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Common fixed points for Hybrid pair of generalized non-expansive mappings by three-step iterative scheme

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Abstract

In this paper, we introduce a three-step iterative scheme, called the MF-iteration process to approximate a common fixed point for a hybrid pair $\{\tau, T\}$ of single-valued and multi-valued maps satisfying a generalized contractive condition defined on uniformly convex Banach spaces. We establish the strong convergence theorem for the proposed process under some basic boundary conditions. We give a numerical example to prove our results' convergence rate. Further, we compare the convergence speed of Sokhuma and Kaewkhao [29] and MF-iterations. we show numerically that the considered iterative scheme converges faster than Sokhuma and Kaewkhao [29] for single-valued and multi-valued non-expansive mappings. Our newly proven results generalize several relevant results in the literature.

Keywords: uniformly convex Banach spaces, Suzuki's Generalized Non-Expansive, Three step iterative scheme, Common fixed point
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1 Introduction

Throughout this paper, $\mathbb{N}$ is the set of all positive integers. We consider that E is a nonempty subset of a Banach space X. We will denote by $FB(X)$ the family of nonempty bounded closed subsets of X and $\mathrm{Fix}(\tau)$ the set of all fixed points of the mapping τ on E. Let A, B be two bounded subsets of X. The Hausdorff distance between A and B is defined by

$$H(A,B) = \max\{\sup_{x\in A} \mathrm{dist}(x,B), \sup_{x\in B} \mathrm{dist}(x,A)\}, A, B \in FB(X),$$

where $\mathrm{dist}(x,B) = \inf\{||x-b|| : b \in B\}$ is the distance from the point x to the subset B.

A mapping $\tau : E \to E$ is said to be non-expansive if

$$||\tau x - \tau y|| \leq ||x-y||, \forall x, y \in E.$$

A point x is called a fixed point of τ if $\tau x = x$. We know that $\mathrm{Fix}(\tau)$ is nonempty when X is uniformly convex, E is bounded closed convex subset of X and τ is non-expansive mapping, (cf. [3]).

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A multi-valued mapping $T : E \to FB(X)$ is said to be non-expansive if

$$H(Tx, Ty) \leq ||x - y||, \forall x, y \in E.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$. We use the notation $\mathrm{Fix}(T)$ to stand for the set of fixed points of the mapping T and $\mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$ stand for the set of common fixed points of τ and T. Precisely, a point x is called a common fixed point of τ and T if $x = \tau x \in Tx$.

In 2006, S. Dhompongsa et al. [6] proved a common fixed point theorem for two non-expansive commuting mappings, (see [6, Theorem 4.2]).

In 2008, Suzuki [31] introduced the concept of generalized non-expansive mappings which is also called condition (C) and is defined as:

A self-mapping τ on E is said to satisfy condition (C) if,

$$\frac{1}{2}||x - \tau x|| \leq ||x - y|| \Rightarrow ||\tau x - \tau y|| \leq ||x - y||, \forall x, y \in E.$$

Suzuki obtained existence of fixed points and convergence results for such mappings. Suzuki also showed that the notion of mappings satisfying condition (C) is more general than the notion of non-expansive mappings.

The following example supports the above claim.

Example 1.1. Define a self mapping τ on $[0, 3]$ by

$$\tau x = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

Here τ satisfy Suzuki's condition (C), but τ is not a non-expansive mapping.

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. However, once the existence of a fixed point of some mappings is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them.

Iterative techniques for approximating fixed points of non-expansive single-valued mappings have been investigated by various authors (e.g., [20, 14, 15, 25, 34]using the Mann iteration scheme or the Ishikawa iteration scheme. Many iterative processes have been developed and it is impossible to cover them all. The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed points. Some of the other well-known iterative processes are those of, Agarwal [2], Noor [23], Abbas [1], SP [8], Normal-S [26], Picard Mann [17], Picard-S [12], Thakur New [35]. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings.

The study of fixed points for multi-valued contractions and non-expansive mappings was initiated by Nadler [22] and Markin [21]. By now, there exists an extensive literature on multi-valued fixed point theory which has applications in convex optimization, differential inclusions, fractals, discontinuous differential equations, optimal control, computing homology of maps, computer-assisted proofs in dynamics, digital imaging and economics (e.g., [11, 16]). Moreover, the existence of fixed points for multi-valued non-expansive mappings in uniformly convex Banach spaces was proved by Lim [19]. Many different iterative processes have been used to approximate the fixed points of multi-valued non-expansive mappings.

In 2005, Sastry and Babu [27] defined Ishikawa iteration scheme for multi-valued mappings in the setting of Hilbert spaces.

In 2007, Panyanak [24] extended the results of sastry and Babu uniformly convex Banach spaces for multi-valued non-expansive mappings.

In 2009, Song and Wang [30], Shahzad and Zegete [28], also in 2011, Gholamjiak and Suantai [10], Gholamjiak et al. [9], introduced the different modified Ishikawa iteration schemes and improved the results Sastry and Babu [27] and Panyanak [24] in many ways.

The case of two mappings in iterative processes has also remained under study since Das and Debata [5] gave and studied a two mappings scheme. Also see, for example, [33] and [18]. Note that two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see, for example, [32].

In 1986, Das and Debata [5] extended Ishikawa scheme for two single-valued mappings while Hu et al. [13] obtained common fixed point theorems for two multi-valued non-expansive mappings satisfying certain contractive conditions via two-step iteration scheme.

In 2010, Sokhuma and Kaewkhao [29], introduced a modified Ishakawa iterative process involving a pair of hybrid mappings in Banach spaces and utilized the same to prove their results.

In this paper, we introduce an iterative process in a new sense, called the "MF-iteration method" with respect to a pair of single-valued Suzuki's generalized non-expansive mapping and multi-valued non-expansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

2 Preliminaries

In this section, to make our exposition self-contained, we present the relevant background material needed to prove our results.

Definition 2.1. Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in E \subset X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} ||x_n - x||.$$

The asymptotic radius of $\{x_n\}$ relative to E is defined by

$$r(E, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in E\}.$$

The asymptotic center of $\{x_n\}$ relative to E is the set

$$A(E, \{x_n\}) = \{x \in E : r(x, \{x_n\}) = r(E, \{x_n\})\}.$$

It is well known that, $A(E, \{x_n\})$ consists exactly one point, in the case when X is uniformly convex Banach space.

A fundamental principle which plays a key role is the demiclosedness principle.

Definition 2.2. Let E be a nonempty, closed and convex subset of a Banach space X. A mapping $\tau : E \to X$ is called demiclosed with respect to $y \in X$, if for each sequence $\{x_n\}$ in E and each $x \in E$, $\{x_n\}$ converges weakly to x and $\{\tau x_n\}$ converges strongly to y imply that $\tau x = y$.

Theorem 2.3. [4] Let E be a nonempty closed convex subset of a uniformly convex Banach space X, and let $\tau : E \to E$ be a non-expansive mapping. If a sequence $\{x_n\}$ in E converges weakly to p and $\{x_n - \tau x_n\}$ converges to 0 as $n \to \infty$, then $p \in \mathrm{Fix}(\tau)$.

The following lemma contains an important property of the uniformly convex Banach space which will be used in the sequel.

Lemma 2.4. ([7]) Let X be a uniformly convex Banach space, $\{u_n\}$ a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \in \mathbb{N}$, $\{x_n\}$ and $\{y_n\}$ sequences of X such that $\limsup_{n\to\infty} ||x_n|| \leq a, \limsup_{n\to\infty} ||y_n|| \leq a$ and $\lim_{n\to\infty} ||u_n x_n + (1 - u_n)y_n|| = a$ for some $a \geq 0$. Then, $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

The following observation will be used in proving our results, and the proof is straightforward.

Lemma 2.5. Let X be a Banach space and E be a nonempty closed convex subset of X. Then,

$$\mathrm{dist}(y, Ty) \leq ||y - x|| + \mathrm{dist}(x, Tx) + H(Tx, Ty) \tag{2.1}$$

where $x, y \in E$ and T is a multi-valued non-expansive mapping from E into $FB(E)$.

In (2005), Sastry and Babu [27] defined Ishikawa iterative scheme for multi-valued mappings as follows.

Let E be a compact convex subset of a Hilbert space X and $T : E \to P(E)$ be a multi-valued mapping and $w \in \text{Fix}(T)$. In this scheme the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ with $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $||z_n - w|| = \text{dist}(w, Tx_n)$ and $||z'_n - w|| = \text{dist}(w, Ty_n)$.

They also proved the strong convergence of the above Ishikawa iterative scheme for a multi-valued non-expansive mapping T with a fixed point w under some certain conditions in a Hilbert space.

In 2007, Panyanak [24] extended the results of Sastry and Babu to a uniformly convex Banach space and also modified the iterative scheme of Sastry and Babu [27] and imposed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method.

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a,b]$, $0 < a < b < 1$, with $z_n \in Tx_n$ such that $||z_n - u_n|| = \text{dist}(u_n, Tx_n)$ and $u_n \in \text{Fix}(T)$ such that $||x_n - u_n|| = \text{dist}(x_n, \text{Fix}(T))$, respectively. Moreover, $z'_n \in Tx_n$ and $v_n \in \text{Fix}(T)$ such that $||z'_n - v_n|| = \text{dist}(v_n, Tx_n)$ and $||y_n - v_n|| = \text{dist}(y_n, \text{Fix}(T))$, respectively.

In 2009, Song and Wang [30] improved the results of [24]. By using the following iteration scheme, they solved and revised the gap and give the partial answer to the question raised by Panyanak [24].

Let $\alpha_n, \beta_n \in [0,1)$ and $\gamma_n \in (0,+\infty)$ such that, $\lim_{n\to\infty} \gamma_n = 0$. In this scheme the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.4}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $||z_n - z'_n|| \leq H(Tx_n, Ty_n) + \gamma_n$ and $||z_{n+1} - z'_n|| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$, respectively. Moreover, $\gamma_n \in (0,+\infty)$ such that, $\lim_{n\to\infty} \gamma_n = 0$.

At the same period, Shahzad and Zegeye [28] modified the Ishikawa iterative scheme and extended the result of Sastry and Babu [27] and Song and Wang [30] to a multi-valued quasi non-expansive mappings. They also, relaxed the end point condition along with compactness of the domain by using the following modified iteration scheme and gave the affirmative answer to the Panyanak's question in a more general setting.

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $\alpha_n, \beta_n \in [0,1]$, $z_n \in P_T x_n$ and $z'_n \in P_T y_n$.

In 2010, Sokhuma and Kaewkhao [29] introduced the following modified Ishikawa iteration scheme for a pair hybrid mappings.

Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ a single-valued non-expansive mapping and $T : E \to FB(E)$ a multi-valued non-expansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6}$$

where $z_n \in Tx_n$, and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

In this paper, we introduce a new iteration process called it "MF-iteration process" as follows:

Let E be a nonempty closed bounded convex subset of a Banach space X, $\tau : E \to E$ a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ a multi-valued non-expansive mapping. The sequence $\{x_n\}$ of the MF-iteration is defined by

$$\begin{cases} x = x_1 \in E, \\ x_{n+1} = (1-\alpha_n)\tau z_n + \alpha_n \tau y_n, \\ y_n = (1-\beta_n)\tau z_n + \beta_n z_n', \\ z_n = (1-\gamma_n)x_n + \gamma_n \tau x_n, \;\; n \in \mathbb{N}, \end{cases} \tag{2.7}$$

where $z_n' \in Tz_n$, and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$.

3 Main Results

In this section, we give the sufficient conditions which imply the existence of common fixed points for single-valued Suzuki's generalized non-expansive mappings and multi-valued non-expansive mappings in uniformly convex Banach spaces. First, we obtain the following useful lemmas which will be used in our main results.

Lemma 3.1. Let E be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\tau : E \to E$ be a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ be a multi-valued non-expansive mapping with $\mathrm{Fix}(\tau) \cap \mathrm{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (2.7). Then, $\lim_{n\to\infty} ||x_n - w||$ exists for all $w \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$.

Proof . Let $w \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$. Since E is convex then for any $x_n, y_n, z_n \in E$ we have

$$\frac{1}{2}||w - \tau w|| = 0 \leq ||w - x_n||$$
$$\frac{1}{2}||w - \tau w|| = 0 \leq ||w - y_n||$$
$$\frac{1}{2}||w - \tau w|| = 0 \leq ||w - z_n||$$

since τ is a Suzuki's generalized non-expansive mapping, we have

$$\begin{aligned} ||\tau x_n - \tau w|| &\leq ||x_n - w|| \\ ||\tau y_n - \tau w|| &\leq ||y_n - w|| \\ ||\tau z_n - \tau w|| &\leq ||z_n - w|| \end{aligned} \tag{3.1}$$

Now from iterative scheme (2.7), we get

$$\begin{aligned} ||z_n - w|| &= ||(1-\gamma_n)x_n + \gamma_n \tau x_n - w|| \\ &= ||(1-\gamma_n)(x_n - w) + \gamma_n(\tau x_n - w)|| \\ &\leq (1-\gamma_n)||x_n - w|| + \gamma_n ||\tau x_n - w|| \\ &\leq (1-\gamma_n)||x_n - w|| + \gamma_n ||x_n - w|| \\ &= ||x_n - w||, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} ||y_n - w|| &= ||(1-\beta_n)\tau z_n + \beta_n z_n' - w|| \\ &= ||(1-\beta_n)(\tau z_n - w) + \beta_n(z_n' - w)|| \\ &\leq (1-\beta_n)||\tau z_n - w|| + \beta_n ||z_n' - w|| \\ &\leq (1-\beta_n)||z_n - w|| + \beta_n \mathrm{dist}(z_n', Tw) \\ &\leq (1-\beta_n)||z_n - w|| + \beta_n \mathrm{H}(Tz_n, Tw) \\ &\leq (1-\beta_n)||z_n - w|| + \beta_n ||z_n - w|| \\ &= ||z_n - w|| \\ &\leq ||x_n - w||. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), we have

$$\begin{aligned} ||x_{n+1}-w|| &= ||(1-\alpha_n)\tau z_n + \alpha_n \tau y_n - w|| \\ &= ||(1-\alpha_n)(\tau z_n - w) + \alpha_n(\tau y_n - w)|| \\ &\leq (1-\alpha_n)||z_n - w|| + \alpha_n||y_n - w|| \\ &\leq (1-\alpha_n)||x_n - w|| + \alpha_n||x_n - w|| \\ &= ||x_n - w||. \end{aligned} \tag{3.4}$$

This shows that $\{||x_n-w||\}$ is bounded and non-increasing sequence for all $w \in \text{Fix}(\tau)\cap\text{Fix}(T)$. Hence, $\lim_{n\to\infty} ||x_n-w||$ exists. □

Lemma 3.2. Let E be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\tau : E \to E$ be a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ be a multi-valued non-expansive mapping with $\text{Fix}(\tau) \cap \text{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \text{Fix}(\tau) \cap \text{Fix}(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (2.7). If $0 < a \leq \alpha_n, \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then, $\lim_{n\to\infty} ||\tau x_n - x_n|| = 0$.

Proof . Let $w \in \text{Fix}(\tau) \cap \text{Fix}(T)$. By Lemma 3.1, $\lim_{n\to\infty} ||x_n - w||$ exists and $\{x_n\}$is bounded. Put

$$\lim_{n\to\infty} ||x_n - w|| = c. \tag{3.5}$$

From (3.1), (3.2), (3.3) and (3.5), we have

$$\begin{aligned} \limsup_{n\to\infty} ||\tau x_n - \tau w|| &\leq \limsup_{n\to\infty} ||x_n - w|| \leq c \\ \limsup_{n\to\infty} ||y_n - w|| &\leq \limsup_{n\to\infty} ||x_n - w|| \leq c \\ \limsup_{n\to\infty} ||z_n - w|| &\leq \limsup_{n\to\infty} ||x_n - w|| \leq c. \end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned} \limsup_{n\to\infty} ||\tau y_n - w|| &\leq \limsup_{n\to\infty} ||y_n - w|| \leq \limsup_{n\to\infty} ||x_n - w|| \leq c \\ \limsup_{n\to\infty} ||\tau z_n - w|| &\leq \limsup_{n\to\infty} ||z_n - w|| \leq \limsup_{n\to\infty} ||x_n - w|| \leq c. \end{aligned} \tag{3.7}$$

Again,

$$\begin{aligned} c &= \lim_{n\to\infty} ||x_{n+1} - w|| \\ &= \lim_{n\to\infty} ||(1-\alpha_n)\tau z_n + \alpha_n \tau y_n - w|| \\ &= \lim_{n\to\infty} ||(1-\alpha_n)(\tau z_n - w) + \alpha_n(\tau y_n - w)||. \end{aligned} \tag{3.8}$$

From (3.7), (3.8) and using Lemma 2.4, we have

$$\lim_{n\to\infty} ||\tau z_n - \tau y_n|| = 0. \tag{3.9}$$

Now,

$$\begin{aligned} ||x_{n+1} - w|| &= ||(1-\alpha_n)\tau z_n + \alpha_n \tau y_n - w|| \\ &\leq ||\tau z_n - w|| + \alpha_n||\tau z_n - \tau y_n||. \end{aligned}$$

Taking the **liminf** on both sides, we get,

$$c = \liminf_{n\to\infty} ||x_{n+1} - w|| \leq \liminf_{n\to\infty} ||\tau z_n - w|| \leq \liminf_{n\to\infty} ||z_n - w||. \tag{3.10}$$

So that, (3.6) and (3.10) we have,

$$\lim_{n\to\infty} ||z_n - w|| = c.$$

Then,

$$c = \lim_{n\to\infty} ||z_n - w|| = \lim_{n\to\infty} ||(1-\gamma_n)x_n + \gamma_n \tau x_n - w|| = \lim_{n\to\infty} ||(1-\gamma_n)(x_n - w) + \gamma_n(\tau x_n - w)||. \quad (3.11)$$

Owing to Equations (3.5), (3.6), (3.11) and Lemma 2.4 we get

$$\lim_{n\to\infty} ||\tau x_n - x_n|| = 0.$$

□

Lemma 3.3. Let E be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\tau : E \to E$ be a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ be a multi-valued non-expansive mapping with $\text{Fix}(\tau) \cap \text{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \text{Fix}(\tau) \cap \text{Fix}(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (2.7). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then, $\lim_{n\to\infty} ||\tau z_n - z'_n|| = 0$.

Proof . Let $w \in \text{Fix}(\tau) \cap \text{Fix}(T)$. We put, as in Lemma 3.2, $\lim_{n\to\infty} ||x_n - w|| = c$. For $n \geq 1$, by using 3.2, we have

$$\begin{aligned} ||x_{n+1} - w|| &= ||(1-\alpha_n)\tau z_n + \alpha_n \tau y_n - w|| \\ &= ||(1-\alpha_n)(\tau z_n - w) + \alpha_n(\tau y_n - w)|| \\ &\leq (1-\alpha_n)||z_n - w|| + \alpha_n ||y_n - w|| \\ &\leq (1-\alpha_n)||x_n - w|| + \alpha_n ||y_n - w||, \end{aligned}$$

and hence

$$\begin{aligned} ||x_{n+1} - w|| - ||x_n - w|| &\leq -\alpha_n ||x_n - w|| + \alpha_n ||y_n - w||, \\ ||x_{n+1} - w|| - ||x_n - w|| &\leq \alpha_n(||y_n - w|| - ||x_n - w||), \\ \frac{||x_{n+1} - w|| - ||x_n - w||}{\alpha_n} &\leq ||y_n - w|| - ||x_n - w||. \end{aligned}$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\frac{||x_{n+1} - w|| - ||x_n - w||}{b} + ||x_n - w|| \leq ||y_n - w||.$$

Thus,

$$\liminf_{n\to\infty} \left(\frac{||x_{n+1} - w|| + ||x_n - w||}{b} + ||x_n - w||\right) \leq \liminf_{n\to\infty} ||y_n - w||.$$

It follows that

$$c \leq \liminf_{n\to\infty} ||y_n - w||. \quad (3.12)$$

From 3.3, we have

$$\limsup_{n\to\infty} ||y_n - w|| \leq \limsup_{n\to\infty} ||x_n - w|| = c. \quad (3.13)$$

Then, from 3.12 and 3.13, we have

$$\begin{aligned} c &= \lim_{n\to\infty} ||y_n - w|| \\ &= \lim_{n\to\infty} ||(1-\beta_n)\tau z_n + \beta_n z_n' - w|| \\ &= \lim_{n\to\infty} ||(1-\beta_n)(\tau z_n - w) + \beta_n(z_n' - w)||, \end{aligned}$$

since

$$\begin{aligned} \limsup_{n\to\infty} ||\tau z_n - w|| &\leq \limsup_{n\to\infty} ||z_n - w|| \\ &\leq \limsup_{n\to\infty} ||x_n - w|| = c, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} ||z_n' - w|| &= \limsup_{n\to\infty} ||z_n' - Tw|| \\ &\leq \limsup_{n\to\infty} H(Tz_n, Tw) \\ &\leq \limsup_{n\to\infty} ||z_n - w|| \\ &\leq \limsup_{n\to\infty} ||x_n - w|| = c, \end{aligned}$$

by using Lemma 2.4, we have

$$\lim_{n\to\infty} ||\tau z_n - z_n'|| = 0.$$

□

Now, we are ready to state and prove our main results.

Theorem 3.4. Let E be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\tau : E \to E$ be a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ be a multi-valued non-expansive mapping with $\mathrm{Fix}(\tau) \cap \mathrm{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (2.7) and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$.

Proof . Assume that $\lim_{n\to\infty} ||x_{n_i} - y|| = 0$. From Lemma 3.2, we have

$$0 = \lim_{n\to\infty} ||\tau x_{n_i} - x_{n_i}|| = \lim_{n\to\infty} ||(I-\tau)x_{n_i}||.$$

Since $I - \tau$ is demiclosed at 0, we have $(I-\tau)(y) = 0$, and hence $y = \tau y$, that is,

$$y \in \mathrm{Fix}(\tau). \tag{3.14}$$

Now we prove $y \in \mathrm{Fix}(T)$. By using iteration (2.7), we can build the subsequence $\{z_{n_i}\}$, such that

$$\begin{aligned} ||z_{n_i} - y|| &= ||(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\tau x_{n_i} - y|| \\ &\leq ||(1-\gamma_{n_i})(x_{n_i} - y) + \gamma_{n_i}(\tau x_{n_i} - y)|| \\ &\leq (1-\gamma_{n_i})||x_{n_i} - y|| + \gamma_{n_i}||\tau x_{n_i} - y|| \\ &\leq (1-\gamma_{n_i})||x_{n_i} - y|| + \gamma_{n_i}||x_{n_i} - y|| \\ &= ||x_{n_i} - y||, \end{aligned}$$

since $\lim_{i\to\infty} ||x_{n_i} - y|| = 0$, we get

$$\lim_{n\to\infty} ||z_{n_i} - y|| = 0. \tag{3.15}$$

On the other hand, for $i \geq 1$, we have

$$\begin{aligned} ||z_{n_i} - z'_{n_i}|| &\leq ||z_{n_i} - \tau z_{n_i}|| + ||\tau z_{n_i} - z'_{n_i}|| \\ &\leq ||z_{n_i} - y|| + ||y - \tau z_{n_i}|| + ||\tau z_{n_i} - z'_{n_i}|| \\ &\leq ||z_{n_i} - y|| + ||y - z_{n_i}|| + ||\tau z_{n_i} - z'_{n_i}|| \\ &= 2||z_{n_i} - y|| + ||\tau z_{n_i} - z'_{n_i}||, \end{aligned}$$

which on using (3.15) and Lemma 3.3, gives that

$$\lim_{i\to\infty} ||z_{n_i} - z'_{n_i}|| = 0. \tag{3.16}$$

By Lemma 2.5, we have

$$\begin{aligned} dist(y, Ty) &\leq ||y - z_{n_i}|| + \mathrm{dist}(z_{n_i}, Tz_{n_i}) + H(Tz_{n_i}, Ty). \\ &\leq ||y - z_{n_i}|| + ||z_{n_i} - z'_{n_i}|| + ||z_{n_i} - y|| \\ &= 2||y - z_{n_i}|| + ||z_{n_i} - z'_{n_i}||. \end{aligned}$$

Taking the **limit** as $i \to \infty$, we obtain that $dist(y, Ty) = 0$. Since Ty is closed it follows that $y \in Ty$, that is, $y \in \mathrm{Fix}(T)$. Therefore $y \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ as desired. □

Hereafter, we arrive at the convergence theorem of the sequence of the "MF-iteration".

Theorem 3.5. Let E be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\tau : E \to E$ be a single-valued Suzuki's generalized non-expansive mapping and $T : E \to FB(E)$ be a multi-valued non-expansive mapping with $\mathrm{Fix}(\tau) \cap \mathrm{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (2.7) and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a common fixed point of τ and T.

Proof . Since $\{x_n\}$ is contained in E which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in E$, that is, $\lim_{i\to\infty} ||x_{n_i} - y|| = 0$. By Theorem 3.4, we have $y \in \mathrm{Fix}(\tau) \cap \mathrm{Fix}(T)$, and by Lemma 3.1, we have that $\lim_{n\to\infty} ||x_n - y||$ exists. It must be the case that $\lim_{n\to\infty} ||x_n - y|| = \lim_{i\to\infty} ||x_{n_i} - y|| = 0$. Therefore, $\{x_n\}$ converges strongly to a common fixed point y of τ and T. □

Remark 3.6. All the results in this paper generalize the corresponding results of Sokhuma and Kaewkhao [29] and many others because mappings here are generalized non-expansive and iterative scheme is more general than the others.

Now, we furnish the following numerical example to support our results.

Example 3.7. Consider $E = [1, 2]$. Define a multi-valued mapping $T : E \to FB(E)$ by $Tx = [x, 2]$, and self-mapping τ on E as

$$\tau x = \begin{cases} 3 - x, & \text{if } x \in [1, 1.1), \\ \frac{x+4}{3}, & \text{if } x \in [1.1, 2]. \end{cases}$$

Since, $H(Tx, Ty) = ||x - y||$, T is a multi-valued non-expansive mapping. Here τ is a Suzuki's generalized non-expansive mapping, but τ is not a non-expansive. Take, $x = \frac{12}{11}$ and $y = \frac{13}{11}$.

$$||x - y|| = ||\frac{12}{11} - \frac{13}{11}|| = \frac{1}{11},$$

and

$$||\tau x - \tau y|| = ||3 - \frac{12}{11} - (\frac{\frac{13}{11} + 4}{3})|| = \frac{2}{11}.$$

Hence τ is not a non-expansive mapping. Now, we verify that the mapping τ is a Suzuki's generalized non-expansive mapping. Here following cases arise:

n	1	2	3	4	5	6	7
MF-iteration	1.2	1.953488	1.997295	1.999842	1.999990	1.999999	2

Table 1:

Case I. If either $x, y \in [1, 1.1)$ or $x, y \in [1.1, 2]$. Then in both the cases τ is non-expansive mapping and hence τ is Suzuki's generalized non-expansive mapping.

Case II. Let $x \in [1, 1.1)$. Then $\frac{1}{2}||x - \tau x|| = \frac{1}{2}||x - (3 - x)|| = \frac{1}{2}||2x - 3|| \in (0.4, 0.5]$. For $\frac{1}{2}||x - \tau x|| \leq ||x - y||$, we must have $\frac{3-2x}{2} \leq y - x$ and then $y \geq 1.5$. Hence, $y \in [1.5, 2]$. Now we have

$$||x - y|| = y - x \geq 1.5 - 1.1 = 0.4,$$

and

$$||\tau x - \tau y|| = ||\frac{y+4}{3} - (3 - x)|| = ||\frac{y + 3x - 5}{3}|| \leq ||\frac{2 + 3.3 - 5}{3}|| = 0.1 < 0.4.$$

Hence

$$\frac{1}{2}||x - \tau x|| \leq ||x - y|| \Rightarrow ||\tau x - \tau y|| \leq ||x - y||.$$

Case III. Let $x \in [1.1, 2]$. Then $\frac{1}{2}||x - \tau x|| = \frac{1}{2}||\frac{x+4}{3} - x|| = ||\frac{4-2x}{6}|| = ||\frac{2-x}{3}|| \in [0, 0.3]$. For $\frac{1}{2}||x - \tau x|| \leq ||x - y||$, we must have $||\frac{2-x}{3}|| \leq |x - y|$, which gives two possibilities:

1) Let $x < y$, we have

$$\frac{2-x}{3} \leq y - x \Rightarrow \frac{2+2x}{3} \leq y \Rightarrow y \in [1.4, 2] \subseteq [1.1, 2].$$

So

$$||\tau x - \tau y|| = ||\frac{x+4}{3} - \frac{y+4}{3}|| = \frac{1}{3}||x - y|| \leq ||x - y||.$$

Hence $\frac{1}{2}||x - \tau x|| \leq ||x - y|| \Rightarrow ||\tau x - \tau y|| \leq ||x - y||$.

2) Let $x > y$,

$$\begin{aligned} \frac{2-x}{3} \leq x - y \quad &\Rightarrow \quad y \leq x - \frac{2-x}{3} = \frac{4x-2}{3} \\ &\Rightarrow \quad y \leq \frac{4x-2}{3}, \end{aligned}$$

since $x \in [1.1, 2]$, we have $y \leq 2$ and $y \leq 0.8$, then $y \in [1, 2]$. Since $y \in [1, 2]$ and $y \leq \frac{4x-2}{3}$, we have

$$\frac{3y+2}{4} \leq x \Rightarrow x \in [1.25, 2].$$

If $x \in [1.25, 2]$ and $y \in [1.1, 2]$ is already included in **Case I**. Therefore consider,$x \in [1.25, 2]$ and $y \in [1, 1.1)$. Then

$$||x - y|| = x - y > 1.25 - 1.1 = 0.15,$$

and

$$||\tau x - \tau y|| = ||\frac{x+4}{3} - (3 - y)|| = ||\frac{x + 3y - 5}{3} < 0.1 < 0.15 < ||x - y||.$$

Hence $\frac{1}{2}||x - \tau x|| \leq ||x - y|| \Rightarrow ||\tau x - \tau y|| \leq ||x - y||$. Thus, τ is Suzuki's generalized non-expansive mapping. Choose $\alpha_n = 0.85, \beta_n = 0.65, \gamma_n = 0.45$, with the initial value $x_1 = 1.2$.

Now, we present a numerical example which shows that our iteration process (2.7) converges at a rate faster than Sokhuma and Kaewkhao (2.6)

Example 3.8. Consider $E = [1, 2]$. Define a multi-valued mapping $T : E \to FB(E)$ by $Tx = [x, 2]$, and self-mapping τ on E as $\tau x = \frac{x+16}{9}$.

Clearly, τ is a non-expansive and since, $H(Tx, Ty) = ||x - y||$, then T is a multi-valued non-expansive mapping. Choose $\alpha_n = 0.85, \beta_n = 0.65, \gamma_n = 0.45$, with the initial value $x_1 = 1.2$.

n	Sokhuma, Kaewkhao iteration	MF-iteration
1	1.2	1.2
2	1.853555	1.990237
3	1.973192	1.999880
4	1.995092	1.999998
5	1.999101	1.999999
6	1.999835	2
7	1.999969	
8	1.999994	
9	1.999998	
10	1.999999	
11	2	

Table 2:

Remark 3.9. The iterative scheme (2.7) converges faster than Sokhuma and Kaewkhao (2.6) schemes for single-valued Suzuki's generalized non-expansive and multi-valued non-expansive mappings as shown in the above table. The class of Suzuki's generalized non-expansive mappings is bigger than the class of non-expansive mappings as shown in Example 3.8.

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