^- - 2}{2}},\ C_q^{q^+}\left(\frac{E(0)}{\beta l}\right)^{\frac{q^+ - 2}{2}}\}\|\nabla u\|^2,\end{aligned}$$

which implies

$$\int_\Omega |u|^{q(x)}dx \leq \gamma_0 \|\nabla u\|^2, \tag{3.7}$$

and therefore by virtue of (3.1) and (3.4), we deduce

$$I(u(t)) > 0, \quad \forall t \in [0, T^*].$$

By repeating this procedure T^* extended to T and proof of Lemma 3.1 is completed. □

Theorem 3.2. Let $u(x,t)$ be the local solution of (1.1)-(1.3). Under the assumption of Lemma 3.1, $u(x,t)$ is global.

Proof . By virtue of (3.3) and (3.5), we obtain

$$\begin{aligned}E(t) =& \frac{1}{2}\|u_t\|^2 + J(u(t)) \\ \geq& \frac{1}{2}\|u_t\|^2 + \beta\Big(l\|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_\Omega |\nabla u|^{p(x)}dx + \Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2\Big).\end{aligned}$$

From the fact that $\beta \leq \frac{q^{-}-2}{2q^{-}} < \frac{1}{2}$, we get

$$\|u_t\|^2 + \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_\Omega |\nabla u|^{p(x)}dx + \left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 \leq \frac{E(t)}{\beta l}. \tag{3.8}$$

Hence by attention to the nonincreasingness of $E(t)$, we have

$$\|u_t\|^2 + \|\nabla u\|^2 + (g \odot \nabla u)(t) + \|\nabla u\|^{2(r+1)} + \int_\Omega |\nabla u|^{p(x)}dx + \left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 \leq \frac{E(0)}{\beta l},$$

and this shows that the local solution $u(x,t)$ of (1.1)-(1.3) is global and bounded. □

4 Asymptotic stability result

This section aims at proving the asymptotic stability of the global solutions of (1.1)-(1.3). To prove this result, we establish the following Lemma:

Lemma 4.1. Assume that assumptions of Lemma 3.1 hold. Then for any solution of (1.1)-(1.3), there exists $\gamma_1 > 0$ such that

$$\int_\Omega |u|^{m(x)}dx \leq \gamma_1 \|\nabla u\|^2. \tag{4.1}$$

Proof . By using (A2), (3.6) and embedding $H_0^1(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ with the constant C_m, we have

$$\begin{aligned}
\int_\Omega |u|^{m(x)}dx \leq& \max\{\left(\int_\Omega |u|^{m(x)}dx\right)^{m^-}, \left(\int_\Omega |u|^{m(x)}dx\right)^{m^+}\} \\
\leq& \max\{C_m^{m^-}\|\nabla u\|^{m^-}, C_m^{m^+}\|\nabla u\|^{m^+}\} \\
=& \max\{C_m^{m^-}\|\nabla u\|^{m^- -2}, C_m^{m^+}\|\nabla u\|^{m^+ -2}\}\|\nabla u\|^2 \\
\leq& \max\{C_m^{m^-}\left(\frac{E(0)}{\beta l}\right)^{\frac{m^- -2}{2}}, C_m^{m^+}\left(\frac{E(0)}{\beta l}\right)^{\frac{m^+ -2}{2}}\}\|\nabla u\|^2 \\
\leq& \gamma_1 \|\nabla u\|^2,
\end{aligned}$$

and therefore the proof is completed. □

We are in a position that states and prove the main result:

Theorem 4.2. Suppose that the assumptions of Lemma 3.1 hold. Moreover, we assume that r is sufficiently small and $b > p^+(1-\gamma_0)$ where γ_0 (c.f. (3.4)) satisfies $\gamma_0 < \frac{2\beta l}{2\beta l+1}$. Then the solutions of problem (1.1)-(1.3) are asymptotically stable.

Proof . Multiplying equation (1.1) by $E(t)^\alpha u$ and integrating it over $\Omega \times (S,T)$, we obtain

$$\begin{aligned}
&\left[E(t)^\alpha \int_\Omega uu_t dx\right]\Big|_S^T - \alpha \int_S^T E(t)^{\alpha-1}E'(t)\int_\Omega uu_t dxdt - \int_S^T E(t)^\alpha \|u_t\|^2 dt \\
&+ \int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + \int_S^T E(t)^\alpha (1 - \int_0^t g(s)ds)\|\nabla u\|^2 dt \\
&+ a\int_S^T E(t)^\alpha \int_\Omega |\nabla u|^{p(x)}dxdt - \int_S^T E(t)^\alpha \int_\Omega |u|^{q(x)}dxdt \\
&+ b\int_S^T E(t)^\alpha \big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)\big(\int_\Omega |\nabla u|^{p(x)}dx\big)dt \\
&- \int_S^T E(t)^\alpha \int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s) - \nabla u)dxdsdt \\
&+ \int_S^T E(t)^\alpha \|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_t dxdt = 0.
\end{aligned} \tag{4.2}$$

Now, we add and subtract the following term in (4.2)

$$\int_S^T E(t)^\alpha\Big((\gamma_0+1)\|u_t\|^2+\gamma_0\big[(1-\int_0^t g(s)ds)\|\nabla u\|^2+(g\odot\nabla u)(t) \\ +\|\nabla u\|^{2(r+1)}+a\int_\Omega|\nabla u|^{p(x)}dx+(\int_\Omega|\nabla u|^{p(x)}dx)^2\big]\Big)dt,$$

then, by using (3.1), we obtain

$$\begin{aligned}(1-\gamma_0)&\int_S^T E(t)^\alpha\Big(\|u_t\|^2+\int_\Omega|u|^{q(x)}dx+I(u(t))\Big)dt\\ =&(2-\gamma_0)\int_S^T E(t)^\alpha\|u_t\|^2dt-\gamma_0\int_S^T E(t)^\alpha\left(1-\int_0^t g(s)ds\right)\|\nabla u\|^2dt\\ &-\gamma_0\int_S^T E(t)^\alpha\|\nabla u\|^{2(r+1)}dt+(1-\gamma_0)\int_S^T E(t)^\alpha(g\odot\nabla u)(t)dt\\ &-a\gamma_0\int_S^T E(t)^\alpha\int_\Omega|\nabla u|^{p(x)}dxdt+\int_S^T E(t)^\alpha\int_\Omega|u|^{q(x)}dxdt\\ &+(1-\gamma_0)\int_S^T E(t)^\alpha\left(\int_\Omega|\nabla u|^{p(x)}dx\right)^2dt\\ &-\Big[E(t)^\alpha\int_\Omega uu_tdx\Big]\Big|_S^T+\alpha\int_S^T E(t)^{\alpha-1}E'(t)\int_\Omega uu_tdxdt\\ &-b\int_S^T E(t)^\alpha\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\left(\int_\Omega|\nabla u|^{p(x)}dx\right)dt\\ &-\int_S^T E(t)^\alpha\|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_tdxdt\\ &+\int_S^T E(t)^\alpha\int_0^t g(t-s)\int_\Omega\nabla u(\nabla u(s)-\nabla u)dxdsdt. \end{aligned}\tag{4.3}$$

Thanks to the assumptions (A1) and (A2), we deduce that

$$\begin{aligned}(1-\gamma_0)\int_S^T E(t)^\alpha\Big(\|u_t\|^2+\int_\Omega|u|^{q(x)}dx+I(u(t))\Big)dt\le&(2-\gamma_0)\int_S^T E(t)^\alpha\|u_t\|^2dt-\gamma_0 l\int_S^T E(t)^\alpha\|\nabla u\|^2dt\\ &-\gamma_0\int_S^T E(t)^\alpha\|\nabla u\|^{2(r+1)}dt+(1-\gamma_0)\int_S^T E(t)^\alpha(g\odot\nabla u)(t)dt\\ &-a\gamma_0\int_S^T E(t)^\alpha\int_\Omega|\nabla u|^{p(x)}dxdt+\int_S^T E(t)^\alpha\int_\Omega|u|^{q(x)}dxdt\\ &+(1-\gamma_0-\frac{b}{p^+})\int_S^T E(t)^\alpha\left(\int_\Omega|\nabla u|^{p(x)}dx\right)^2dt\\ &-\Big[E(t)^\alpha\int_\Omega uu_tdx\Big]\Big|_S^T+\alpha\int_S^T E(t)^{\alpha-1}E'(t)\int_\Omega uu_tdxdt\\ &-\int_S^T E(t)^\alpha\|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_tdxdt\\ &+\int_S^T E(t)^\alpha\int_0^t g(t-s)\int_\Omega\nabla u(\nabla u(s)-\nabla u)dxdsdt. \end{aligned}\tag{4.4}$$

By using the assumption of Theorem 4.2, i.e $b > p^+(1-\gamma_0)$ and inequality (3.7), we get

$$
\begin{aligned}
&(1-\gamma_0)\int_S^T E(t)^\alpha\Big(\|u_t\|^2+\int_\Omega |u|^{q(x)}dx+I(u(t))\Big)dt\\
&\leq(2-\gamma_0)\int_S^T E(t)^\alpha\|u_t\|^2dt+\gamma_0(1-l)\int_S^T E(t)^\alpha\|\nabla u\|^2dt\\
&\quad-\gamma_0\int_S^T E(t)^\alpha\|\nabla u\|^{2(r+1)}dt+(1-\gamma_0)\int_S^T E(t)^\alpha(g\odot\nabla u)(t)dt\\
&\quad-\Big[E(t)^\alpha\int_\Omega uu_tdx\Big]\Big|_S^T+\alpha\int_S^T E(t)^{\alpha-1}E'(t)\int_\Omega uu_tdxdt\\
&\quad-\int_S^T E(t)^\alpha\|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_tdxdt\\
&\quad+\int_S^T E(t)^\alpha\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxdsdt. \qquad (4.5)
\end{aligned}
$$

At this point, we estimate the last four terms on the right-hand side of (4.5). Firstly, by using Hölder, Poincaré and Young inequalities, and since $E(t)$ is a nonincreasing and nonnegative function, we obtain

$$
\begin{aligned}
|-\Big[E(t)^\alpha\int_\Omega uu_tdx\Big]\Big|_S^T| &= |E(S)^\alpha\int_\Omega u(x,S)u_t(x,S)dx-E(T)^\alpha\int_\Omega u(x,T)u_t(x,T)dx|\\
&\leq E(S)^\alpha\Big|\int_\Omega u(x,S)u_t(x,S)dx-\int_\Omega u(x,T)u_t(x,T)dx\Big|\\
&\leq E(S)^\alpha\Big[\|u(x,S)\|\|u_t(x,S)\|+\|u(x,T)\|\|u_t(x,T)\|\Big]\\
&\leq C_1E(S)^{\alpha+1}\\
&\leq C_1E(0)^\alpha E(S)\\
&\leq C_2E(S), \qquad (4.6)
\end{aligned}
$$

where (3.8) has been used. Also, we have

$$
\begin{aligned}
\Big|\alpha\int_S^T E(t)^{\alpha-1}E'(t)\int_\Omega uu_tdxdt\Big| &\leq \frac{\alpha}{2}\int_S^T |E(t)^{\alpha-1}E'(t)|(\|u\|^2+\|u_t\|^2)dt\\
&\leq C_3\int_S^T E(t)^\alpha(-E'(t))dt\\
&= -\frac{C_3}{\alpha+1}\int_S^T d(E(t)^{\alpha+1})\\
&= \frac{C_3}{\alpha+1}(E(S)^{\alpha+1}-E(T)^{\alpha+1})\\
&\leq C_3E(S)^{\alpha+1}\\
&\leq C_3E(0)^\alpha E(S)\\
&\leq C_4E(S). \qquad (4.7)
\end{aligned}
$$

Moreover, by using Hölder, Poincaré and Young inequalities, we get

$$
\begin{aligned}
|\int_S^T E(t)^\alpha\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxdsdt| &\leq \int_S^T E(t)^\alpha|\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxds|dt\\
&\leq\int_S^T E(t)^\alpha\big(\gamma_0 l\|\nabla u\|^2+\frac{1-l}{2\gamma_0 l}(g\odot\nabla u)(t)\big)dt\\
&=\gamma_0 l\int_S^T E(t)^\alpha\|\nabla u\|^2dt+\frac{1-l}{2\gamma_0 l}\int_S^T E(t)^\alpha(g\odot\nabla u)(t)dt. \qquad (4.8)
\end{aligned}
$$

On the other hand, by using Hölder, Poincaré, Young inequalities, Lemma 4.1 and (2.5), we obtain for any $\delta_0 > 0$:

$$\begin{aligned}
\Big|\int_S^T E(t)^\alpha \|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_t dxdt\Big| \leq& \int_S^T E(t)^\alpha \|\nabla u\|^{2r}\Big|\int_\Omega u|u_t|^{m(x)-2}u_t dx\Big|dt \qquad (4.9)\\
\leq& \int_S^T E(t)^\alpha \|\nabla u\|^{2r}\Big(\delta_0\int_\Omega |u|^{m(x)}dx + \int_\Omega C_{\delta_0}(x)|u_t|^{m(x)}dx\Big)dt\\
\leq& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + \overline{C}_{\delta_0}\int_S^T E(t)^\alpha\|\nabla u\|^{2r}\int_\Omega |u_t|^{m(x)}dxdt\\
\leq& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + \overline{C}_{\delta_0}\int_S^T E(t)^\alpha(-E'(t))dt\\
=& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + \frac{\overline{C}_{\delta_0}}{\alpha+1}\int_T^S d(E(t)^{\alpha+1})\\
\leq& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + C_5E(S)^{\alpha+1}\\
\leq& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + C_5E(0)^\alpha E(S)\\
\leq& \delta_0\gamma_1\int_S^T E(t)^\alpha \|\nabla u\|^{2(r+1)}dt + C_6E(S), \qquad (4.10)
\end{aligned}$$

where $\overline{C}_{\delta_0}$ is an upper bound of $C_{\delta_0}(x) = \frac{m(x)-1}{m(x)}(\delta_0 m(x))^{-\frac{1}{m(x)-1}}$. Utilizing (4.6)-(4.9) into (4.5) and setting $\delta_0 := \frac{\gamma_0}{\gamma_1}$, then we have

$$\begin{aligned}
(1-\gamma_0)\int_S^T E(t)^\alpha\Big(\|u_t\|^2 + \int_\Omega |u|^{q(x)}dx + I(u(t))\Big)dt \leq& (2-\gamma_0)\int_S^T E(t)^\alpha\|u_t\|^2dt + \gamma_0\int_S^T E(t)^\alpha\|\nabla u\|^2dt\\
&+ (1+\frac{1-l}{2\gamma_0 l}-\gamma_0)\int_S^T E(t)^\alpha(g\odot\nabla u)(t)dt + C_7E(S), \qquad (4.11)
\end{aligned}$$

where inequality (3.7) and the assumptions of Theorem 4.2 about γ_0 and p^+ have been used.

Therefore, by using the definition of $E(t)$, inequality (3.8) and since $\xi(t)$ is a positive differentiable function, we deduce

$$\begin{aligned}
[2(1-\gamma_0)-\frac{\gamma_0}{\beta l}]\int_S^T E(t)^{\alpha+1}dt \leq& (2-\gamma_0)\int_S^T E(t)^\alpha\|u_t\|^2dt\\
&+ (1+\frac{1-l}{2\gamma_0 l}-\gamma_0)\int_S^T E(t)^\alpha\xi(t)(g\odot\nabla u)(t)dt + C_7E(S). \qquad (4.12)
\end{aligned}$$

At this point, we estimate the first term on the right hand side of (4.12). For this goal, by using the inequality (3.6) we have

$$\|\nabla u\|^{2r} \leq \left(\frac{E(0)}{\beta l}\right)^r, \qquad (4.13)$$

thus, by using (2.5), (4.13) and for sufficiently small r ($r < \frac{m^-}{2}$), we obtain

$$
\begin{aligned}
(2-\gamma_0)\int_S^T E(t)^\alpha \|u_t\|^2 dt \leq & (2-\gamma_0)\int_S^T E(t)^\alpha \left(\int_{\Omega_-}|u_t|^2dx+\int_{\Omega_+}|u_t|^2dx\right)dt \\
\leq & C_8\int_S^T E(t)^\alpha \Big[\left(\int_{\Omega_-}|u_t|^{m^+}dx\right)^{\frac{2}{m^+}}+\left(\int_{\Omega_+}|u_t|^{m^-}dx\right)^{\frac{2}{m^-}}\Big]dt \\
\leq & C_8\int_S^T E(t)^\alpha \Big[\left(\int_{\Omega_-}|u_t|^{m(x)}dx\right)^{\frac{2}{m^+}}+\left(\int_{\Omega_+}|u_t|^{m(x)}dx\right)^{\frac{2}{m^-}}\Big]dt \\
\leq & C_8\int_S^T E(t)^\alpha \Big[\left(\frac{\|\nabla u\|^{2r}}{\left(\frac{E(0)}{\beta l}\right)^r}\int_\Omega |u_t|^{m(x)}dx\right)^{\frac{2}{m^+}}+\left(\frac{\|\nabla u\|^{2r}}{\left(\frac{E(0)}{\beta l}\right)^r}\int_\Omega |u_t|^{m(x)}dx\right)^{\frac{2}{m^-}}\Big]dt \\
\leq & C_9\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^+}}dt+C_9\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^-}}dt. \qquad (4.14)
\end{aligned}
$$

Thanks to the Young's inequality, we get for any $\varepsilon > 0$

$$
\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^+}}dt \leq \frac{\alpha\varepsilon}{\alpha+1}\int_S^T E(t)^{\alpha+1}dt+\frac{1}{(\alpha+1)\varepsilon^\alpha}\int_S^T \left(-E'(t)\right)^{\frac{2(\alpha+1)}{m^+}}dt.
$$

We take $\alpha = \frac{m^+-2}{2}$ to find

$$
\begin{aligned}
\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^+}}dt \leq & \varepsilon C_{10}\int_S^T E(t)^{\alpha+1}dt+C_{\varepsilon,m^+}\int_S^T \left(-E'(t)\right)dt \\
\leq & \varepsilon C_{10}\int_S^T E(t)^{\alpha+1}dt+C_{\varepsilon,m^+}E(S). \qquad (4.15)
\end{aligned}
$$

On the other hand, if $m^- = 2$, then we have

$$
\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^-}}dt = \int_S^T E(t)^\alpha \left(-E'(t)\right)dt \leq C_\varepsilon E(S).
$$

Moreover, if $m^- > 2$, then by using the Young's inequality, we obtain

$$
\begin{aligned}
\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^-}}dt \leq & \frac{\varepsilon(m^--2)}{m^-}\int_S^T E(t)^{\frac{\alpha m^-}{m^--2}}dt+\frac{2\varepsilon^{-\frac{m^--2}{2}}}{m^-}\int_S^T \left(-E'(t)\right)dt \\
\leq & \varepsilon C_{11}\int_S^T E(t)^{\frac{\alpha m^-}{m^--2}}dt+C_{\varepsilon,m^-}E(S), \qquad (4.16)
\end{aligned}
$$

it is easy to see that $\frac{\alpha m^-}{m^--2} = \alpha+1+\frac{m^+-m^-}{m^--2}$ (note that we had $\alpha = \frac{m^+-2}{2}$), thus the inequality (4.16) can be rewrite as

$$
\begin{aligned}
\int_S^T E(t)^\alpha \left(-E'(t)\right)^{\frac{2}{m^-}}dt \leq & \varepsilon C_{11}\int_S^T E(t)^{\alpha+1}E(t)^{\frac{m^+-m^-}{m^--2}}dt+C_{\varepsilon,m^-}E(S) \\
\leq & \varepsilon C_{11}E(S)^{\frac{m^+-m^-}{m^--2}}\int_S^T E(t)^{\alpha+1}dt+C_{\varepsilon,m^-}E(S) \\
\leq & \varepsilon C_{12}\int_S^T E(t)^{\alpha+1}dt+C_{\varepsilon,m^-}E(S). \qquad (4.17)
\end{aligned}
$$

By combining (4.15) and (4.17) with (4.14), we obtain

$$
(2-\gamma_0)\int_S^T E(t)^\alpha \|u_t\|^2 dt \leq \varepsilon C_{13}\int_S^T E(t)^{\alpha+1}dt+C_\varepsilon E(S), \qquad (4.18)
$$

where $C_\varepsilon = \max\{(2-\gamma_0)C_{\varepsilon,m^-},\ (2-\gamma_0)C_{\varepsilon,m^+}\}$. Now, by using the condition (A1), we have

$$\begin{aligned}\int_S^T E(t)^\alpha \xi(t)(g\odot\nabla u)(t)dt \leq& \int_S^T E(t)^\alpha(-g'\odot\nabla u)(t)dt\\ \leq& \int_S^T E(t)^\alpha\left(-E'(t)\right)dt\\ \leq& E(S)^{\alpha+1}\\ \leq& E(0)^\alpha E(S)\\ \leq& C_{14}E(S).\end{aligned}\tag{4.19}$$

Finally, suppose that ε is sufficiently small and since $\gamma_0 < \frac{2\beta l}{2\beta l+1}$, thus by utilizing (4.18) and (4.19) into the inequality (4.12), we get

$$\int_S^T E(t)^{\alpha+1}dt \leq C_{15}E(S).\tag{4.20}$$

Now, if we choose $\alpha = 0$, then we have $\gamma = 1$ and $A = C_{15}$ in Lemma 2.4 and therefore, by letting T goes to infinity, we get the result.
Moreover, by choosing $\alpha = \frac{m^+-2}{2} = \frac{\gamma-1}{2}$ $(m^+ > 2)$, then with $\gamma > 1$ hypotheses of Lemma 2.4 hold and the proof of Theorem 4.2 is completed. □

5 Blow-up

In this section, we shall prove the blow-up of solutions with positive as well as negative initial energy. Under an appropriate range of initial data and variable exponents, we show that there exists a finite time t^* such that any local solution of the problem (1.1)-(1.3) blows up at this time, that is

$$\lim_{t\to t^*} u(x,t) = +\infty.$$

The following Lemma will be used in the proof of blow-up results:

Lemma 5.1. Assume that assumptions of Theorem 2.5 hold. Moreover suppose that for sufficiently large q^-, we have $I(u(t)) < 0$. Then for any weak solution of (1.1)-(1.3), there exists a constant $\gamma_2 > 1$ such that

$$\int_\Omega |u|^{m(x)}dx \leq \gamma_2\|\nabla u\|^2.\tag{5.1}$$

Proof . By using (A2) and embedding $H_0^1(\Omega)\hookrightarrow L^{m(\cdot)}(\Omega)$, one could easily get the inequality (5.1) with positive as well as negative initial energy similar to Lemma 4.1. □

5.1 Blow-up with positive initial energy

Our blow-up result for problem (1.1)-(1.3) with positive initial energy $E(0) \geq 0$ reads as follows:

Theorem 5.2. Let $u(x,t)$ is a local solution of problem (1.1)-(1.3) and the conditions of Theorem 2.5 hold. Moreover, suppose that $E(0) > 0$ (maybe sufficiently large) and
(B1)

$$\max\{\frac{\sqrt{2}}{l},\ \frac{(r+1)(m^-+\gamma_2)}{m^-},\ \frac{(p^+)^2}{p^-}\} \leq m^+ \leq \frac{q^-}{2},$$

$$\int_\Omega u_0(x)u_1(x)dx > m^+C_*E(0) > 0,$$

Then there exists a finite time which the solutions of problem (1.1)-(1.3) blow up at this time.

Proof . Suppose that u is a global solution of the problem (1.1)-(1.3). Let define

$$A(t)=\int_\Omega uu_t dx. \tag{5.2}$$

Differentiate (5.2), we get

$$\begin{aligned} A'(t) &= \|u_t\|^2+\int_\Omega uu_{tt}dx \\ &= \|u_t\|^2-(1-\int_0^t g(s)ds)\|\nabla u\|^2-\|\nabla u\|^{2(r+1)}-a\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad -b\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)\big(\int_\Omega |\nabla u|^{p(x)}dx\big)+\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxds \\ &\quad -\|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_t dx+\int_\Omega |u|^{q(x)}dx. \end{aligned} \tag{5.3}$$

By using definitions of $A(t)$ and $E(t)$, we have for any $\delta,\varepsilon>0$:

$$\begin{aligned} A'(t) &\geq \delta(A(t)-\varepsilon E(t))-\delta\int_\Omega uu_t dx+(1+\frac{\delta\varepsilon}{2})\|u_t\|^2 \\ &\quad +(\frac{\delta\varepsilon}{2}-1)(1-\int_0^t g(s)ds)\|\nabla u\|^2+(\frac{\delta\varepsilon}{2(r+1)}-1)\|\nabla u\|^{2(r+1)} \\ &\quad +a(\frac{\delta\varepsilon}{2p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx+b\big(\frac{\delta\varepsilon}{2(p^+)^2}-\frac{1}{p^-}\big)\big(\int_\Omega |\nabla u|^{p(x)}dx\big)^2 \\ &\quad +(1-\frac{\delta\varepsilon}{q^-})\int_\Omega |u|^{q(x)}dx+\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxds \\ &\quad +\frac{\delta\varepsilon}{2}(g\odot\nabla u)(t)-\|\nabla u\|^{2r}\int_\Omega u|u_t|^{m(x)-2}u_t dx, \end{aligned} \tag{5.4}$$

where condition (A2) has been used. At this point, by using the Hölder and Young inequalities, we estimate the terms on the right hand side of (5.4) as follows

$$\int_0^t g(t-s)\big|\int_\Omega \nabla u(\nabla u(s)-\nabla u)dx\big|ds\leq (1-l)\|\nabla u\|^2+\frac{1}{4}(g\odot\nabla u)(t). \tag{5.5}$$

Also, by using (5.1) and (2.5), we have

$$\begin{aligned} \|\nabla u\|^{2r}\big|\int_\Omega u|u_t|^{m(x)-2}u_t dx\big| &\leq \|\nabla u\|^{2r}\left(\int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx+\int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx\right) \\ &\leq \frac{\gamma_2}{m^-}\|\nabla u\|^{2(r+1)}+\frac{m^+-1}{m^+}\|\nabla u\|^{2r}\int_\Omega |u_t|^{m(x)}dx \\ &\leq \frac{\gamma_2}{m^-}\|\nabla u\|^{2(r+1)}-\frac{m^+-1}{m^+}E'(t). \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) with (5.4), we deduce

$$\begin{aligned} A'(t) &\geq \delta(A(t)-\varepsilon E(t))-\delta\int_\Omega uu_t dx+(1+\frac{\delta\varepsilon}{2})\|u_t\|^2+(\frac{\delta\varepsilon l}{2}-1)\|\nabla u\|^2 \\ &\quad +(\frac{\delta\varepsilon}{2(r+1)}-\frac{\gamma_2}{m^-}-1)\|\nabla u\|^{2(r+1)}+a(\frac{\delta\varepsilon}{2p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad +b\big(\frac{\delta\varepsilon}{2(p^+)^2}-\frac{1}{p^-}\big)\big(\int_\Omega |\nabla u|^{p(x)}dx\big)^2+(1-\frac{\delta\varepsilon}{q^-})\int_\Omega |u|^{q(x)}dx \\ &\quad +\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxds \\ &\quad +(\frac{\delta\varepsilon}{2}-\frac{1}{4})(g\odot\nabla u)(t)+\frac{m^+-1}{m^+}E'(t), \end{aligned} \tag{5.7}$$

where the fact that $1-\int_0^t g(s)ds > 1-\int_0^\infty g(s)ds = l$, has been used. Now if we choose $\delta\varepsilon := 2m^+$, then we obtain

$$A'(t) \geq \frac{2m^+}{\varepsilon}(A(t)-\varepsilon E(t)) - \frac{2m^+}{\varepsilon}\int_\Omega uu_t dx + (m^++1)\|u_t\|^2 + (lm^+-1)\|\nabla u\|^2 + \frac{m^+-1}{m^+}E'(t), \tag{5.8}$$

where (B1) has been used. On the other hand, by using Cauchy-Schwartz and Young inequalities, we have

$$|\int_\Omega uu_t dx| \leq \frac{\varepsilon(lm^+-1)}{2m^+C_*^2}\|u\|^2 + \frac{m^+C_*^2}{2\varepsilon(lm^+-1)}\|u_t\|^2,$$

Thus by using Poincaré inequality, we have

$$\frac{2m^+}{\varepsilon}|\int_\Omega uu_t dx| \leq (lm^+-1)\|\nabla u\|^2 + \frac{(m^+)^2C_*^2}{\varepsilon^2(lm^+-1)}\|u_t\|^2. \tag{5.9}$$

Now we choose $\varepsilon := m^+C_*$, then we get

$$\begin{aligned} \frac{2m^+}{\varepsilon}|\int_\Omega uu_t dx| &\leq (lm^+-1)\|\nabla u\|^2 + \frac{1}{lm^+-1}\|u_t\|^2 \\ &\leq (lm^+-1)\|\nabla u\|^2 + (m^++1)\|u_t\|^2, \end{aligned} \tag{5.10}$$

where we used this fact that if $m^+ > \frac{\sqrt{2}}{l}$ then $(lm^+-1)^{-1} < lm^++1 < m^++1$. Applying (5.10) into (5.8), we obtain

$$A'(t) \geq \frac{2}{C_*}(A(t)-m^+C_*E(t)) + \frac{m^+-1}{m^+}E'(t),$$

Therefore it is easy to see that

$$\begin{aligned} A'(t) - m^+C_*E'(t) &\geq -(m^+C_* - \frac{m^+-1}{m^+})E'(t) + \frac{2}{C_*}(A(t)-m^+C_*E(t)) \\ &\geq \frac{2}{C_*}(A(t)-m^+C_*E(t)), \end{aligned}$$

Now if we set $H(t) = A(t) - m^+C_*E(t)$ then we have

$$H'(t) \geq \frac{2}{C_*}H(t). \tag{5.11}$$

By (B1), it holds that $H(0) = A(0) - m^+C_*E(0) > 0$. Therefore, by (5.11), we conclude that

$$H(t) \geq e^{2C_*^{-1}t}H(0), \quad \forall t \geq 0,$$

this shows that if time goes to infinity then $H(t)$ grows exponentially. From definitions of $H(t)$ and $A(t)$, it is easy to see that

$$H(t) = A(t) - m^+C_*E(t) \leq \int_\Omega uu_t dx \leq \frac{C_*^2}{2}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2. \tag{5.12}$$

Thus inequality (5.12) shows that since $H(t)$ grows exponentially then must $\|\nabla u\|$ or $\|u_t\|$ grow exponentially. Monotonicity of energy yields

$$E(0) = E(t) - \frac{1}{2}\int_0^t (g'\odot\nabla u)(s)ds + \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|^2 ds + \int_0^t \|\nabla u\|^{2r}\int_\Omega |u_t(s)|^{m(x)}dxds, \tag{5.13}$$

thus by (A1) and assumption of Theorem 5.2 that $0 < E(t) \leq E(0)$, therefore we obtain from (5.13)

$$\int_0^t \|\nabla u\|^{2r}\int_\Omega |u_t(s)|^{m(x)}dxds \leq E(0). \tag{5.14}$$

Finally, since $m(x) > 2$, we have

$$\int_0^t \|\nabla u\|^{2r}\int_\Omega |u_t|^2 dxds \leq C_8E(0), \tag{5.15}$$

which contradicts the previous result that $\|\nabla u\|$ or $\|u_t\|$ are exponentially growing. Therefore there exists a finite time T^* such that solutions of problem (1.1)-(1.3) blow up and proof of Theorem 5.2 is completed. □

5.2 Blow-up with negative initial energy

At this point, we shall prove the blow-up of solutions for problem (1.1)-(1.3) when the initial energy is negative. For this goal, we make the following condition on variable exponents:
(B2)

$$\max\{\frac{2(p^+)^2}{p^-}, 2(r+1)(1+\frac{\gamma_2}{m^-})\} < m^+ < q^-.$$

Now, we are in a position to state and prove our blow-up result.

Theorem 5.3. Let the assumptions of Theorem 2.5 and (B2) be satisfied and assume that $E(0) < 0$. Then the solution to the problem (1.1)-(1.3) blows up in finite time T^* such that for a positive constant η

$$T^* \leq \frac{1-\theta}{\eta\theta\psi^{\frac{\theta}{1-\theta}}(0)},$$

where $0 < \theta < 1$, and ε is sufficiently small such that

$$\psi(t) = (-E(t))^{1-\theta} + \varepsilon \int_\Omega u u_t dx.$$

Proof . Define $H(t) = -E(t)$ and thus by using (2.5), we arrive at energy relation

$$H'(t) = -E'(t) = \|\nabla u\|^{2l} \int_\Omega |u_t|^{m(x)} dx \geq 0. \tag{5.16}$$

Since $E(0) < 0$, then (5.16) gives $H(t) \geq H(0) > 0$. Also

$$H(t) \leq \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx \leq \frac{1}{q^-} \int_\Omega |u|^{q(x)} dx. \tag{5.17}$$

Define

$$\psi(t) = H^{1-\theta}(t) + \varepsilon \int_\Omega u u_t dx, \tag{5.18}$$

where $0 < \theta < 1$ and ε is sufficiently small. By taking a derivation of (5.18) and using equation (1.1), we obtain

$$\begin{aligned}
\psi'(t) &= (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon \int_\Omega u u_{tt} dx \\
&= (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon M(\|\nabla u\|^2)\|\nabla u\|^2 - \varepsilon \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)} dx \\
&\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u \nabla u(s) dx ds + \varepsilon \int_\Omega |u|^{q(x)} dx - \varepsilon\sigma(\|\nabla u\|^2) \int_\Omega u |u_t|^{m(x)-2} u_t dx.
\end{aligned}$$

Utilizing (A1) gives

$$\begin{aligned}
\psi'(t) \geq &(1-\theta)H^{-\theta}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon(1+\|\nabla u\|^{2r})\|\nabla u\|^2 - \varepsilon a \int_\Omega |\nabla u|^{p(x)} dx - \frac{\varepsilon b}{p^-}\Big(\int_\Omega |\nabla u|^{p(x)} dx\Big)^2 + \varepsilon \int_\Omega |u|^{q(x)} dx \\
&+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u \nabla u(s) dx ds - \varepsilon\sigma(\|\nabla u\|^2) \int_\Omega u |u_t|^{m(x)-2} u_t dx.
\end{aligned} \tag{5.19}$$

Using the definition of $H(t)$, it follows

$$\begin{aligned}
-\varepsilon m^+ H(t) &= \frac{\varepsilon m^+}{2}\|u_t\|^2 + \frac{\varepsilon m^+}{2}(1-\int_0^t g(s)ds)\|\nabla u\|^2 + \frac{\varepsilon m^+}{2}(g\odot\nabla u)(t) \\
&+\varepsilon a m^+ \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{\varepsilon m^+ b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 \\
&-\varepsilon m^+ \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx + \frac{\varepsilon m^+}{2(r+1)}\|\nabla u\|^{2(r+1)} \\
&\geq \frac{\varepsilon m^+}{2}\|u_t\|^2 + \frac{\varepsilon m^+}{2}(1-\int_0^t g(s)ds)\|\nabla u\|^2 + \frac{\varepsilon m^+}{2}(g\odot\nabla u)(t) \\
&+\frac{\varepsilon m^+ a}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{\varepsilon m^+ b}{2(p^+)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
&-\frac{\varepsilon m^+}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{\varepsilon m^+}{2(r+1)}\|\nabla u\|^{2(r+1)}, \qquad (5.20)
\end{aligned}$$

where the condition (A1) has been used. By combining (5.20) and (5.19), we arrive at

$$\begin{aligned}
\psi'(t) &\geq \varepsilon m^+ H(t) + (1-\theta)H^{-\theta}(t)H'(t) + \frac{\varepsilon(m^++2)}{2}\|u_t\|^2 + \left(\frac{\varepsilon m^+ l}{2}-\varepsilon\right)\|\nabla u\|^2 + \varepsilon\big(\frac{m^+}{2(r+1)}-1\big)\|\nabla u\|^{2(r+1)} \\
&+\varepsilon a(\frac{m^+}{p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx + \frac{\varepsilon m^+}{2}(g\odot\nabla u)(t) \\
&+\varepsilon b(\frac{m^+}{2(p^+)^2}-\frac{1}{p^-})\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon(1-\frac{m^+}{q^-})\int_\Omega |u|^{q(x)}dx \\
&+\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u\nabla u(s)dxds - \varepsilon\sigma(\|\nabla u\|^2)\int_\Omega u|u_t|^{m(x)-2}u_t dx. \qquad (5.21)
\end{aligned}$$

To estimate the last two terms in (5.21), by applying Hölder's inequality, Young's inequality, (5.1), (A1) and (5.16), we obtain

$$\begin{aligned}
\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u\nabla u(s)dxds =&\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(\nabla u(s)-\nabla u)dxds + \varepsilon\int_0^t g(s)ds\|\nabla u\|^2 \\
\leq&\varepsilon l\|\nabla u\|^2 + \frac{\varepsilon(1-l)}{4l}(g\odot\nabla u)(t) + \varepsilon\int_0^{+\infty} g(s)ds\|\nabla u\|^2 \\
=&\varepsilon\|\nabla u\|^2 + \frac{\varepsilon(1-l)}{4l}(g\odot\nabla u)(t). \qquad (5.22)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(\|\nabla u\|^2)|\int_\Omega u|u_t|^{m(x)-2}u_t dx| &\leq \|\nabla u\|^{2r}\left(\int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx\right) \\
&\leq \frac{\gamma_2}{m^-}\|\nabla u\|^{2(r+1)} + \frac{m^+-1}{m^+}\|\nabla u\|^{2r}\int_\Omega |u_t|^{m(x)}dx \\
&= \frac{\gamma_2}{m^-}\|\nabla u\|^{2(r+1)} + \frac{m^+-1}{m^+}H'(t). \qquad (5.23)
\end{aligned}$$

Utilizing (5.22) and (5.23) into (5.21), we get

$$\begin{aligned}
\psi'(t) &\geq \varepsilon m^+ H(t) + (1-\theta)H^{-\theta}(t)H'(t) - \frac{\varepsilon(m^+-1)}{m^+}H'(t) + \frac{\varepsilon(m^++2)}{2}\|u_t\|^2 \\
&+\varepsilon\left(\frac{m^+ l}{2}-2\right)\|\nabla u\|^2 + \varepsilon\left(\frac{m^+}{2(r+1)}-\frac{\gamma_2}{m^-}-1\right)\|\nabla u\|^{2(r+1)} \\
&+\frac{\varepsilon}{2}\left(m^+-\frac{1-l}{2l}\right)(g\odot\nabla u)(t) + \varepsilon a(\frac{m^+}{p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx \\
&+\varepsilon b\left(\frac{m^+}{2(p^+)^2}-\frac{1}{p^-}\right)\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 + \varepsilon(1-\frac{m^+}{q^-})\int_\Omega |u|^{q(x)}dx. \qquad (5.24)
\end{aligned}$$

At this point, applying (B2) into (5.24) to obtain

$$\psi'(t) \geq \varepsilon m^+ H(t) + (1-\theta)H^{-\theta}(t)H'(t) - \frac{\varepsilon(m^+-1)}{m^+}H'(t). \tag{5.25}$$

Thanks to (5.16) and the definition of $H(t)$, there exists a positive constant K (specified later) such that

$$H'(t) = -E'(t) \leq KH^{-\theta}(t)H'(t),$$

substituting this inequality into (5.25) and since $H(t) \geq 0$, we arrive at

$$\psi'(t) \geq \big[1-\theta - \frac{\varepsilon(m^+-1)}{m^+}K\big]H^{-\theta}(t)H'(t). \tag{5.26}$$

At this moment, by choosing K large enough and picking ε sufficiently small such that $1-\theta-\frac{\varepsilon(m^+-1)}{m^+}K > 0$, we consequently get

$$\psi(t) \geq \psi(0) = H^{1-\theta}(0) + \varepsilon \int_\Omega u_0 u_1 dx > 0, \quad \forall\, t \geq 0.$$

Suppose that C is a generic constant. By using the Hölder and Young inequalities, we have

$$|\int_\Omega u u_t dx|^{\frac{1}{1-\theta}} \leq C(\|u\|_{q^-}^{\frac{1}{1-\theta}}\|u_t\|^{\frac{1}{1-\theta}}) \leq C(\|u\|_{q^-}^{q^-} + \|u_t\|^2 + H(t)). \tag{5.27}$$

Combining (5.27) with (5.18), we obtain for some $\eta > 0$

$$\begin{aligned}\psi^{\frac{1}{1-\theta}}(t) =& \Big[H^{1-\theta}(t) + \varepsilon \int_\Omega u u_t dx\Big]^{\frac{1}{1-\theta}}\\ \leq& 2^{\frac{1}{1-\theta}}\Big(H(t) + \varepsilon^{\frac{1}{1-\theta}}|\int_\Omega u u_t dx|^{\frac{1}{1-\theta}}\Big)\\ \leq& C(\|u\|_{q^-}^{q^-} + \|u_t\|^2 + H(t)) \leq \eta^{-1}\psi'(t),\end{aligned}$$

therefore

$$\psi'(t) \geq \eta \psi^{\frac{1}{1-\theta}}(t). \tag{5.28}$$

Integrating (5.28) from 0 to t, we deduce

$$\psi^{\frac{\theta}{1-\theta}}(t) \geq \frac{1}{\psi^{-\frac{\theta}{1-\theta}}(0) - \frac{\eta\theta t}{1-\theta}}.$$

This shows that solutions blow up in finite time $t^* = \frac{1-\theta}{\eta\theta\psi^{\frac{\theta}{1-\theta}}(0)}$, and proof of Theorem 5.3 has been completed. □

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