Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 315–325
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24682.2796

Unified space-time fractional cable equation

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(Communicated by Mohammad Bagher Ghaemi)

Abstract

Recently fractional cable equation has been investigated by many authors who have applied it in various areas. Here we introduce and investigate a generalized space-time fractional cable equation associated with Riemann-Liouville and Hilfer fractional derivatives. By mainly applying both Laplace and Fourier transforms, we express the solution of the proposed generalized fractional cable equation as H-functions. The main results here are general enough to be specialized to yield many new and known results, only several of which are demonstrated in corollaries. Finally, we consider the moment of the Green function with its several asymptotic formulas.

Keywords: Space-time fractional cable equation, Riemann-Liouville fractional derivatives, Caputo fractional derivative, Hilfer operator, Mittag-Leffler function, Green function, H-function, Laplace transform, Fourier transform, Moments of the Green function
2020 MSC: 26A33, 44A10, 33C60

1 Introduction and Preliminaries

Fractional calculus has had a large spectrum of possible applications in different experimental scenarios with the aid of extensive developments of its theories. Many existing works show the modelling potentiality of the fractional calculus as well as a vision of the associated many open fractional questions deserving deeper investigations. In recent years fractional cable equation has attracted many authors who have applied in such various areas as models for spiny dendrites [8] and models for anomalous electro-diffusion in nerve cells [17]. Time-space fractional cable equation that describes the anomalous transport of electro-diffusion in nerve cells has also been investigated (see, *e.g.*, [22]). For many other studies of fractional cable equation, one may be referred, for example, to the recent works [1, 7, 8, 21, 23, 25, 32, 38, 40].

The left-sided and right-sided Riemann-Liouville fractional integrals $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ of a function f defined, respectively, by (see, e.g., [24, 25, 32])

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \quad (x > a) \tag{1.1}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt \quad (x < b), \tag{1.2}$$

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Received: September 2021 *Accepted:* November 2021

whenever each of the right sides exists. Here $\Gamma(\alpha)$ is the familiar Gamma function (see, e.g., [35, Section 1.1]). Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be sets of complex numbers, real numbers, positive real numbers, and positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The left-sided and right-sided Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ of a function $f(x)$ are defined, respectively, by

$$\begin{aligned} D_{a+}^{\alpha} f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} \, dt \\ =& \frac{d^n}{dx^n} I_{a+}^{n-\alpha} f(x) \quad (n = [\Re(\alpha)] + 1, \ x > a) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} D_{b-}^{\alpha} f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}} \, dt \\ =& \frac{d^n}{dx^n} I_{b-}^{n-\alpha} f(x) \quad (n = [\Re(\alpha)] + 1, \ x < b) \end{aligned} \tag{1.4}$$

whenever each of the right sides exists and where $[\Re(\alpha)]$ denotes the greatest integer less than or equal to $\Re(\alpha)$. In particular, we find (see [24, 28])

$$D_{0+}^{1-\delta} f(x) = \frac{1}{\Gamma(\delta)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^{1-\delta}} \, du \quad (0 < \delta < 1, \ x > 0). \tag{1.5}$$

The Caputo fractional derivative of order α of a function $f(t)$ is defined and denoted by (see, e.g., [4, 11, 31])

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \tag{1.6}$$

$$(\Re(\alpha) > 0, \Re(\alpha) \notin \mathbb{N}; \ m = [\Re(\alpha)] + 1; \ t > 0)$$

and

$$ {}_0^C D_t^{\alpha} f(t) = \frac{d^m f(t)}{dt^m} \quad (\alpha = m \in \mathbb{N}). \tag{1.7}$$

The Laplace transform of Caputo derivative is given as follows (see [24, 28]):

$$\mathcal{L}\left\{ {}_0^C D_t^{\alpha} f(t) \right\} = s^{\alpha} f^{\sim}(s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(0+) \tag{1.8}$$

$$(m \in \mathbb{N}, \ m-1 < \alpha \leq m),$$

where $f^{\sim}(s)$ is the Laplace transform of $f(t)$ (see also, [33, 36]).

The (left-sided) fractional derivative $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ of a function f is defined as follows (see [9, 10, 37])

$$D_{a+}^{\mu,\nu} f(x) = I_{a+}^{\nu(1-\mu)} \frac{d}{dx} I_{a+}^{(1-\nu)(1-\mu)} f(x), \tag{1.9}$$

whenever the right side exists. Obviously $D_{a+}^{\mu,0} f(x) = D_{a+}^{\mu} f(x)$. The operator (1.9) is often called Hilfer fractional derivative. The Hilfer operator in (1.9) is rewritten in a more general form (see [10]):

$$D_{a+}^{\mu,\nu} f(x) = I_{a+}^{\nu(n-\mu)} \frac{d^n}{dx^n} I_{a+}^{(1-\nu)(n-\mu)} f(x) = I_{a+}^{\nu(n-\mu)} D_{a+}^{\mu+\nu n-\mu\nu} f(x) \tag{1.10}$$

$$(0 \leq \ \nu \leq 1; \ n \in \mathbb{N}, n-1 < \mu \leq n).$$

The Laplace transform of the above operator in (1.10) is given in the following form (see, Tomovski [37, eqn. (7.1), p. 3380]):

$$\begin{aligned} &\mathcal{L}\left\{ D_{0+}^{\mu,\nu} f(x); s \right\} \\ &= s^{\mu} f^{\sim}(s) - \sum_{k=0}^{n-1} s^{n-k-\nu(n-\mu)-1} \left\{ \frac{d^k}{dx^k} I_{0+}^{(1-\nu)(n-\mu)} f(x) \right\} (0+) \end{aligned} \tag{1.11}$$

$$(0 \le \ \nu \le 1;\ n \in \mathbb{N},\ n-1 < \mu \le n),$$

where

$$\left\{ \frac{d^k}{dx^k} I_{0+}^{(1-\nu)(n-\mu)} f(x) \right\}(0+) = \lim_{x \to 0+} \frac{d^k}{dx^k} I_{0+}^{(1-\nu)(n-\mu)} f(x).$$

The classical cable equation which models the membrane potential $V = V(x,t)$ along the axial x-direction of a dendrite with diameter d, relative to the resting membrane potential V_{rest}, is given by (see [34])

$$\frac{V - V_{\text{rest}}}{r_m} - r_m\, i_e\,(x,t) + c_m r_m \frac{\partial V\,(x,t)}{\partial t} = \frac{\mathrm{d}\, r_m}{4\, r_L} \frac{\partial^2 V\,(x,t)}{\partial x^2}, \tag{1.12}$$

where r_m denotes the specific membrane resistance, r_L is the longitudinal resistivity, c_m denotes the membrane capacitance per unit area, and i_e is the external injected current per unit area. The product $\tau = c_m r_m$ is the time constant for the dendrite.

The macroscopic model (1.12) can be obtained by combining the standard current continuity equation

$$c_m \frac{\partial V\,(x,t)}{\partial t} = -\frac{\mathrm{d}}{4} \frac{\partial I_L\,(x,t)}{\partial x} - i_m\,(x,t) + i_e\,(x,t)\,, \tag{1.13}$$

$i_m\,(x,t)$ and $i_e\,(x,t)$ being, respectively, the total ionic trans membrane current density and the injected current density, and the constitutive equation

$$I_L\,(x,t) = \frac{1}{r_L} \frac{\partial V\,(x,t)}{\partial x}\, i_m\,(x,t) = \frac{V - V_{\text{rest}}}{r_m} \tag{1.14}$$

occurring from the temporal memory and spatial and nonlocal effects. The Ohm's law [8] is therefore modified as generalized fractional Ohm's law [21]:

$$I_L\,(x,t) = \frac{1}{r_L\,(\alpha,\mu)} \frac{d^{1-\alpha}}{dt^{1-\alpha}} \left(\Delta^{\mu-1} V\,(x,t) \right), \tag{1.15}$$

where $r_L\,(\alpha,\mu)$ is a parameter depending on α and μ. Here Δ^μ is the Riesz fractional operator defined by (see [11, 31])

$$\Delta^\mu = -\frac{1}{2\cos\left(\frac{\pi\mu}{2}\right)} \left(D^\mu_{-\infty} + D^\mu_{\infty} \right) \quad (1 < \mu \le 2), \tag{1.16}$$

where the left-sided and right-sided Riemann-Liouville fractional derivatives are given in (1.3) and (1.4). The Δ^μ in (1.16) is the nonlocal operator in the fractional flux of cells (1.15) and related to the Lévy flights [2, 5].

Remark 1.1. The case $\nu = 0$ of the Hilfer fractional derivative (1.9) reduces to the left-sided Riemann-Liouville fractional derivative (1.3):

$$D^{\mu,0}_{a+} f(x) = \frac{d}{dx} I^{1-\mu}_{a+} f(x) = D^\mu_{a+} f(x) \quad (0 < \mu < 1). \tag{1.17}$$

The case $\nu = 1$ of the Hilfer fractional derivative (1.9) reduces to the Caputo fractional derivative (1.6):

$$D^{\mu,1}_{0+} f(x) = I^{1-\mu}_{0+} f'(x) = {}^C D^{1-\mu}_x f(x) \quad (0 < \mu < 1). \tag{1.18}$$

For clarity and simplicity, we also use the following notations:

$$D^\alpha_x f(x) := D^\alpha_{0+} f(x);\ \ {}^C D^\alpha_x f(x) = {}^C_0 D^\alpha_x f(x);\ \ D_x^{\mu,\nu} f(x) := D_{0+}^{\mu,\nu} f(x). \tag{1.19}$$

A generalization of the Mittag-Leffler function (see [6, 26, 27]):

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \quad (\alpha \in \mathbb{C},\ \Re(\alpha) > 0) \tag{1.20}$$

was introduced by Wiman [39] in the following form:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)} \quad (\alpha,\ \beta \in \mathbb{C},\ \Re(\alpha) > 0). \tag{1.21}$$

For more results involving the functions in (1.20) and (1.21), one may be referred, for example, to [6, Section 18.1].

By considering the generalized fractional Ohm's law (1.15) and spatial non-local effects, Li and Deng [18] introduced and investigated the following space-time fractional cable equation

$$\frac{\partial}{\partial t}V(x,t) = D_t^{1-\alpha}\ \left(\Delta^{\mu}V(x,t)\right) - \lambda^2\ D_t^{1-\beta}V\left(x,t\right) + u\left(x,t\right) \tag{1.22}$$

$$\left(x \in \mathbb{R},\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1\right),$$

where x and t are dimensionless parameters, $\lambda = \sqrt{\frac{\mathrm{d}\, r_m}{4\, r_L}}$ is the space dendrite for the dendrite (the cable), $u\left(x,t\right)$ is the external source (external injected current) given by

$$u\left(x,t\right) = \lambda^2\ D_t^{1-\beta}\ \left(V_{rest} + r_m\, i_e(x,t)\right),$$

$D_t^{1-\nu}$ and Δ^{μ} are defined, respectively, in (1.5) and (1.16). Saxena et al. [34] generalized the space-time fractional cable equation (1.22) to investigate the following equation

$$\tau_{\gamma}\ {}^{C}D_t^{\gamma}V(x,t) = D_t^{1-\alpha}\ \left(\Delta^{\mu}V(x,t)\right) - \lambda^2\ D_t^{1-\beta}V\left(x,t\right) + u\left(x,t\right), \tag{1.23}$$

where ${}^{C}D_t^{\gamma}$ is the Caputo fractional derivative in (1.6) and τ_{γ} is a time parameter introduced for dimensional purpose. Without loss of generality, one may set $\tau_{\gamma} = 1$.

In this paper, instead of the Caputo fractional derivative in (1.23), we consider the following generalized space-time fractional cable equation

$$D_t^{\gamma,\delta}V(x,t) = D_t^{1-\alpha}\ \left(\Delta^{\mu}V(x,t)\right) - \lambda^2\ D_t^{1-\beta}V\left(x,t\right) + u\left(x,t\right), \tag{1.24}$$

where the notations and conditions are the same in (1.22). Then we aim to derive the fundamental solution (Green function) as well as an analytic solution of the generalized fractional space-time cable equation in (1.24), by mainly using Laplace transform and Fourier transform. The complete solution of the equation (1.24) is expressed in terms of an infinite series of the H-functions (see [24]). The main results here are general enough to be specialized to yield many new and known results, only several of which are demonstrated in corollaries. Finally we also consider the moment of the Green function with its several asymptotic formulas.

2 Generalized fractional space-time cable equation

Here our first main result is given in the following theorem.

Theorem 2.1. Let $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Also let α, β, γ, δ, and μ be real parameters such that

$$0 < \alpha \leq 1,\quad 0 < \beta \leq 1,\ 0 < \gamma < 1,\ 0 \leq \delta \leq 1,\ 1 < \mu \leq 2. \tag{2.1}$$

Then the fundamental solution (Green function) $G(x,t)$ of the generalized fractional cable equation (1.24) with the initial conditions

$$\lim_{|x|\to\infty} V\left(x,t\right) = 0 \quad \textit{and} \quad V(x,0) = g(x), \tag{2.2}$$

and without the external injected current, that is, $u(x,t) = 0$ is given by

$$G(x,t) = \frac{1}{\mu|x|\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-\lambda^2 t^{\beta})^n t^{(\gamma-1)n+\ \delta(1-\gamma)+\gamma-1}}{n!} \tag{2.3}$$

$$\times H_{2,3}^{2,1}\left[\frac{|x|}{2t^{(\alpha+\gamma-1)/\mu}}\ \middle|\ \begin{matrix}\left(1,\ \frac{1}{\mu}\right),\left(\delta\left(1-\gamma\right) - n + \beta n + \gamma + \gamma n,\ \frac{\gamma+\alpha-1}{\mu}\right)\\ \left(\frac{1}{2},\frac{1}{2}\right),\left(1+n,\frac{1}{\mu}\right),\left(1,\ \frac{1}{2}\right)\end{matrix}\right],$$

where $\Re\left(\mu\right) > 0$ and $H_{p,q}^{m,n}\left[z\ \middle|\ \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right]$ is the H-function (see also, [24, 30]).

Proof . Firstly, we will derive the Green function of the equation (1.24) without external injected current, that is, $u(x,t)=0$. If we apply the Laplace transform with respect to the time derivative t and use the initial conditions (2.2) and the following formula for the Laplace transform of the Riemann-Liouville fractional derivative (see [24, 28]):

$$\mathcal{L}\{D_t^{1-\nu}V(x,t);s\}=s^{1-\nu}V^{\sim}(x,s)-\left[D_t^{1-\nu}V(x,t)\right]_{t=0},$$

with another condition:

$$\left[D_t^{1-\nu}V(x,t)\right]_{t=0}=0 \quad (\nu=\alpha \text{ or } \beta), \tag{2.4}$$

we obtain

$$s^{\gamma}V^{\sim}(x,s)-s^{-\delta(1-\gamma)}g(x)=-s^{1-\alpha}\left(\Delta^{\mu}V^{\sim}(x,s)\right)-\lambda^2 s^{1-\beta}V^{\sim}(x,s). \tag{2.5}$$

Taking the Fourier transform (denoted by $\circledast$) on the above equation with respect to the space variable x, we have

$$s^{\gamma}V^{\sim\circledast}(k,s)-s^{-\delta(1-\gamma)}g^{\circledast}(k)=-s^{1-\alpha}|k|^{\mu}V^{\sim\circledast}(k,s)-\lambda^2 s^{1-\beta}V^{\sim\circledast}(k,s). \tag{2.6}$$

Solving (2.6) for $V^{\sim\circledast}(k,s)$, we get

$$V^{\sim\circledast}(k,s)=\frac{s^{-\delta(1-\gamma)}g^{\circledast}(k)}{s^{\gamma}+s^{1-\alpha}|k|^{\mu}+\lambda^2 s^{1-\beta}}:=G^{\sim\circledast}(k,s)\,g^{\circledast}(k), \tag{2.7}$$

where

$$G^{\sim\circledast}(k,s):=\frac{s^{-\delta(1-\gamma)+\alpha-1}}{s^{\gamma+\alpha-1}+|k|^{\mu}+\lambda^2 s^{\alpha-\beta}}. \tag{2.8}$$

Expanding the right hand side of (2.8) in a power series gives

$$G^{\sim\circledast}(k,s)=\sum_{n=0}^{\infty}\frac{(-\lambda^2)^n s^{\alpha-\delta(1-\gamma)+(\alpha-\beta)n-1}}{(s^{\gamma+\alpha-1}+|k|^{\mu})^{n+1}}. \tag{2.9}$$

Using the following known formula for the Laplace transform of the derivative of the Mittag-Leffler function in (1.21) (see also, [15, 16, 28]):

$$\mathcal{L}\left\{t^{\gamma^n+\delta-1}E_{\gamma,\delta}^{(n)}(-at^{\gamma}):s\right\}=\frac{n!s^{\gamma-\delta}}{(s^{\gamma}+a)^{n+1}} \quad \left(\Re(s)>|a|^{1/\gamma}\right) \tag{2.10}$$

and taking the inverse Laplace transform on (2.9), we obtain

$$\begin{aligned}G^{\circledast}(k,t)=\sum_{n=0}^{\infty}&\frac{(-\lambda^2t^{\beta})^n t^{n\gamma-n+\delta(1-\gamma)+\gamma-1}}{n!}\\ &\times E_{\gamma+\alpha-1,\delta(1-\gamma)+(\beta-\alpha)n+\gamma}^{(n)}\left(-a|k|^{\mu}t^{\gamma+\alpha-1}\right).\end{aligned} \tag{2.11}$$

Here the m^{th} derivative of the Mittag-Leffler function is given as follows (see [20]):

$$E_{\rho,\sigma}^{(m)}(z)=\frac{d^m}{dz^m}\left[E_{\rho,\sigma}(z)\right]=\sum_{n=0}^{\infty}\frac{(m+n)!z^n}{n!\,\Gamma(\rho(m+n)+\sigma)} \quad (m\in\mathbb{N}_0). \tag{2.12}$$

Using the following known relation between the derivative of the Mittag-Leffler function and the H-function (see [34]):

$$E_{\rho,\sigma}^{(m)}(z)=H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(-m,1)\\(0,1),\ (1-(\rho m+\sigma),\rho)\end{matrix}\right.\right] \quad (m\in\mathbb{N}_0) \tag{2.13}$$

in (2.11), we obtain

$$\begin{aligned}G^{\circledast}(k,t)=&\sum_{n=0}^{\infty}\frac{(-\lambda^2t^{\beta})^n t^{(\gamma-1)n+\delta(1-\gamma)+\gamma-1}}{n!}\\ &\times H_{1,2}^{1,1}\left[|k|^{\mu}t^{\gamma+\alpha-1}\left|\begin{matrix}(-n,1)\\(0,1),(1-(\gamma-1)n+\delta(1-\gamma)+\beta n+\gamma,\gamma+\alpha-1)\end{matrix}\right.\right].\end{aligned} \tag{2.14}$$

Taking the inverse Fourier transform on the expression (2.14) yields (see [14])

$$G(x,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-\lambda^2t^\beta)^n t^{(\gamma-1)n+\delta(1-\gamma)+\gamma-1}}{n!}\int_{-\infty}^{\infty}\exp(-ikx)$$

$$\times H_{1,2}^{1,1}\left[|k|^\mu t^{\gamma+\alpha-1}\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(1+n+\beta n-\gamma n+\delta(1-\gamma)+\gamma,\gamma+\alpha-1)\end{matrix}\right]dk. \tag{2.15}$$

Making use of the cosine transform of the H-function (see [24]):

$$\int_0^\infty t^{\rho-1}\cos(kt)\,H_{p,q}^{m,n}\left[at^\mu\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]dt$$

$$=\frac{2^{\rho-1}\sqrt{\pi}}{k^\rho}H_{p+2,q}^{m,n+1}\left[a\,(2/k)^\mu\,\middle|\,\begin{matrix}\left(\frac{2-\rho}{2},\frac{\mu}{2}\right),(a_p,A_p),\left(\frac{1-\rho}{2},\frac{\mu}{2}\right)\\(b_q,B_q)\end{matrix}\right] \tag{2.16}$$

$$\left(\Re(\rho)+\mu\min_{1\le j\le m}\Re\left[\frac{b_j}{B_j}\right]>0,\ \rho=\mu\max_{1\le j\le n}\Re\left[\frac{a_j-1}{A_j}\right]<0,\right.$$

$$\left.\Omega:=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j>0,\ |\arg a|<\frac{\pi\Omega}{2}\right),$$

we find

$$G(x,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-\lambda^2t^\beta)^n t^{(\gamma-1)n+\delta(1-\gamma)+\gamma-1}}{n!|x|}$$

$$\times H_{3,2}^{1,2}\left[t^{\gamma+\alpha-1}(2/|x|)^\mu\,\middle|\,\begin{matrix}\left(\frac{1}{2},\frac{\mu}{2}\right),(-n,1),\left(0,\frac{\mu}{2}\right)\\(0,1),(1-(\gamma-1)n+\delta(1-\gamma)+\beta n+\gamma,\gamma+\alpha-1)\end{matrix}\right]. \tag{2.17}$$

If we use the following property of the H-function (see [24, p. 12]):

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]=\frac{1}{\sigma}H_{p,q}^{m,n}\left[z^{1/\sigma}\,\middle|\,\begin{matrix}(a_p,A_p/\sigma)\\(b_q,B_q/\sigma)\end{matrix}\right]\quad(\sigma>0) \tag{2.18}$$

in (2.17), we get

$$G(x,t)=\frac{1}{\mu|x|\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-\lambda^2t^\beta)^n t^{(\gamma-1)n+\delta(1-\gamma)+\gamma-1}}{n!|x|}$$

$$\times H_{3,2}^{1,2}\left[\frac{2t^{(\alpha+\gamma-1)/\mu}}{|x|}\,\middle|\,\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),\left(-n,\frac{1}{\mu}\right),\left(0,\frac{1}{2}\right)\\\left(0,\frac{1}{\mu}\right),\left(1+n-\gamma n+\delta(1-\gamma)+\beta n+\gamma,\frac{\gamma+\alpha-1}{\mu}\right)\end{matrix}\right]. \tag{2.19}$$

Finally applying the following transformation formula for the H-function (see [24, p. 11]):

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]=H_{q,p}^{n,m}\left[\frac{1}{z}\,\middle|\,\begin{matrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{matrix}\right] \tag{2.20}$$

to the expression (2.19) is seen to yield the desired result (2.3). □

Setting $\gamma=\delta=1$ in the result given in Theorem 2.1, we obtain a known result due to Li and Deng [18], which is given by Corollary 2.2.

Corollary 2.2. Let $t\in\mathbb{R}^+$ and $x\in\mathbb{R}$. Also let α, β, γ, δ, and μ be real parameters such that

$$0<\alpha\le 1,\quad 0<\beta\le 1,\ 1<\mu\le 2.$$

Then the fundamental solution (Green function) of the following one-dimensional fractional reaction-diffusion equation of fractional order

$$\frac{\partial}{\partial t}V(x,t)=D_t^{1-\alpha}\left(\Delta^\mu V(x,t)\right)-\lambda^2\ D_t^{1-\beta}V(x,t)+u(x,t) \tag{2.21}$$

$$\lim_{|x|\to\infty} V(x,t)=0 \quad and \quad V(x,0)=g(x),$$

and without the external injected current, that is, $u(x,t)=0$ is given by

$$G(x,t)=\frac{1}{\mu|x|\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-\lambda^2t^\beta)^n}{n!} \times H_{2,3}^{2,1}\left[\frac{|x|}{2t^{\alpha/\mu}}\,\middle|\,\begin{matrix}\left(1,\frac{1}{\mu}\right),\left(1+\beta n,\frac{\alpha}{\mu}\right)\\ \left(\frac{1}{2},\frac{1}{2}\right),\left(1+n,\frac{1}{\mu}\right),\left(1,\frac{1}{2}\right)\end{matrix}\right], \tag{2.22}$$

where $H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]$ is the H-function (see [24]).

The second main result is given in the following theorem.

Theorem 2.3. Let $t\in\mathbb{R}^+$ and $x\in\mathbb{R}$. Also let α, β, γ, δ, and μ be real parameters such that

$$0<\alpha\le 1,\;\; 0<\beta\le 1,\;\; 0<\gamma<1,\;\; 0\le\delta\le 1,\;\; 1<\mu\le 2. \tag{2.23}$$

Then the fundamental solution (Green function) of the generalized fractional cable equation (1.24) with the initial conditions

$$\lim_{|x|\to\infty} V(x,t)=0 \quad and \quad V(x,0)=g(x) \tag{2.24}$$

is given by

$$\begin{aligned}V(x,t)&=\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-\lambda^2t^\beta)^n\,g^{\circledast}(k)\exp(-ikx)}{n!}\\ &\times H_{1,2}^{1,1}\left[|k|^\mu\, t^{\gamma+\alpha-1}\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(1+\gamma+(\beta-\gamma)n+n+\delta(1-\gamma),\gamma+\alpha-1)\end{matrix}\right]dk\\ &+\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\xi^{\gamma-1}}{n!}\int_0^t(-\lambda^2\xi^{\beta+\gamma-1})^n\int_{-\infty}^{\infty}U^{\circledast}(k,t-\xi)\exp(-ikx)\\ &\times H_{1,2}^{1,1}\left[|k|^\mu t^{\gamma+\alpha-1}\,\middle|\,\begin{matrix}(-n,1)\\(0,1),(1-\gamma-n(\beta+\gamma-1),\gamma+\alpha-1)\end{matrix}\right]dk\,d\xi,\end{aligned} \tag{2.25}$$

where $H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]$ is the H-function (see [24]).

Proof . A similar argument as in the proof of Theorem 2.1 will establish the desired result. We omit the details.
□

Among many special cases of the result in Theorem 2.3, we consider only two cases, which are given in Corollaries 2.4 and 2.5.

The case $\delta=0$ of Hilfer fractional derivative $D_t^{\gamma,\delta}$ in (1.9) is seen to reduce to the Riemann-Liouville fractional derivative D_t^γ. So, setting $\delta=0$ in Theorem 2.3 gives the following result in Corollary 2.4.

Corollary 2.4. Let $t\in\mathbb{R}^+$ and $x\in\mathbb{R}$. Also let α, β, γ, and μ be real parameters such that

$$0<\alpha\le 1,\;\; 0<\beta\le 1,\;\; 0<\gamma<1,\;\; 1<\mu\le 2.$$

Then the solution of the following generalized fractional cable equation

$$D_t^\gamma V(x,t)=D_t^{1-\alpha}\;(\Delta^\mu V(x,t))-\lambda^2\;D_t^{1-\beta}V(x,t)+U(x,t) \tag{2.26}$$

with the initial conditions

$$\lim_{|x|\to\infty} V(x,t) = 0 \quad and \quad V(x,0) = g(x) \tag{2.27}$$

is given by

$$\begin{aligned} V(x,t) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-\lambda^2 t^\beta)^n g^{\circledast}(k) \exp(-ikx)}{n!} \\ &\times\ H_{1,2}^{1,1}\left[|k|^\mu\ t^{\gamma+\alpha-1} \middle| \begin{matrix} (-n,1) \\ (0,1), (1+\gamma+(\beta-\gamma)n+n, \gamma+\alpha-1) \end{matrix} \right] dk \\ &+ \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\xi^{\gamma-1}}{n!} \int_0^t (-\lambda^2 \xi^{\beta+\gamma-1})^n \int_{-\infty}^{\infty} U^{\circledast}(k, t-\xi) \exp(-ikx) \\ &\times H_{1,2}^{1,1}\left[|k|^\mu t^{\gamma+\alpha-1} \middle| \begin{matrix} (-n,1) \\ (0,1), (1-\gamma-n(\beta+\gamma-1), \gamma+\alpha-1) \end{matrix} \right] dk\, d\xi \end{aligned} \tag{2.28}$$

where $\Re(\mu) > 0$ and $H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]$ is the H-function (see [24]).

The case $\delta = 1$ of the Hilfer fractional derivative $D_t^{\gamma,\delta}$ reduces to the Caputo fractional derivative ${}^{C}D_t^{\gamma}$. Setting $\delta = 1$ in Theorem 2.3 yields the following result in Corollary 2.5.

Corollary 2.5. Let $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Also let α, β, γ, and μ be real parameters such that

$$0 < \alpha \leq 1, \quad 0 < \beta \leq 1,\ 0 < \gamma < 1,\ 1 < \mu \leq 2. \tag{2.29}$$

Then the solution of the following generalized fractional cable equation

$${}^{C}D_t^{\gamma} V(x,t) = D_t^{1-\alpha}\ (\Delta^\mu V(x,t)) - \lambda^2\ D_t^{1-\beta} V(x,t) + u(x,t) \tag{2.30}$$

with the initial conditions

$$\lim_{|x|\to\infty} V(x,t) = 0 \quad and \quad V(x,0) = g(x) \tag{2.31}$$

is given by

$$\begin{aligned} V(x,t) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-\lambda^2 t^\beta)^n g^{\circledast}(k) \exp(-ikx)}{n!} \\ &\times H_{1,2}^{1,1}\left[|k|^\mu t^{\gamma+\alpha-1} \middle| \begin{matrix} (-n,1) \\ (0,1), (2+\gamma+(\beta-\gamma)n+n, \gamma+\alpha-1) \end{matrix} \right] dk \\ &+ \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\xi^{\gamma-1}}{n!} \int_0^t (-\lambda^2 \xi^{\beta+\gamma-1})^n \int_{-\infty}^{\infty} U^{\circledast}(k, t-\xi) \exp(-ikx) \\ &\times H_{1,2}^{1,1}\left[|k|^\mu t^{\gamma+\alpha-1} \middle| \begin{matrix} (-n,1) \\ (0,1), (1-\gamma-n(\beta+\gamma-1), \gamma+\alpha-1) \end{matrix} \right] dk\, d\xi, \end{aligned} \tag{2.32}$$

where $H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]$ is the H-function (see [24]).

3 Moments of the Green Function

The moment of the Green function $G(x,t)$ in (2.3) is defined as follows (see [25]):

$$< x^\sigma(t) >:= 2 \int_0^\infty |x|^\sigma\ G(x,t) dx. \tag{3.1}$$

Using the Mellin transform (see [24, 25, 31]):

$$\mathfrak{M}\{u(x);\sigma\} = \int_0^\infty x^{\sigma-1} u(x)\, dx, \tag{3.2}$$

we find that, for $0 < \sigma < \mu$,

$$< x^{\sigma}(t) >= \sum_{n=0}^{\infty} \frac{(-\lambda^2 t^{\beta})^n}{n!} \frac{2^{\sigma+1} t^{\gamma n - n + \delta(1-\gamma)+\gamma-1+\frac{\sigma(\alpha+\gamma-1)}{\mu}}}{\mu\sqrt{\pi}} \times \frac{\Gamma\left(-\frac{\sigma}{\mu}\right)\Gamma\left(\frac{1}{2}+\frac{\sigma}{\mu}\right)\Gamma(1+n+\frac{\sigma}{\mu})}{\Gamma\left(\gamma n - n + \delta(1-\gamma) + \beta n + \gamma + \frac{\sigma(\alpha+\gamma-1)}{\mu}\right)\Gamma(-\frac{\sigma}{2})}. \tag{3.3}$$

Here consider certain asymptotic behavior of the moment in (3.3).

Example 1. Setting $\mu = 2$ in (3.3) and recalling the following asymptotic formula:

$$\frac{1}{\Gamma(z)} \sim z \quad (z \ll 1), \tag{3.4}$$

we obtain

$$\lim_{\sigma\to 0} < x^{\sigma}(t) >= t^{\delta(1-\gamma)+\gamma-1} E_{\beta+\gamma-1,\delta(1-\gamma)+\gamma}(-\lambda^2 t^{\beta+\gamma-1}) \tag{3.5}$$

$$= t^{\delta(1-\gamma)+\gamma-1}\ H^{1,1}_{1,2}\left[-\lambda^2 t^{\beta+\gamma-1} \middle| \begin{matrix} (0,1) \\ (0,1),(1-\gamma-\delta(1-\gamma),\beta+\gamma-1) \end{matrix}\right]. \tag{3.6}$$

Example 2. Using the asymptotic formula of the Mittag-Leffler function (see [24, 28, 29]): For $\omega > 0$ and $\rho < 1$,

$$E_{\tau,\rho}(-\omega t^{\tau}) \sim \frac{t^{-\tau}}{\omega\Gamma(\rho-\tau)} \quad (t\to\infty), \tag{3.7}$$

we get

$$\lim_{\sigma\to 0}\ < x^{\sigma}(t) >\sim \frac{t^{\delta(1-\gamma)-\beta-(\beta+\gamma-1)}}{\lambda^2\Gamma(\delta(1-\gamma)-\beta+1)} \quad (t\to\infty). \tag{3.8}$$

Example 3. Taking the limits $\sigma \to 2$ and $\mu \to 2$ in (3.3) gives the temporal behavior of the mean-square displacement:

$$\lim_{\sigma\to 2,\mu\to 2}\ < x^{\sigma}(t) >= t^{\alpha+\gamma-1}\sum_{n=0}^{\infty}\frac{(n+1)!\left(-\lambda^2 t^{\beta+\gamma-1}\right)^n}{n!\ \Gamma((\gamma+\beta-1)n+\gamma+\alpha)},$$

which can also be expressed in terms of the first derivative of the generalized Mittag-Leffler function as follows (see [26, 27, 39]):

$$\lim_{\sigma\to 2,\mu\to 2}\ < x^{\sigma}(t) >= t^{\alpha+\gamma-1}E^{(1)}_{\gamma+\alpha-1,\alpha-\beta+1}\left(-\lambda^2 t^{\beta+\gamma-1}\right). \tag{3.9}$$

4 Concluding Remarks

Here, the generalization of the space-time fractional cable equation, as a presumably new mathematical model, takes into account the temporal memory effects and spatial non-locality. By applying both Laplace and Fourier transforms, we expressed the Green function of the generalized space-time fractional cable equation as an infinite series of H-functions. It is also shown that the main results are general enough to be specialized to yield many known or new results.

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