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The problem of the network flow interdiction

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Abstract

In this paper, we state the problem of the network flow interdiction in a set of initial and destination nodes so that each initial is capable of only delivering products to certain pre-determined destinations. The network user's purpose is to deliver the highest value of flow from the sources to the sinks and the network interdictor's purpose is to reduce the highest value of flow being used. In this paper, the networks flow interdiction in multi-source and multi-sink conditions is addressed in a way that the parameters of arc capacity are trapezoidal fuzzy sets.

Keywords: interdiction problem, network flow, fuzzy sets, simplex method, ranking function, duality, bi-level programming, decomposition
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1 Introduction

Wood [12] has proposed the network flow interdiction. He showed the network flow interdiction with absolute parameters in a single product mode, and then he demonstrated the NPH. Generally speaking, network flow interdiction is an issue with two distinct decision makers (defender and attacker). Examples such as drug delivery, antidote delivery, narcotics systems and issues like these in which two decision makers with different goals use the network flow simultaneously are a kind of network flow interdiction. This matter was first taken under investigation by Wollmer [11].

Many decision making problems implicate uncertainty data (fuzzy, interval, etc.) and the decision making process is dealt with uncertainty using appropriate methods to find a solution of the problem. Fuzzy set theory is one of the widely used and acceptable tools to deal with non-probabilistic uncertainty. For example, Network flow data such as capacity of arcs, the costs of sending the flow and flow interdiction on each arc are expressed in an interval, fuzzy or stochastic form. Cormican et al. [2] formulated stochastic version of the network flow interdiction problem. They solved it with interdiction variables binary and random.

The literature exhibits several methods to solve such type of constrained optimization problems consisting of uncertain parameters such as fuzzy, stochastic, etc. Chance-constrained programming (CCP) is one of the critical methods to solve constrained optimization problems with fuzzy parameters (Liu and Iwamura1998 [8]; Yang and Liu2007 [13]; Kundu et al.2014a [6]).

The theory of fuzzy sets is used in many decision making problems. Actually, uncertain data can be captured applying fuzzy quantities. Tanaka [10] suggested fuzzy mathematical programming in control theory, management sciences, mathematical modeling, industrial uses, etc. Zimmerman [14] presented the first mathematical formulation

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of linear programming. Several researchers have also presented different fuzzy linear programming and approximations to solve them. Some researchers have used the comparative concept of fuzzy sets in approximation of solving fuzzy linear programming. Most numerical solutions are based on comparing fuzzy sets, using ranking function. In this paper, we have proposed a chance constraint programming method to solve the problem of network flow interdiction in fuzzy set.

2 Preliminaries

In this section, we briefly discuss the fuzzy variables. Additionally, we introduce the concept of generalized credibility measure, which takes the central role in developing the proposed methodologies in the paper. We review the fundamental notions of the fuzzy set theory, which was initiated by Bellman and Zadeh [1].

Definition 2.1. A convex fuzzy set $\widetilde{A}$ on $\mathbb{R}$ is a fuzzy number, if the following conditions hold:
a) Its membership function is piecewise continuous.
b) There exist three intervals, increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

Definition 2.2. Let $\widetilde{A} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $(a^L - \alpha, a^U + \beta)$ is the support of $\widetilde{A}$ and $[a^L, a^U]$ is its core. The set of all trapezoidal fuzzy numbers is denoted by $F(\mathbb{R})$ and the arithmetic on trapezoidal fuzzy numbers is defined as follow:
Let $\widetilde{a} = (a^L, a^U, \alpha, \beta)$ and $\widetilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers:

$$\begin{cases} x \geq 0,\ x \in \mathbb{R}: & x\widetilde{a} = (xa^L, xa^U, x\alpha, x\beta) \\ x < 0,\ x \in \mathbb{R}: & x\widetilde{a} = (xa^U, xa^L, -x\beta, -x\alpha) \\ \widetilde{a} + \widetilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta) \end{cases}$$

2.1 Generalized trapezoidal fuzzy variable

A generalized trapezoidal fuzzy variable $\widetilde{\xi}$ is a trapezoidal fuzzy variable (TrF) which may or may not be normalized. It is determined by $(a, b, c, d; \omega)$ with $a < b \leq c < d$, $0 < \omega \leq 1$ and its membership function is given by

$$\mu_{\widetilde{\xi}} = \begin{cases} \dfrac{\omega(x-a)}{b-a}, & \text{if } \ a \leq x \leq b \\ \omega, & \text{if } \ b \leq x \leq c, \quad 0 < \omega \leq 1 \\ \dfrac{\omega(d-x)}{d-c}, & \text{if } \ c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

Here ω is called its height, and in particular if $\omega = 1$, then $\widetilde{\xi}$ is usual(i.e. normalized) trapezoidal fuzzy variable. Also if $b = c$, then $\widetilde{\xi}$ becomes a triangular fuzzy variable [7].

2.2 Ranking functions

One convenient approach to solve Fuzzy Linear Programming problems is based on the concept of comparison of fuzzy numbers by using ranking functions [9]. An approach to order the elements of the set of all trapezoidal fuzzy numbers is to define a ranking function R which maps each fuzzy number into the real line, we present R by:

if $\widetilde{a}, \widetilde{b} \in F(\mathbb{R}) \implies$
a) $\widetilde{a} \succ \widetilde{b}$ if and only if $R(\widetilde{a}) > R(\widetilde{b})$,
b) $\widetilde{a} = \widetilde{b}$ if and only if $R(\widetilde{a}) = R(\widetilde{b})$,

c) $R(k\widetilde{a}+\widetilde{b}) = kR(\widetilde{a}) + R(\widetilde{b})$,
d) If $\widetilde{a} \succ \widetilde{b}$ and $\widetilde{c} \succ \widetilde{d}$, then $\widetilde{a}+\widetilde{c} \succ \widetilde{b}+\widetilde{d}$.

We consider the fundamentals of trapezoidal fuzzy sets and accept the primary functions of fuzzy sets [4]. The linear ranking functions on trapezoidal fuzzy sets, i.e. $\widetilde{a} = (a^l, a^u, \alpha, \beta)$, can be considered as follow:

$$R(\widetilde{a}) = (c_l a^l, c_u a^u, c_\alpha \alpha, c_\beta \beta).$$

So that c_l, c_u, c_α, c_β are constant numbers, one of which is at least non-zero.

In special situations, the linear ranking function is considered as follow [3]:

$$R(\widetilde{a}) = \frac{a^l + a^u}{2} + \frac{1}{4}(\beta - \alpha). \tag{2.1}$$

This function is valid for all conditions of linear ranking function [3].

The problem of network flow interdiction let, $G(V, L)$ is denoted a directed network with V and L as node set and arc set, respectively.

We will usually refer to an arc as an ordered pair (i, j) here $\{i, j\} \in V$, although we can also refer to it by its number k. It is assumed that G contains no self-loops, i.e., no arcs of the form (i, i). If $L' \subseteq L$, then, $G - L'$ indicated G with edges L' deleted and if $V' \subseteq V$, then, $G - V'$ denotes G with all nodes in V' deleted along with all arcs incident into or from nodes in V'. It will be useful to distinguish two nodes s and t as $s \neq t$. Maximum flow from s to t will be the same as maximum flow along an extra return of arc (t, s) added to L. An $s - t$ cut set is a partition of $\vee$ into two subsets V_s and V_t such that $s \in V_s$ and $t \in V_t$. Concerning that cut, an arc is a forward arc if it is directed from a node in V_s to a node in V_t; whereas it is backward if it is directed from a node in V_t to a node in V_s. If each arc (i, j) has a capacity u_{ij}, then the capacity of the cut is the sum of the capacities of forwards arcs associated with the cut.

Max flow model:

$$\text{Max } \nu$$

$$\text{s.t :}$$

$$\sum_{j:(i,j)\in L} x_{ij} - \sum_{j:(j,i)\in L} x_{ji} = \begin{cases} \nu & \text{for } i = s \\ 0 & \text{for all } i \in V - \{s, t\} \\ -\nu & \text{for } i = t \end{cases}$$

$$0 \leq x_{ij} \leq u_{ij} \text{ for each } (i, j) \in L.$$

The problem is considered on the non-directional network of $G(V, L)$, with V being the vertex set and L the network arc set. In the max flow model, the amount of sending flow from ith node to jth node is shown by x_{ij}, and the amount of all middle nodes (the ones that are not from the source and the sink) is considered as 0.

The issue of the network flow interdiction, which is generalized from the issue of max flow, has two decision-makers with different goals. The interdictor's problem is first represented as a bi-level, min-max program and then converted into a Mixed Integer Programming (MIP) problem. The interdictor's decision variable w_{ij} takes the value of 1 if arc (i, j) is interdicted and 0 otherwise.

This issue is introduced with network flow interdiction with multiple initials and multiple destinations, which is shown as MSNFI [5]. In this issue, x_{ijk} is the amount of sending flow on arc (i, j) which has been sent from kth initial.

The problem consists of two decision makers with opposing purposes, which simultaneously make decisions in the network. The first decision maker or network user aims to deliver the highest possible flow from pre-determined initials $(M = \{m_1, \ldots, m_n\})$ to specified destinations for each initial $(T_i = \{t_j \mid t_j \text{ is sin } k \text{ for } m_i\})$. The second decision maker or interdictor, with the purpose of reducing the delivered flow by the network user, deals with ignoring the flow network arcs. Ignoring each arc of $(i, j) \in L$ costs c_{ij}. The VP of middle nodes(the nodes which are not source nor sink) and $T = \{T_i \mid i = 1, \ldots, n\}$ is destinations set.

MSNFI:

$$\underset{w}{\text{Min}}\ \text{Max} \sum_{k=1}^{n} \sum_{(m_k,j)\in L} x_{m_k jk} \tag{2.2}$$

$$\text{s.t :} \quad \sum_{k=1}^{n} (x_{ijk} + x_{jik}) \leq u_{ij}(1 - w_{ij}) \qquad \forall\ (i,j) \in L \tag{2.3}$$

$$\sum_{j:(i,j)\in L} x_{ijk} - \sum_{j:(j,i)\in L} x_{jik} = 0 \qquad \forall\ i \in VP, \qquad \forall\ k = 1,\dots,n \tag{2.4}$$

$$\sum_{(i,j)\in L\ \&\ j\in T_k} x_{ijk} = \sum_{(m_k,j)\in L} x_{n_k jk} \qquad \forall\ k = 1,\dots,n \tag{2.5}$$

$$\sum_{(i,j)\in L} c_{ij} w_{ij} \leq C \tag{2.6}$$

$$x_{ijk} \geq 0 \qquad \forall\ (i,j) \in L,\ k = 1,\dots,n \tag{2.7}$$

$$w_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in L$$

The above model minimizes the maximum flow achievable by user of network. For a fixed W, the inner maximization tries to maximize the user's flow. In other words, the objective function minimizes the maximum out-flow from all sources in M. Constraints (2.3) sets all flows on arc (i,j) to zero when $w_{ij} = 1$ and at most to u_{ij} when $w_{ij} = 0$. Constraints (2.6) limit the expenditure of interdiction resource and require interdiction variables to be binary, $w_{ij} \in \{0,1\}$.

The interdictor considers the cost of the interdiction of arc (i,j) and its deletion from the network amount c_{ij}. Further, C is considered as a complete budget for the interdiction of network amount.

3 Main results

In this paper, a problem in flow networks, considering fuzzy sets' rules [4], is presented. Each arc of $(i,j) \in L$ has the capacity of $\widetilde{u}_{ij}$ and the cost of arc interdiction is c_{ij}. The arc capacities' set of $U = \{\widetilde{u}_{ij} \mid (i,j) \in L\}$ are considered as trapezoidal fuzzy sets and the costs of arc interdiction of $C = \{c_{ij} \mid (i,j) \in L\}$ are real numerical.

Consider a problem of network flow with constraints involving trapezoidal fuzzy variables:

Model1

$$\underset{w}{\text{Min}}\ \text{Max} \sum_{k=1}^{n} \sum_{(m_k,j)\in L} x_{m_k jk}$$

$$s.t \quad \sum_{k=1}^{n} x_{ijk} + x_{jik} \leq \widetilde{u}_{ij}(1 - w_{ij}) \qquad \forall\ (i,j) \in L \tag{3.1}$$

$$\sum_{j:(i,j)\in L} x_{ijk} - \sum_{j:(j,i)\in L} x_{ijk} = 0 \qquad \forall\ i \in VP,\ \ \forall\ k = 1,\dots,n \tag{3.2}$$

$$\sum_{(i,j)\in L\ \&\ j\in T_k} x_{ijk} = \sum_{(m_k,j)\in L} x_{m_k jk} \qquad \forall\ k = 1,\dots,n \tag{3.3}$$

$$\sum_{(i,j)\in L} c_{ij} w_{ij} \leq C \tag{3.4}$$

$$w_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in L$$

$$x_{ijk} \geq 0 \qquad \forall\ (i,j) \in L,\ \ \forall\ k = 1,\dots,n$$

The first model's hypotheses are as follow: x_{ijk} the output value from k initial on the arc (i,j), w_{ij} is the variable interdictor if $w_{ij} = 1$, then arc (i,j) is ignored from the network and if $w_{ij} = 0$ then this arc with the capacity of $\widetilde{u}_{ij}$ stays in the network. The VP of middle nodes (the nodes which are not source nor sink), and the condition:

Condition (2.7) shows the capacity of each arc. Condition (3.2) is balance conditions which guarantee that supply and demand numbers of middle nodes are zero. Condition (3.3) the total output value from a specific initial equals the total input value to specified destination from that initial. Condition (3.4) shows the limitation of the interdiction for the problem's interdictor.

4 Step of solving the problem of network flow interdiction in fuzzy sets

Step1. As the stated problem consists of two simultaneous target functions with different goals, model 1 cannot be directly solved. To solve this dual model, the internal problem, by introducing dual variables of γ_k, α_{ik}, β_{ij} for condition (3.1), (3.2), and (3.3) respectively, is written the Model 2 as follow (the numbers γ_k, α_{ik}, β_{ij} are written as real numbers).

Model2

$$\operatorname*{Min}_{w,\alpha,\beta} \sum_{(i,j)\in L} \widetilde{u}_{ij}(1-w_{ij})\beta_{ij}$$

$$\text{s.t} \quad -\alpha_{ik}+\alpha_{jk}+\beta_{ij}\geq 0, \qquad \forall\, k=1,\dots,n, \quad \forall\,(i,j)\in L, \quad \forall\, i,j\in VP \tag{4.1}$$

$$-\alpha_{jk}+\alpha_{ik}+\beta_{ij}\geq 0, \qquad \forall\, k=1,\dots,n, \quad \forall\,(i,j)\in L, \quad \forall\, i,j\in VP \tag{4.2}$$

$$\beta_{ij}-\gamma_k\geq 1, \qquad \forall\, i\in V, \quad \forall\,(i,j)\in L, \quad \forall\, k=1,\dots,n \tag{4.3}$$

$$\beta_{ij}-\gamma_k\geq 0, \qquad \forall\, i\in T, \quad \forall\,(i,j)\in L, \quad \forall\, k=1,\dots,n \tag{4.4}$$

$$\sum_{(i,j)\in L} c_{ij}w_{ij}\leq C$$

$$w_{ij}\in\{0,1\}, \quad \forall\,(i,j)\in L$$

$$\alpha_{ik} \quad \text{free} \quad \forall\, i\in VP, \quad \forall\, k=1,\dots,n$$

$$\beta_{ij}\geq 0, \quad \forall\,(i,j)\in L$$

$$\gamma_k \quad \text{free} \quad \forall\, k=1,\dots,n$$

Lemma 4.1. There is an optimal solution for Model2, which has the following conditions:

$$-1\leq\alpha_{ik}\leq 0, \qquad \forall\, i\in VP, \quad \forall\, k=1,\dots,n$$

$$0\leq\beta_{ij}\leq 1, \qquad \forall\,(i,j)\in L$$

$$-1\leq\gamma_k\leq 0, \qquad \forall\, k=1,\dots,n$$

Proof . First, we translate all variables and parameters on Model2 to real number by linear ranking function $R(\widetilde{u})$ and then, note that

(i) the coefficients of β_{ij} in the objective function are positive so that making each β_{ij} as small as permitted by the constraints, decreases the objective function value.

(ii) No two variables α_{ik} and $\alpha_{ik'}$ with the same node index i and different source indices k and k' appear in the same constraint. Accordingly, the restriction of a variable α_{ik} to the interval $[-1,0]$ does not affect any other $\alpha_{ik'}$ for $k\neq k'$.

(iii) Constraints (4.1) and (4.2) imply that, for each arc $(i,j)\in L$, $i,j\in VP$, the variable β_{ij} is bounded below by $\max_k\{-\alpha_{jk}+\alpha_{ik},-\alpha_{ik}+\alpha_{jk}\}$. The restriction of the variables α_{ik} and α_{jk} to the interval $[-1,0]$ implies that the lower bound on β_{ij}, enforced by constraints (4.1) and (4.2), is at most 1.

Accordingly, restricting β_{ij} to the interval $[0,1]$ for such arcs, maintains feasibility without loss of optimality. Similarly, constraint (4.3) and (4.4) imply that β_{ij} is bounded below by $1+\gamma_k$ for $(i,j)\in L$, $i\in V$, $k=1,\dots,n$ and by γ_k for $(i,j)\in L$, $j\in T$, $k=1,\dots,n$. Restriction of γ_k to the interval $[-1,0]$ for the related arcs implies that the maximum of these lower bounds is again at most 1. Therefore, we can conclude that $\beta_{ij}=1$ (for the corresponding arcs) in optimal solution. This completes the proof. □

Step2. Since the objective function of this model is nonlinear, in order to make it linear, the new variable of P_{ij} is introduced in the following lemma:

Lemma 4.2. Let $\rho_{ij} = (1 - w_{ij})\beta_{ij}$. Then the following relations hold:

$$0 \leq \rho_{ij} \leq 1 \quad \text{and} \quad \beta_{ij} - w_{ij} \leq \rho_{ij}$$

Proof . Since $w_{ij} \in \{0,1\}$, $\forall\,(i,j) \in L$, $0 \leq 1 - w_{ij} \leq 1$, $\forall\,(i,j) \in L$. According to $\rho_{ij} = \beta_{ij} - w_{ij}\beta_{ij}$ and $0 \leq \beta_{ij} \leq 1$, we have $0 \leq \rho_{ij} \leq 1$ and $\rho_{ij} \geq \beta_{ij} - w_{ij}$. □

By changing the applied variable in Lemma 4.2, Model 2 is rewritten as Model 3:

Model 3

$$\begin{aligned}
\min_{w,\alpha,\beta} \ & \sum_{(i,j)\in L} \widetilde{u}_{ij}\rho_{ij} \\
\text{s.t} \quad & -\alpha_{ik} + \alpha_{jk} + \beta_{ij} \geq 0, && \forall\, k = 1,\dots,n, \quad \forall\,(i,j) \in L, \quad \forall\, i,j \in VP \\
& -\alpha_{jk} + \alpha_{ik} + \beta_{ij} \geq 0, && \forall\, k = 1,\dots,n, \quad \forall\,(i,j) \in L, \quad \forall\, i,j \in VP \\
& \beta_{ij} - \gamma_k \geq 1, && \forall\, i \in V, \quad \forall\,(i,j) \in L, \quad \forall\, k = 1,\dots,n \\
& \beta_{ij} - \gamma_k \geq 0, && \forall\, i \in T, \quad \forall\,(i,j) \in L, \quad \forall\, k = 1,\dots,n \\
& \sum_{(i,j)\in L} c_{ij}w_{ij} \leq C \\
& 0 \leq \rho_{ij} \leq 1, && \forall\,(i,j) \in L \\
& \beta_{ij} - w_{ij} \leq \rho_{ij}, && \forall\,(i,j) \in L \\
& w_{ij} \in \{0,1\}, && \forall\,(i,j) \in L \\
& \alpha_{ik} \quad \text{free} && \forall\, i \in VP, \quad \forall\, k = 1,\dots,n \\
& \beta_{ij} \geq 0, && \forall\,(i,j) \in L \\
& \gamma_k \quad \text{free} && \forall\, k = 1,\dots,n
\end{aligned}$$

Step 3. Considering the linear ranking function for Model 3, we construct Model 4 based on characteristic ranking function as follow:
Model 4

$$\begin{aligned}
\min_{w,\alpha,\beta} \ & \sum_{(i,j)\in L} R\{\widetilde{u}_{ij}\}\rho_{ij} \\
\text{s.t} \quad & -\alpha_{ik} + \alpha_{jk} + \beta_{ij} \geq 0, && \forall\, k = 1,\dots,n, \quad \forall\,(i,j) \in L, \quad \forall\, i,j \in VP \\
& -\alpha_{jk} + \alpha_{ik} + \beta_{ij} \geq 0, && \forall\, k = 1,\dots,n, \quad \forall\,(i,j) \in L, \quad \forall\, i,j \in VP \\
& \beta_{ij} - \gamma_k \geq 1, && \forall\, i \in V, \quad \forall\,(i,j) \in L, \quad \forall\, k = 1,\dots,n \\
& \beta_{ij} - \gamma_k \geq 0, && \forall\, i \in T, \quad \forall\,(i,j) \in L, \quad \forall\, k = 1,\dots,n \\
& \sum_{(i,j)\in L} c_{ij}w_{ij} \leq C \\
& 0 \leq \rho_{ij} \leq 1, && \forall\,(i,j) \in L \\
& \beta_{ij} - w_{ij} \leq \rho_{ij}, && \forall\,(i,j) \in L \\
& w_{ij} \in \{0,1\}, && \forall\,(i,j) \in L \\
& \alpha_{ik} \quad \text{free} && \forall\, i \in VP, \quad \forall\, k = 1,\dots,n \\
& \beta_{ij} \geq 0, && \forall\,(i,j) \in L \\
& \gamma_k \quad \text{free} && \forall\, k = 1,\dots,n
\end{aligned}$$

Now transform the Model 3 into deterministic one. Suppose the corresponding deterministic model is obtained.

Corollary 4.3. the stated problem in model 1 is an NPH problem. As the relations between primal and dual problems have been proven with regard to fuzzy number sets [3], the optimal value and optimal solution for Model 3 can be calculated with linear ranking function (2.1).

To simplify the solution for this model, we present the below model and apply planning with integers.

5 An equivalent model

The arcs and nodes of the network are divided into three groups including:

(i) S_1 includes all sources and arcs that connect sources together,

(ii) S_2 all middle nodes (no source and no sink) and edges which connect middle nodes together and edges connect middle nodes to the sources and the sinks,

(iii) S_3 includes all sinks and edges that connect sinks together.
S_1, S_2, S_3 may be formulated as following:

$$S_1 = M \cup \{(i,j) \ : \ i,j \in M\}$$

$$S_2 = VP \cup \{(i,j) \ : \ (i,j) \in L \ \& \ (i,j) \notin S_1 \cup S_3\}$$

$$S_3 = T \cup \{(i,j) \ : \ i,j \in T\}$$

Now we can assume S_1 as a pseudo-source and S_3 as a pseudo-sink. The user tries to maximize the network flow value from S_1 to S_3 and the interdictor tries to decrease this value as much as his accessible resource allows. According to the new definition of source and sink, the problem may be formulated as following:

$$\begin{aligned}
&\min_{w} \ \text{Max} \ X \\
&\text{s.t} \quad \sum_{k=1}^{n} (x_{ijk} + x_{jik}) \leq \widetilde{u}_{ij}(1 - w_{ij}), \qquad \forall\ (i,j) \in L \\
&\sum_{k=1}^{n} \Bigg(\sum_{\substack{j\,:\,(i,j) \in L \\ i \in S_1 \\ j \notin S_1}} x_{ijk} - \sum_{\substack{j\,:\,(j,i) \in L \\ j \in S_1 \\ i \notin S_1}} x_{ijk} \Bigg) = X \\
&\sum_{j:(i,j)\in L} x_{ijk} - \sum_{j:(j,i)\in L} x_{jik} = 0, \qquad \forall\ i \in S_2, \quad \forall\ k = 1, \ldots, n \\
&\sum_{t_j \in T_k} \Bigg(\sum_{(i,t_j)\in L} x_{it_jk} - \sum_{(t_j,i)\in L} x_{t_jik} \Bigg) - \sum_{(m_k,j)\in A} x_{m_kjk} = 0, \qquad \forall\ k = 1, \ldots, n \\
&\sum_{(i,j)\in L} c_{ij} w_{ij} \leq C \\
&w_{ij} \in \{0,1\}, \qquad \forall\ (i,j) \in L \\
&x_{ijk} \geq 0, \qquad \forall\ (i,j) \in L, \quad \forall\ k = 1, \ldots, n
\end{aligned}$$

We call this form of the problem as grouped multi-source-sinks network flow interdiction problem with fuzzy variables.

Now, for fixed w_{ij}, if the inner maximization problem is replaced by its dual the problem's model becomes as following:

$$
\begin{aligned}
\underset{\alpha,\beta,w}{\text{Min}} \sum_{(i,j)\in L} & R\{\widetilde{u}_{ij}\}(1-w_{ij})\beta_{ij} \\
\text{s.t} \quad \beta_{ij}+\beta_{ji}-\gamma_i \geq 0, & \quad \forall\ (i,j)\in L, \quad \forall\ i,j\in S_1 \\
f_1+\beta_{ij}+\beta_{ji}+\alpha_{jk} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_1 \text{ and } j\in S_2 \\
f_1+\beta_{ij}+\beta_{ji} \geq 0, & \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_1 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_1 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik}-\alpha_{jk} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_2 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik}+\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}-\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_3 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}-\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_3 \text{ and } j\in S_2 \\
-f_1 \geq 1 & \\
\sum_{(i,j)\in L} c_{ij}w_{ij} \leq C & \\
w_{ij}\in\{0,1\}, & \quad \forall\ (i,j)\in L \\
-1\leq\gamma_k\leq 0 \quad \text{free} & \quad k=1,\dots,n \\
f_1 \quad \text{free} & \\
0\leq\beta_{ij}\leq 1, & \quad \forall\ (i,j)\in L \\
-1\leq\alpha_{ik}\leq 0, & \quad i\in S_2 \text{ and } k=1,\dots,n
\end{aligned}
$$

The latter constraint allows us to replace $(1-w_{ij})\beta_{ij}$ by $\rho_{ij}\geq 0$ and add the set of constraints $\rho_{ij}\geq\beta_{ij}-w_{ij}$, $(i,j)\in L$, to the model.

$$
\begin{aligned}
\underset{\alpha,\beta,w,\rho}{\text{Min}} \sum_{(i,j)\in L} & R\{\widetilde{u}_{ij}\}\rho_{ij} \\
\text{s.t} \quad \beta_{ij}+\beta_{ji}-\gamma_i \geq 0, & \quad \forall\ (i,j)\in L, \quad \forall\ i,j\in S_1 \\
f_1+\beta_{ij}+\beta_{ji}+\alpha_{jk} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_1 \text{ and } j\in S_2 \\
f_1+\beta_{ij}+\beta_{ji} \geq 0, & \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_1 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_1 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik}-\alpha_{jk} \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_2 \\
\beta_{ij}+\beta_{ji}+\alpha_{ik}+\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_2 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}-\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_3 \text{ and } j\in S_3 \\
\beta_{ij}+\beta_{ji}-\gamma_k \geq 0, & \quad \forall\ k=1,\dots,n, \quad \forall\ (i,j)\in L, \quad \forall\ i\in S_3 \text{ and } j\in S_2 \\
-f_1 \geq 1 & \\
\sum_{(i,j)\in L} c_{ij}w_{ij} \leq C & \\
w_{ij}\in\{0,1\}, & \quad \forall\ (i,j)\in L \\
\rho_{ij}\geq\beta_{ij}-x_{ij}, & \quad \forall\ (i,j)\in L \\
0\leq\rho_{ij}\leq 1, & \quad \forall\ (i,j)\in L \\
\gamma_k \quad \text{free} & \quad k=1,\dots,n \\
f_1 \quad \text{free} & \\
\beta_{ij}\in\{0,1\}, & \quad \forall\ (i,j)\in L \\
\alpha_{ik}\in\{0,1\}, & \quad i\in S_2 \text{ and } k=1,\dots,n
\end{aligned}
$$

Lemma 5.1. The optimal value of ρ_{ij} is β_{ij}.

Proof . If $w_{ij} = 0$ is optimal solution, the corresponding term in the objective function is equal to $R\{\widetilde{u}_{ij}\}\beta_{ij}$. If $w_{ij} = 1$ is optimal then the corresponding term in objective function is 0. Thus, to linearize the model, it must be true that $\rho_{ij} = 0$ when $w_{ij} = 1$ and $\rho_{ij} = \beta_{ij}$ when $w_{ij} = 0$. When $w_{ij} = 1$, constraints $\rho_{ij} \geq \beta_{ij} - w_{ij},\ \forall\ (i,j) \in L$ are satisfied for $0 \leq \beta_{ij} \leq 1$. However, because setting ρ_{ij} to any value greater than 0 increases the objective function, the value of ρ_{ij} must be zero. When $x_{ij} = 0$, constraints $\rho_{ij} \geq \beta_{ij} - w_{ij},\ \forall\ (i,j) \in L$ are satisfied for $\rho_{ij} \geq \beta_{ij}$. However, due to the minimization goal $\left(\underset{\alpha,\beta,w,\rho}{\mathrm{Min}} \sum_{(i,j)\in L} R\{\widetilde{u}_{ij}\}\rho_{ij}\right)$, it must be true that $\rho_{ij} = \beta_{ij}$. This justifies the correctness of the linear objective function with binary and real variables of binary or real. □

6 Results

Interest in the interdiction network was developed from trafficking drugs, medicine, and chemical materials in the transportation system in the south of America. It was when the enemy was attempting to deliver the highest amount of flow during the network and interdictor.

This problem can be used in delivering medicine, drugs, etc [12]. The problem in which the interdictor and user compete in a non-directional network with the capacity of the arcs of trapezoidal fuzzy sets. Dual propositions and relations and rules of fuzzy sets can be used to solve this problem and satisfy both decision-makers.

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