Int. J. Nonlinear Anal. Appl. 14 (2023) 3, 231–243
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.22027.2319

Some fixed point theorems on algebraic cone metric spaces

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(Communicated by Ali Ebadian)

Abstract

In this paper, we prove some fixed point theorems for self-mappings on an algebraic cone metric space. These results are related to the product of the cone, and improve some well-known results by inserting an algebraic cone $\mathcal{P}$ instead of $\mathbb{R}^+$.

Keywords: cone metric space, algebraic cone, Riesz space, α-property, property (C), property (E)
2020 MSC: Primary 55M20; Secondary 47H10, 54H25, 46A40

1 Introduction

Cone metric spaces were initiated by Huang and Zhang [6]. They described the convergence and completness in cone metric spaces, and proved some fixed point theorems for contractive mappings on such spaces. Subsequently, many articles and generalizations of cone metric spaces are presentated. One of these generalizations is the concept of algebraic cone metric spaces which was introduced by M. Akbari Tootkaboni and A. Bagheri Salec in [2]. They provided some fixed point theorems to the algebraic cone metric spaces. In algebraic cone metric spaces corresponding cones are defined on algebras, and in addition to their conical properties, they are closed relative to multiplication. This additional property of the algebraic cone metric spaces gives the ability to replacing vectors with scalars in fixed point theorems. For example, in [2], the contraction functions are defined by the following property

$$d(F(x), F(y)) \leq \alpha d(x, y)$$

for each $x, y \in X$, where α is a vector in the algebraic cone such that $\|\alpha\| < 1$ (or more generally $\Sigma_{n=1}^{\infty}\|\alpha\|^n < \infty$). In this paper, we tried to use more of the multiplication properties in algebraic cones, and to derive the results of cone metric spaces. In this regard, the concept of α-property is defined in vector mode, and then the results are proved with respect to the fixed points of the functions on the algebras with α-property. These results generalize some known theorems in metric spaces. Also, mappings with property (C) and mappings with property (E) in vector state are defined and some results are presented to them.

2 Preliminaries

In the following we recall some notions and facts related to cone metric spaces. Let $(\mathcal{A}, \|\cdot\|)$ be a real Banach space. A non-empty closed subset $\mathcal{P}$ of $\mathcal{A}$ is called a *cone* whenever the following conditions are satisfied:

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Received: December 2020 *Accepted:* January 2022

i) $\mathcal{P} \neq \{0\}$,
ii) if $x, y \in \mathcal{P}$, $a, b \in \mathbb{R}$ and $a, b \geq 0$, then $ax + by \in \mathcal{P}$,
iii) for every $x \in \mathcal{P}$, $-x \in \mathcal{P}$ if and only if $x = 0$.

Given a cone $\mathcal{P} \subseteq \mathcal{A}$, a partial ordering $\preceq$ with respect to $\mathcal{P}$ is defined by $x \preceq y$ if $y - x \in \mathcal{P}$. Furthermore, we write $x \prec y$ if $x \preceq y$ and $x \neq y$; While $x \ll y$ will stand for $y - x \in \text{int}\mathcal{P}$, where $\text{int}\mathcal{P}$ is the interior of $\mathcal{P}$. $x \in \mathcal{A}$ is called positive if $0 \prec x$ i.e. $x \in \mathcal{P}$ and $x \neq 0$.
The cone $\mathcal{P}$ is called *normal* if there is a number M> 0 such that for all $x, y \in \mathcal{A}$, $0 \preceq x \preceq y$, implies $\|x\| \leq M\|y\|$. The least positive number M satisfying the recent condition is called the *normal constant* of $\mathcal{P}$.

Lemma 2.1. Let $\mathcal{P}$ be a cone in a Banach space $(\mathcal{A}, \|\cdot\|)$. Then the following conditions are equivalent
a) $\inf\{\|x + y\| : x, y \in \mathcal{P},\ \|x\| = \|y\| = 1\} > 0$.
b) $\mathcal{P}$ is a normal cone.
c) For arbitrary sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in $\mathcal{A}$, if $x_n \preceq y_n \preceq z_n$ for each n, and $lim_{n\to\infty} x_n = lim_{n\to\infty} z_n = x$, then $lim_{n\to\infty} y_n = x$.
d) There exists a norm $\|\cdot\|_1$ on $\mathcal{A}$, equivalent with $\|\cdot\|$, such that the cone $\mathcal{P}$ is monotone with respect to $\|\cdot\|_1$ i.e. if $0 \preceq x \preceq y$ then $\|x\|_1 \leq \|y\|_1$.

Proof . See [4], [7], [9] and [14]. □

Definition 2.2. Let X be a nonempty set, $\mathcal{A}$ be a real Banach space and $\mathcal{P} \subset \mathcal{A}$ be a cone. A mapping $d : X \times X \to \mathcal{P}$ satisfiying
i) $d(x, y) = 0$ if and only if $x = y$;
ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. $\{x_n\}$ is said to be *convergent* to x, if for every $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$. Likewise, $\{x_n\}$ is called a Cauchy sequence in X if, for every $0 \ll c$ there is $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$. A cone metric space X is said to be *complete* if every *Cauchy sequence* in X is convergent in X. To replace standard properties of a metric, the following lemma is often useful while dealing with cone metrics when the cone is not normal.

Lemma 2.3. Let (X, d) be a cone metric space corresponding to a given cone $\mathcal{P}$. Let $x \in \mathcal{P}$, and $\{x_n\}$ and $\{a_n\}$ be sequences in X and $\mathcal{A}$, respectively. Then,
a) If $0 \preceq x \ll c$ for all $c \in int\mathcal{P}$, then $x = 0$.
b) If $0 \preceq d(x_n, x) \preceq a_n$ and $a_n \to 0$, then for each $c \in int\mathcal{P}$ there exists $n_0 \in \mathbb{N}$ such that, $d(x_n, x) \ll c$ for all $n > n_0$.
c) If $c \in int\mathcal{P}$, $0 \preceq a_n$, and $a_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that, for each $n > n_0$, $a_n \ll c$.

Proof . See [7, page 2598]. □

It follows from the part (c) of Lemma 2.1 that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$, and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$, as $n, m \to \infty$. The converses are true if $\mathcal{P}$ is a normal cone.

Definition 2.4. Let $\mathcal{P}$ be a cone in a Banach space $\mathcal{A}$, X be a vector space over $\mathbb{C}$, and the mapping $\|\cdot\| : X \to \mathcal{P}$ satisfies:
i) $\|x\| = 0$ if and only if $x = 0$,
ii) $\|x + y\| \preceq \|x\| + \|y\|$ for all $x, y \in X$,
iii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in X$.
Then, $\|\cdot\|$ is called a *cone norm* on X, and the pair $(X, \|\cdot\|)$ is called a *cone normed space* (CNS).
Sometimes, for emphasizing on the cone $\mathcal{P}$, we write $\|\cdot\|_{\mathcal{P}}$ instead of the above $\|\cdot\|$.

In [2, definitios 2.2 and 2.3], were initiated the concepts of algebraic cones and the cone Banach algebras as follows.

Definition 2.5. Let $\mathcal{A}$ be a Banach algebra with identity element $e_{\mathcal{A}}$. A cone $\mathcal{P} \subseteq \mathcal{A}$ is called an algebraic cone if $e_{\mathcal{A}} \in \mathcal{P}$ and for each $a, b \in \mathcal{P}$, $ab \in \mathcal{P}$.

If $\mathcal{P}$ is an algebraic cone, then for all $x, y \in \mathcal{A}$ if $x \preceq y$ and $a \in \mathcal{P}$, then $ax \preceq ay$.

Definition 2.6. Let X be an algebra, $\mathcal{A}$ be a Banach algebra, $\mathcal{P}$ be an algebraic cone in $\mathcal{A}$, and $\|\cdot\| : X \to \mathcal{P}$ be a cone norm. $(X, \|\cdot\|)$ is called a *Banach cone algebra* if $(X, \|\cdot\|)$ induces a complete cone metric space, and $\|xy\| \preceq \|x\|\|y\|$ for all $x, y \in X$.

3 α-Property in Totally Orderd cones and Fxed Point Theorems

In this section, first we introduce α-property for a cone Banach algebra. Let's remind that an ordered set $(E, \leq)$ is called a *lattice* if any two elements $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup\{x, y\}$, and a greatest lower bound denoted by $x \wedge y = \inf\{x, y\}$. A subset F of ordered set $(E, \leq)$ is called totally orderd whenever any elements $x, y \in F$ are comparable, i.e. one of the conditions $x \preceq y$ or $y \preceq x$ are hold. Obviously any totally orderd subset of E is a lattice. A real vector space E which is also an ordered set is called an ordered vector space if the order and vector space structure are compatible in the following sense:
If $x, y \in E$ such that $y \leq x$ then $y + z \leq x + z$ for all $z \in E$ and $ay \leq ax$ for all a which $a \geq 0$.
By Definition (2.5) algebraic cone metrices are ordered vector spaces. An ordered vector space which is also a lattice, called a *Riesz space* (or vector lattice).

In the following theorem X is an algebra with the identity e_X and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a unital Banach algebra with the identity $e_{\mathcal{A}}$.

Theorem 3.1. Suppose that $(X, \|\cdot\|)$ is a normal Banach cone algebra, $x \in X$ and $\|\|x\|\|_{\mathcal{A}} < 1$. Then
(a) $e_X - x$ is invertible.
(b) $\|\|(e_X - x)^{-1} - e_X - x\|\|_{\mathcal{A}} \leq \frac{\|\|x\|\|^2_{\mathcal{A}}}{1-\|\|x\|\|_{\mathcal{A}}}$.
(c) $\|\|\phi(x)\|\|_{\mathcal{A}} < 1$, for every homomorphism $\phi : X \to \mathcal{A}$ with $\|\|\phi(e_X)\|\|_{\mathcal{A}} = 1$.

Proof . See [2, Theoerm 2.4]. □

Recall that if X is an algebra and $x \in X$, then the spectrum $\sigma(x)$ of x is the set of all complex numbers λ, such that $\lambda e_X - x$ is not invertible. The spectral radius of each $x \in \mathcal{A}$ is the number $\rho(x) := \sup\{|\lambda| : \lambda \in \sigma(x)\}$. For every $x \in X$ and $\lambda \in \mathbb{C}$ if $|\lambda| \geq \|\|x\|\|_{\mathcal{A}}$ then $e_X - \lambda^{-1}x$ is invertible by Theorem 2.5 in [2], and so does $\lambda e_X - x$. This proves that $\rho(x) \leq \|\|x\|\|_{\mathcal{A}}$.

As in [12] a binary operation $\diamond$ which is associative and continuous is said to satisfy *α-property* if there exists a positive real number α such that

$$a \diamond b \leq \alpha \max\{a, b\}$$

for all $a, b \in \mathbb{R}^+$.

Definition 3.2. Let $\mathcal{P}$ be an algebraic cone in a Riesz space A or an algebraic totally orderd cone. We say that $\mathcal{P}$ satisfy *α-property* if there exists an element $0 \neq \alpha \in \mathcal{P}$ such that

$$ab \preceq \alpha \sup\{a, b\}$$

for all $a, b \in \mathcal{P}$.

Example 3.3. Let X be a non-empty set, and consider $\mathbb{C}^X$ equipped with pointwise convergence topology. Obviously, $\mathbb{C}^X$ under pointwise multiplication is a topological algebra. Define

$$\mathcal{P} := \{f \in \mathbb{C}^X : f(x) \geq 0, \ \textit{for each } x \in X\}.$$

Then $\mathcal{P}$ is an algebraic cone in $\mathbb{C}^X$. Now it is obvious that, if for every $f, g \in \mathbb{C}^X$ we define

$$(f \diamond g)(x) = \frac{f(x)g(x)}{\max\{f(x), g(x), 1\}} \quad (x \in X),$$

then $\diamond$ on $\mathbb{C}^X$ have the α-property for each function $\alpha \in \mathcal{P}$, with $\alpha \geq 1$ where 1 is the constant function.

Definition 3.4. Let T and S are two self-mappings on a set X. We say that T and S are weakly compatible if they commute at their coincidence points; on the other words if $x \in X$ and $Tx = Sx$, then $TSx = STx$.

The proof of the following lemma is trivial by the definition of weakly compatible mappings.

Lemma 3.5. If two mappings T and S on a set X are weakly compatible and T and S have a unique coincidence point, then T and S have a unique fixed point.

The following theorem is a generalization of [12, Theorem 2.1] to the algebraic cones. The main idea of the proof comes from above references but the proof here has been shorter.

Theorem 3.6. Let (X,d) be a complete algebraic cone metric space with totally orderd cone $\mathcal{P}$, satisfing α-property with $0 \neq \alpha \in \mathcal{P}$. Let A, B, S and T be self-mappings on X such that the pairs (A,S) and (B,T) are weakly compatible. Let $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ be a closed subset of X. If for all $x, y \in X$,

$$\begin{aligned} d(Ax,By) &\preceq k_1 d(Sx,Ty)d(Ax,Sx) + k_2 d(Sx,Ty)d(By,Ty) \\ &+ k_3 d(Sx,Ty)\frac{d(Sx,By)+d(Ax,Ty)}{2}, \end{aligned}$$

where $k_1, k_2, k_3 \in \mathcal{P}$, $\alpha(k_1+k_2+k_3)x \prec x$ for each $x \in \mathcal{P}$ and $\lim_{n\to\infty}((k_1+k_2+k_3)\alpha)^n = 0$, then A, B, S and T have a uniqe common fixed point in X.

Proof . Under the assumptions, put

$$\begin{aligned} \mathcal{E}(x,y) &:= k_1 d(Sx,Ty)d(Ax,Sx) + k_2 d(Sx,Ty)d(By,Ty) \\ &+ k_3 d(Sx,Ty)\frac{d(Sx,By)+d(Ax,Ty)}{2}, \end{aligned}$$

where $x, y \in X$. Fix an element x_0 in X and define inductively a sequence $\{x_n\}$ in X by $y_{2n} := Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} := Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, ...$. Setting $\alpha_n := d(y_n, y_{n+1})$ for $n = 0, 1, 2, ...$, we have

$$\begin{aligned} \alpha_{2n} &= d(Ax_{2n}, Bx_{2n+1}) \\ &\preceq \mathcal{E}(x_{2n}, x_{2n+1}) \\ &= k_1\alpha_{2n-1}^2 + k_2\alpha_{2n-1}\alpha_{2n} + k_3\alpha_{2n-1}\frac{d(y_{2n-1}, y_{2n+1})}{2}. \end{aligned}$$

Therefore, since $\mathcal{P}$ have α-property, we get

$$\alpha_{2n} \preceq k_1\alpha\alpha_{2n-1} + k_2\alpha\max\{\alpha_{2n-1}, \alpha_{2n}\} + k_3\alpha\max\{\alpha_{2n-1}, \frac{\alpha_{2n-1}+\alpha_{2n}}{2}\}.$$

If $\alpha_{2n-1} \prec \alpha_{2n}$, we attend the following contraction

$$\alpha_{2n} \preceq (k_1+k_2+k_3)\alpha\alpha_{2n} \prec \alpha_{2n}.$$

Therefore, since $\mathcal{P}$ is totally orderd, $\alpha_{2n} \preceq \alpha_{2n-1}$ holds for $n = 0, 1, 2, ...$. Similarly, we have $\alpha_{2n+1} \preceq \alpha_{2n}$ for $n = 1, 2, ...$. So by taking $k := (k_1+k_2+k_3)\alpha$, we see $\alpha_n \preceq k^n\alpha_0$ for $n = 1, 2, ...$, and if $n \leq m$, then

$$\begin{aligned} d(y_n, y_m) &\preceq \alpha_n + \alpha_{n+1} + ... + \alpha_{m-1} \\ &\preceq k^n\alpha_0 + k^{n+1}\alpha_0 + ... + k^{m-1}\alpha_0 \\ &\preceq (e_{\mathcal{A}} - k)^{-1}k^n\alpha_0 \to 0, \end{aligned}$$

as $m, n \to 0$. It follows that the sequence $\{y_n\}$ is a Cauchy sequence and by completeness of X, $\{y_n\}$ converges to an element, like $y \in X$. So,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2}.$$

Now, assume that $T(X)$ is closed. Then, there exists an element $z \in X$ such that $y = T(z)$. By the inequality $d(Ax_{2n}, Bz) \preceq \mathcal{E}(x_{2n}, z)$ $(n \in \mathbb{N})$, we have $d(y, Bz) = 0$. So, $y = Bz$. Since B and T are weakly compatible, we have $BTz = TBz$ and so $By = Ty$. If $By \neq y$, then

$$\begin{aligned} d(y, By) &= \lim_{n\to\infty} d(Ax_{2n}, By) \\ &\preceq \lim_{n\to\infty} \mathcal{E}(x_{2n}, y) \\ &= k_3 d(y, By)^2 \\ &\preceq k_3 \alpha d(y, By) \\ &\preceq k\alpha d(y, By) \\ &\prec d(y, By), \end{aligned}$$

a contradiction. Hence, $By = y$. On the other hand, since $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$. Hence,

$$d(Au, y) = d(Au, By) \preceq \mathcal{E}(u, y) = 0.$$

Therefore, $Au = y$. Since A and S are weakly compatible, we have $ASu = SAu$ and so $Ay = Sy$. Finally, if $Ay \neq y$

$$d(Ay, y) = d(Ay, By) \preceq \mathcal{E}(y, y) \preceq k_3\alpha d(Ay, y) \prec d(Ay, y),$$

a contradiction. Thus,

$$Ay = By = T(y) = S(y) = y.$$

Similarly if $S(X)$ is closed then y is a common fixed point for A, B, S and T. □

Corollary 3.7. Let (X, d) be a complete algebraic cone metric space with respect to a totally orderd cone $\mathcal{P}$, such that satisfies α-property with $0 \neq \alpha \in \mathcal{P}$. Let A and T be two weakly compatible self-mappings on X. Let $A(X) \subseteq T(X)$ and $T(X)$ be a closed subset of X. If for all $x, y \in X$,

$$\begin{aligned} d(Ax, Ay) &\preceq k_1 d(Tx, Ty) d(Ax, Tx) + k_2 d(Tx, Ty) d(Ay, Ty) \\ &+ k_3 d(Tx, Ty)(\frac{d(Tx, Ay) + d(Ax, Ay)}{2}), \end{aligned}$$

where $k_1, k_2, k_3 \in \mathcal{P}$, $\alpha(k_1 + k_2 + k_3)x \prec x$ for each $x \in \mathcal{P}$ and $\lim_{n\to\infty}((k_1 + k_2 + k_3)\alpha)^n = 0$, then A and T have a uniqe common fixed point in X.

Proof . In the Theorem 3.6, put $A = B$ and $T = S$. □

Remark 3.8. a) If $\mathcal{P}$ is a totally orderd cone (not necessarily algebraic) with a binary operation $\diamond$ which is associative and continuous, satisfy α-property for some $0 \neq \alpha \in \mathcal{P}$, provided that the relation in Theorem 3.6 converted to the following relation:

$$\begin{aligned} d(Ax, By) &\preceq k_1(d(Sx, Ty) \diamond d(Ax, Sx)) + k_2(d(Sx, Ty) \diamond d(By, Ty)) \\ &+ k_3(d(Sx, Ty) \diamond (\frac{d(Sx, By) + d(Ax, By)}{2})), \end{aligned}$$

for all $x, y \in X$, Then the proof of the Theorem 3.6 is not valid in this case.

b) If $\mathcal{A}$ is a unital algebra that is also without order, i.e. $ab = 0$ to conclude $a = 0$ or $b = 0$, then for every $k \in \mathcal{P}$, $k \prec e_{\mathcal{A}}$ implies that $kx \prec x$ for every $x \in \mathcal{P}$.

c) Considering $\mathcal{A} = \mathbb{R}$ and $\mathcal{A} = [0, \infty)$, Theorem 3.6 is a generalization of Theorem 2.1 in [12].

Algebraic cones, in addition to being able to generalize some theorems in metric spaces, can also generalize theorems in cone metric spaces by replacing the scalar constant with the vector constant. In the following result we generalize a main theorem of [8] to a fact in the context of algebraic cone metric spaces.

Theorem 3.9. Let (X,d) be a complete algebraic cone metric space with respect to an algebraic cone $\mathcal{P}$ in a without order unital Banach algebra $\mathcal{A}$ with identity element $e_{\mathcal{A}}$. Let T and S be self-mappings on X such that $S(X) \subseteq T(X)$, and $T(X)$ or $S(X)$ is a closed subset of X. Suppose that

$$\alpha d(Sx,Sy) \preceq d(Tx,Ty)$$

for some invertible element $\alpha \in \mathcal{P}$ with $e_{\mathcal{A}} \prec \alpha$ and all $x,y \in X$. If $\lim_{n\to\infty}(\alpha^{-1})^n = 0$ then T and S have a unique point of coincidence. If T and S are also weakly compatible, then S and T have a uniqe point common fixed point in X.

Proof . Fix an element x_0 in X. We define inductively the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n := Sx_n = Tx_{n+1}$ for $n = 0,1,2,\dots$. Then, we have

$$\alpha d(y_{n+1},y_n) = \alpha d(Sx_{n+1},Sx_n) \preceq d(Tx_{n+1},Tx_n) = d(y_{n+1},y_n)$$

for all $n \in \mathbb{N}$. Hence,

$$d(y_{n+1},y_n) \preceq \alpha^{-1}d(y_n,y_{n-1}) \preceq \dots \preceq (\alpha^{-1})^n d(y_1,y_0).$$

Since $\lim_{n\to\infty}(\alpha^{-1})^n = 0$; $\{y_n\}$ is a Cauchy sequence and by completeness of X, it converges to an element $y \in X$. Assume that $T(X)$ is closed. Then, there exists $z \in X$ such that $y = T(z)$. By the inequality $\alpha d(Sx_n,Sz) \preceq d(Tx_n,Tz)$, and the fact $\lim_{n\to\infty} y_{n-1} = \lim_{n\to\infty} Tx_n = T(z)$, we have $S(z) = T(z)$. So, $y = T(z) = S(z)$ is a point of coincidence for T and S. If $w = T(z_1) = S(z_1)$ is another point of coincidence for T and S, since $(\alpha)^{-1} \prec e_{\mathcal{A}}$, the assumption $d(y,w) \neq 0$ leads to the following contradiction

$$d(y,w) \prec \alpha d(y,w) \preceq \alpha d(S(z),S(z_1)) \preceq d(T(z),T(z_1)) = d(y,w).$$

Thus, the point of coincidence for T and S is unique. If T and S are weakly compatible, then by Lemma (3.5), T and S have a unique fixed point. By assuming weakly compatibility of the pair (S,T), we have $STz = TSz$ and so $Sy = Ty$. If $Sy \neq y$, then

$$\begin{aligned} d(y,Sy) &= \lim_{n\to\infty} d(y_n,Sy) \\ &\preceq \lim_{n\to\infty} d(Sx_n,Sy) \\ &\prec \lim_{n\to\infty} \alpha d(Sx_n,Sy) \\ &\preceq \lim_{n\to\infty} \alpha d(Tx_n,Ty) \\ &\preceq \lim_{n\to\infty} \alpha d(y_{n-1},Sy) \\ &\preceq d(y,Sy). \end{aligned}$$

Therefore, $T(y) = S(y) = y$. □

The following corollary is obtained from Theorem 3.9. Comparing this result with the Corollary 3.7 is interesting.

Corollary 3.10. Let (X,d) be a complete algebraic cone metric space with respect to a cone $\mathcal{P}$ in Banach algebra $\mathcal{A}$. Moreover let $\mathcal{A}$ ba a without order Riesz space with respect to an order $\preceq$, such that for some $0 \neq \alpha \in \mathcal{P}$

$$ab \preceq \alpha \inf\{a,b\}$$

for all $a,b \in \mathcal{P}$. Let A and T be two weakly compatible self-mappings on X. Let $A(X) \subseteq T(X)$, and $T(X)$ be a closed subset of X. If for all $x,y \in X$,

$$\begin{aligned} d(Ax,Ay) &\preceq k_1 d(Tx,Ty)d(Ax,Tx) + k_2 d(Tx,Ty)d(Ay,Ty) \\ &+ k_3 d(Tx,Ty)\left(\frac{d(Tx,Ay)+d(Ax,Ay)}{2}\right), \end{aligned}$$

where $k_1,k_2,k_3 \in \mathcal{P}$, $(k_1+k_2+k_3)\alpha$ is invertible and $\lim_{n\to\infty}((k_1+k_2+k_3)\alpha)^n = 0$, then A and T have a uniqe common fixed point in X.

Proof . Under assumptions, of the theorem we have

$$\begin{aligned} d(Ax,Ay) &\preceq k_1 d(Tx,Ty)d(Ax,Tx) + k_2 d(Tx,Ty)d(Ay,Ty) \\ &+ k_3 d(Tx,Ty)(\frac{d(Tx,Ay)+d(Ax,Ay)}{2}) \\ &\preceq k_1\alpha \inf\{d(Tx,Ty),d(Ax,Tx)\} + k_2\alpha \inf\{d(Tx,Ty),d(Ay,Ty)\} \\ &+ k_3\alpha \inf\{d(Tx,Ty),(\frac{d(Tx,Ay)+d(Ax,Ay)}{2})\} \\ &\preceq k_1\alpha d(Tx,Ty) + k_2\alpha d(Tx,Ty) + k_3\alpha d(Tx,Ty) \\ &\preceq (k_1+k_2+k_3)\alpha d(Tx,Ty) \end{aligned}$$

for all $x, y \in X$. So putting $\beta = (k_1 + k_2 + k_3)\alpha$, we have $\lim_{n\to\infty} \beta^n = 0$, and

$$\beta^{-1} d(Ax,Ay) \preceq d(Tx,Ty),$$

for all $x, y \in X$. Hence, by Theorem 3.9, A and T have a unique common fixed point. □

Theorem 3.11. Let (X, d) be an algebraic cone metric space, with cone $\mathcal{P}$ in a unital Banach algebra $\mathcal{A}$ and let T and S be two self-mappings on X such that $S(X) \subseteq T(X)$ and that at least one of these subspaces is complete. Suppose that there exists $0 \neq \alpha \in \mathcal{P}$ such that $\lim_{n\to\infty}((e_{\mathcal{A}} - \alpha)\alpha^{-1})^n = 0$, and

$$\alpha(d(Sx,Tx) + d(Sy,Ty)) \preceq d(Tx,Ty), \tag{3.1}$$

holds for all $x, y \in X$ with $x \neq y$. Then, T and S have a point of coincidence. If T and S are weakly compatible then the point of coincidence of T and S is unique, and so T and S have a uniqe common fixed point in X.

Proof . Fix an element x_0 in X and define inductively the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n := Sx_n = Tx_{n+1}$ for $n = 0, 1, 2, \ldots$. Then by the inequality (3.1) we have

$$\begin{aligned} \alpha(d(y_{n+1},y_n) + d(y_n,y_{n-1})) &= \alpha(d(Sx_{n+1},Tx_{n+1}) + d(Sx_n,Tx_n)) \\ &\preceq d(Tx_{n+1},Tx_n) \\ &= d(y_n,y_{n-1}), \end{aligned}$$

for all $n \in \mathbb{N}$. Hence,

$$d(y_{n+1},y_n) \preceq (e_{\mathcal{A}} - \alpha)\alpha^{-1} d(y_n,y_{n-1}) = \beta d(y_n,y_{n-1}),$$

where $\beta = (e_{\mathcal{A}} - \alpha)\alpha^{-1}$. Therefore,

$$d(y_{n+1},y_n) \preceq \beta^n d(y_1,y_0),$$

and the sequence $\{y_n\}$ is a Cauchy sequence, because $\lim_{n\to\infty} \beta^n = 0$. Now assume that $T(X)$ is closed. Then, there exisits $z \in X$ such that $\lim_{n\to\infty} T(x_n) = T(z)$. Let $0 \ll c$. The inequality (3.1) implies that

$$\alpha d(Sz,Tz) \preceq \alpha d(Sx_n,Tx_n) + \alpha d(Sz,Tz) \preceq d(Tx_n,Tz) \ll \alpha c$$

for large enough n. Since $0 \ll c$ is arbitary, the last inequality concludes that $S(z) = T(z)$. So, z is a point of coincidence for T and S. The rest of theorem follows from Lemma 3.5. □

Corollary 3.12. Let (X, d) be a complete algebraic cone metric space and $T : X \to X$ be surjective. Suppose that there exists $0 \neq \alpha \in \mathcal{P}$ such that $\lim_{n\to\infty}((e_{\mathcal{A}} - \alpha)\alpha^{-1})^n = 0$ and

$$\alpha(d(x,Tx) + d(y,Ty)) \preceq d(Tx,Ty) \tag{3.2}$$

holds for all $x, y \in X$, $x \neq y$. Then, T has a fixed point. Moreover, if $e_{\mathcal{A}} \prec \alpha$ and every positive element of $\mathcal{A}$ is invertible, then the set of fixed points of T and T^n are same for all $n \in \mathbb{N}$.

Proof . Putting $S = I_X$ then (3.1) holds by (3.2). Note that $S(X) = X \subseteq T(X)$ since T is onto. So by Theorem 3.11 one obtains that there exists $z \in X$ such that $T(z) = z$. If x is a fixed point for $T^2 = T \circ T$, then

$$\alpha d(x, Tx) = \alpha(d(x, Tx) + d(Tx, T^2x)) \preceq d(Tx, T^2x) = d(x, Tx).$$

Now if $d(x, Tx) \neq 0$ by positivity of $d(x, Tx)$, it is invertible and so, $\alpha \preceq e_{\mathcal{A}}$. This contradiction prove that $Tx = x$. Inductively, it is proved that for each $n \in \mathbb{N}$, every fixed point of T^n is a fixed point of T. □

4 Property (C) and property (E) in algebraic cone metric spaces

In [13] Suzuki defined a class of generalized non-expansive mappings as follows.

Definition 4.1. Let Y be a non-empty subset of a Banach space X. We say that a mapping $T : Y \to X$ satisfies condition (C) on Y if for all $x, y \in Y$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$.

Of course, every non-expansive mapping $T : Y \to X$ satisfies the condition (C) on Y, but in [13] some examples of non-continuous mappings satisfying condition (C) are given. The above definition has been generalized in [5] as follows.

Definition 4.2. Let Y be a non-empty subset of a Banach space X. For $\alpha \in (0, 1)$ we say that a mapping $T : Y \to X$ satisfy condition (C_α) on Y if for all $x, y \in Y$, $\alpha\|x - Tx\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$.

Given the closedness of multiplication in cones in algebraic cone metric space, we can define the following generalization of two above definitions.

Definition 4.3. Let (X, d) be an algebraic cone normed space corresponding to a cone $\mathcal{P}$ and Y be a nonempty subset of X. For $0 \neq \alpha \in \mathcal{P}$ we say that a mapping $T : Y \to X$ satisfies condition (C_α) on Y if for all $x, y \in Y$, $\alpha\|x - Tx\|_{\mathcal{P}} \preceq \|x - y\|_{\mathcal{P}}$ implies $\|Tx - Ty\|_{\mathcal{P}} \preceq \|x - y\|_{\mathcal{P}}$. We say that T satisfies condition (C) on Y whenever T satisfies (C_α) for some $0 \neq \alpha \in \mathcal{P}$.

Another generalization of property (C) is also given in [5] as follows.

Definition 4.4. Let Y be a non-empty subset of a Banach space X. For $\alpha \geq 1$ we say that a mapping $T : Y \to X$ satisfies condition (E_α) on Y if for all $x, y \in Y$, $\|x - Ty\| \leq \alpha\|x - Tx\| + \|x - y\|$. We say that T satisfies condition (E) on Y whenever T satisfies condition (E_α) for some $\alpha \geq 1$.

We can also generalize this definition as follows for algebraic cone normed spaces.

Definition 4.5. Let (X, d) be an algebraic cone normed space corresponding a cone $\mathcal{P}$, and Y be a nonempty subset of X. For $e_{\mathcal{A}} \preceq \alpha \in \mathcal{P}$ we say that a mapping $T : Y \to X$ satisfies condition (E_α) on Y if for all $x, y \in Y$,

$$\|x - Ty\|_{\mathcal{P}} \preceq \alpha\|x - Tx\|_{\mathcal{P}} + \|x - y\|_{\mathcal{P}}.$$

We say that T satisfies condition (E) on Y whenever T satisfies the condition (E_α) for some $e_{\mathcal{A}} \preceq \alpha \in \mathcal{P}$.

The following lemma provides a generalization of [13, Lemma 7].

Lemma 4.6. Let T be a mapping on a subset Y of an algebraic cone normed space X. Assume that T satisfies the conditions $(C_{\lambda e_{\mathcal{A}}})$ and $(C_{\gamma e_{\mathcal{A}}})$ whenever λ and γ are positive real numbers and $\lambda + \gamma \in (0, 1]$. Then the following statements hold:
a) For all $x, y \in Y$ we have, $\lambda\|x - Tx\|_{\mathcal{P}} \preceq \|x - y\|_{\mathcal{P}}$ or $\gamma\|Tx - T^2x\|_{\mathcal{P}} \preceq \|Tx - y\|_{\mathcal{P}}$.
b) T satisfies the condition $(E_{3e_{\mathcal{A}}})$ on Y.
c) (Lemma 7 in [13]) If T satisfies condition $(C_{\frac{1}{2}e_{\mathcal{A}}})$ on Y, then is satisfies condition $(E_{3e_{\mathcal{A}}})$ on Y.

Proof . a) First, notice that since T satisfies the conditions $(C_{\lambda e_{\mathcal{A}}})$ and $\lambda\|x - Tx\|_{\mathcal{P}} \preceq \|x - Tx\|_{\mathcal{P}}$ we have, $\|Tx - T^2x\|_{\mathcal{P}} \preceq \|x - Tx\|_{\mathcal{P}}$. If we assume that

$$\lambda\|x - Tx\|_{\mathcal{P}} \succ \|x - y\|_{\mathcal{P}} \quad \text{and} \quad \gamma\|Tx - T^2x\|_{\mathcal{P}} \succ \|Tx - y\|_{\mathcal{P}},$$

then we have

$$\begin{aligned} \|x - Tx\|_{\mathcal{P}} &\preceq \|x - y\|_{\mathcal{P}} + \|Tx - y\|_{\mathcal{P}} \\ &\prec \lambda\|x - Tx\|_{\mathcal{P}} + \gamma\|Tx - T^2x\|_{\mathcal{P}} \\ &\preceq (\lambda + \gamma)\|x - Tx\|_{\mathcal{P}} \preceq \|x - Tx\|_{\mathcal{P}}. \end{aligned}$$

This is a contradiction. Therefore, we obtain the desired result. b) According to Part (a) either $\|Tx - Ty\|_{\mathcal{P}} \preceq \|x - y\|_{\mathcal{P}}$ or $\|T^2x - Ty\|_{\mathcal{P}} \preceq \|Tx - y\|_{\mathcal{P}}$ holds. In the first case, we have

$$\begin{aligned} \|x - Ty\|_{\mathcal{P}} &\preceq \|x - Tx\|_{\mathcal{P}} + \|Tx - Ty\|_{\mathcal{P}} \\ &\preceq \|x - Tx\|_{\mathcal{P}} + \|x - y\|_{\mathcal{P}}. \end{aligned}$$

In the second case, by Part (a) again we have

$$\begin{aligned} \|x - Ty\|_{\mathcal{P}} &\preceq \|x - Tx\|_{\mathcal{P}} + \|Tx - T^2x\|_{\mathcal{P}} + \|T^2x - Ty\|_{\mathcal{P}} \\ &\preceq 2\|x - Tx\|_{\mathcal{P}} + \|Tx - y\|_{\mathcal{P}} \\ &\preceq 3\|x - Tx\|_{\mathcal{P}} + \|x - y\|_{\mathcal{P}}. \end{aligned}$$

c) The proof is obtained by putting $\lambda = \gamma = \frac{1}{2}$ in (b). □

Before we extend some results about metric spaces to algebraic cone metric spaces, we remained that a sequence $\{x_n\}$ in a cone normed space Y is called an *almost fixed point sequence* (a.f.p.s. in short) for a mapping $T : Y \to Y$, whenever $\lim_{n\to\infty} \|Tx_n - x_n\|_{\mathcal{P}}$. Also, the subset of Y of X is called bounded, whenever there is an element $b \in \mathcal{P}$, called upper bound for Y, such that $\|x\|_{\mathcal{P}} \preceq b$ for all $x \in Y$.

Lemma 4.7. Let Y be a bounded subset of an algebraic cone normed space X. Let $T : Y \to Y$ be an arbitrary mapping. Then at least one of the following statements holds:
a) There exists an a.f.p.s. for T in Y.
b) T satisfies condition (E) on Y.

Proof . Suppose that T does not satisfy the condition (E). Then, for every positive integer n there exist $x_n, y_n \in Y$ such that

$$ne_{\mathcal{A}}\|x_n - Tx_n\|_{\mathcal{P}} + \|x_n - y_n\|_{\mathcal{P}} \prec \|x_n - Ty_n\|_{\mathcal{P}}.$$

Hence, if b is an upper bound for Y, then for every positive integer n,

$$\|x_n - Tx_n\|_{\mathcal{P}} \prec \frac{4b}{n},$$

and we obtain that $\|x_n - Tx_n\|_{\mathcal{P}} \to 0$, whenever $n \to \infty$, i.e. (a) is holds. □

The following lemma is useful tool to provide some results. This lemma has been proved for hyperbolic spaces in [10] (see also [13, Lemma 3]).

Lemma 4.8. Let X be a cone normed space with cone $\mathcal{P}$. Let $0 \leq \lambda \leq 1$, and two sequences $\{x_n\}$ and $\{y_n\}$ in X satisfy, $x_{n+1} = (1 - \lambda)x_n + \lambda y_n$ and $\|y_{n+1} - y_n\|_{\mathcal{P}} \preceq \|x_{n+1} - x_n\|_{\mathcal{P}}$ for all $n \in \mathbb{N}$. Then,

$$(1 - \lambda)^{-n}[\|y_{i+n} - x_{i+n}\|_{\mathcal{P}} - \|y_i - x_i\|_{\mathcal{P}}] + (1 + n\lambda)\|y_i - x_i\|_{\mathcal{P}} \quad \preceq \quad \|y_{i+n} - x_i\|_{\mathcal{P}}, \tag{4.1}$$

for all $i \in \mathbb{N}$.

Proof . The proof is by induction on n. Put $n = 0$. Trivially the inequality (4.1) is true for all $i \in \mathbb{N}$. Now, let the inequality (4.1) holds for a $n \in \mathbb{N}$. Replacing i by $i+1$ in (4.1),

$$(1-\lambda)^{-n}[\|y_{i+n+1} - x_{i+n+1}\|_{\mathcal{P}} - \|y_{i+1} - x_{i+1}\|_{\mathcal{P}}] + (1+n\lambda)\|y_{i+1} - x_{i+1}\|_{\mathcal{P}} \preceq \|y_{i+n+1} - x_{i+1}\|_{\mathcal{P}}.$$

Also, by the relation $x_{n+1} = (1-\lambda)x_n + \lambda y_n$ ($n \in \mathbb{N}$),

$$\begin{aligned}
\|y_{i+n+1} - x_{i+1}\|_{\mathcal{P}} &\preceq (1-\lambda)\|y_{i+n+1} - x_i\|_{\mathcal{P}} + \lambda\|y_{i+n+1} - y_i\|_{\mathcal{P}} \\
&\preceq (1-\lambda)\|y_{i+n+1} - x_i\|_{\mathcal{P}} + \lambda\sum_{k=0}^{n}\|y_{i+k+1} - y_{i+k}\|_{\mathcal{P}} \\
&\preceq (1-\lambda)\|y_{i+n+1} - x_i\|_{\mathcal{P}} + \lambda\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|_{\mathcal{P}}.
\end{aligned}$$

Combining two last inequalities we have,

$$\begin{aligned}
\|y_{i+n+1} - x_i\|_{\mathcal{P}} &\succeq (1-\lambda)^{-1}\|y_{i+n+1} - x_{i+1}\|_{\mathcal{P}} \\
&- \lambda(1-\lambda)^{-1}\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|_{\mathcal{P}} \\
&\succeq (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\|_{\mathcal{P}} - \|y_{i+1} - x_{i+1}\|_{\mathcal{P}}] \\
&+ (1-\lambda)^{-1}(1+n\lambda)\|y_{i+1} - x_{i+1}\|_{\mathcal{P}} \\
&- \lambda(1-\lambda)^{-1}\sum_{k=0}^{n}\|x_{i+k+1} - x_{i+k}\|_{\mathcal{P}}.
\end{aligned}$$

On the other hand, by construction of the sequence $\{x_n\}$ for each $n \in \mathbb{N}$ we have, $\|x_n - x_{n+1}\|_{\mathcal{P}} = \lambda\|x_n - y_n\|_{\mathcal{P}}$. Futhermore by the assumption $\|y_{n+1} - y_n\|_{\mathcal{P}} \preceq \|x_{n+1} - x_n\|_{\mathcal{P}}$ we have,

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\|_{\mathcal{P}} &\preceq \|x_{n+1} - y_n\|_{\mathcal{P}} + \|y_{n+1} - y_n\|_{\mathcal{P}} \\
&\preceq \|(1-\lambda)(x_n - y_n)\|_{\mathcal{P}} + \|x_{n+1} - x_n\|_{\mathcal{P}} \\
&= [(1-\lambda) + \lambda]\|x_n - y_n\|_{\mathcal{P}} = \|x_n - y_n\|_{\mathcal{P}}
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\|y_{i+n+1} - x_i\|_{\mathcal{P}} &\succeq (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\|_{\mathcal{P}} - \|y_{i+1} - x_{i+1}\|_{\mathcal{P}}] \\
&+ (1-\lambda)^{-1}(1+n\lambda)\|y_{i+1} - x_{i+1}\|_{\mathcal{P}} \\
&- \lambda^2(1-\lambda)^{-1}(n+1)\|y_i - x_i\|_{\mathcal{P}} \\
&= (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\|_{\mathcal{P}} - \|y_i - x_i\|_{\mathcal{P}}] \\
&+ [(1-\lambda)^{-1}(1+n\lambda) - (1-\lambda)^{-(n+1)}])\|y_{i+1} - x_{i+1}\|_{\mathcal{P}} \\
&+ [(1-\lambda)^{-(n+1)} - \lambda^2(1-\lambda)^{-1}(n+1)])\|y_i - x_i\|_{\mathcal{P}} \\
&\succeq (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\|_{\mathcal{P}} - \|y_i - x_i\|_{\mathcal{P}}] \\
&+ [1 + (n+1)\lambda]\|y_i - x_i\|_{\mathcal{P}}.
\end{aligned}$$

This completes the proof. □

In the next theorem, we need a version of the *Archimedean property* for cone metric spaces. We recall that an ordered vector space $(L, \leq)$ is said to have Archimedean property whenever it follows from $y \in L$, $x \in L_+ = \{l \in L : l \geq 0\}$, and $ny \leq x$ for all $n = 1, 2, ...$ that $y \leq 0$. By [1, Lemma 2.4], if L is an ordered Hausdorff topological vector space whose cone L_+ has a non-empty interior, then the cone L_+ is Archimedean if and only if it is closed. Therefore, in the cone metric space (X, d), closedness of the cone $\mathcal{P}$, depending on the Archimedean property. We also recall that an ordered vector space $(L, \leq)$ is said to be *Dedekind complete* if every non-empty subset of L that is bounded from above in L has the supremum in L. Using [1, Lemma 5.3] we see that an ordered vector space L is Dedekind complete if every non-empty subset of L that is bounded from below in L has the infimum in L. It is easy to prove that if an ordered vector space $(L, \leq)$ is Dedekind complete, then every increasing sequence in L that is bounded from above is convergent, and also every decreasing sequence in L that is bounded from below is convergent. In addition, we can define the liminf and limsup of bounded sequences in Dedekind complete cone metric spaces similar to real sequences:

$$\begin{aligned}\liminf_{n\to\infty} x_n &= \bigwedge_{n=1}^{\infty}\bigvee_{i=n}^{\infty} x_i = \lim_{n\to\infty}\bigvee_{i=n}^{\infty} x_i\\ \limsup_{n\to\infty} x_n &= \bigvee_{n=1}^{\infty}\bigwedge_{i=n}^{\infty} x_i = \lim_{n\to\infty}\bigwedge_{i=n}^{\infty} x_i\end{aligned}$$

Finally, similar to normed spaces we call a cone normed space X is satisfy the *Opial condition* whenever for every sequence $\{x_n\}$ with $x_n \to z$ weakly,

$$\liminf_{n\to\infty} \|x_n - z\| \prec \liminf_{n\to\infty} \|x_n - y\|$$

holds, whenever $y \neq z$. Now we are now ready to generalize some results from [5].

Theorem 4.9. Let X be a Dedekind complete cone normed space with cone $\mathcal{P}$. Let $0 \leq \lambda \leq 1$, $x_0 \in X$, and $\{x_n\}$ be a sequence satisfies $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$ for all $n \in \mathbb{N}$, where $T : X \to X$ is a non-expansive mapping. Then if $\{x_n\}$ is bounded, then it is an almost fixed point sequence for T.

Proof . Let $y_n := Tx_n$ for all $n \in \mathbb{N}$. By the assumption there is an element $d \in X$ such that $\|y_{i+n} - x_i\| \preceq d$ for all $n, i \in \mathbb{N}$. By proof of Lemma 4.8 the sequence $\{\|y_n - x_n\|\}$ is decreasing and therefore, since X is Dedekind complete, this sequence is convergent. Let $\lim_{n\to\infty} \|y_n - x_n\| = r$. If $r \succ 0$, then by Archimedean property we can choose an integer N such that $Nr\lambda \succeq d$. Let $0 \neq c \in \mathcal{P}$ satisfy $c(1-\lambda)^{-N} \succ r$. On the other hand, since $\{\|y_n - x_n\|\}$ is Cauchy, there exists $i \in \mathbb{N}$ such that $\|y_i - x_i\| - \|y_{i+N} - x_{i+N}\| \preceq c$. Combined with relation (4.1) in Lemma 4.8 these choices of N, i, and c yield:

$$\begin{aligned} d + r &\preceq (1+N\lambda)r\\ &\preceq (1+N\lambda)\|y_i - x_i\|\\ &\preceq \|y_{i+n} - x_i\| + (1-\lambda)^{-N}c\\ &\prec d + r,\end{aligned}$$

a contradiction. This complete the proof. □

Theorem 4.10. Let Y be a bounded convex subset of a Dedekind complete algebraic cone normed space X. Assume that $T : Y \to Y$ satisfies condition (C_α) as defined by Definition 3.3 for $\alpha = \gamma e_{\mathcal{A}}$, where $\gamma \in (0,1)$. For $\lambda \in [\gamma, 1)$ define a sequence $\{x_n\}$ in Y by tacking $x_1 \in Y$ and $x_{n+1} := (1-\lambda)x_n + \lambda T x_n$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an a.f.p.s. for T.

Proof . For each $n \in \mathbb{N}$, we have

$$\alpha\|x_n - Tx_n\| = \gamma\|x_n - Tx_n\| \preceq \lambda\|x_n - Tx_n\|.$$

From the condition (C_α), it is concluded that $\|Tx_{n+1} - Tx_n\| \preceq \|x_{n+1} - x_n\|$, and we can apply Theorem 4.9 to conclude that $\|x_n - Tx_n\| \to 0$. □

Remark 4.11. By hypothesis of Theorem 4.10 if $T : Y \to Y$ satisfies condition E_β for some $\beta \in \mathcal{P}$, then T has an almost fixed point sequence as x_n. Now if x_n have a subsequence x_{n_k} such that $\lim_{k\to\infty} x_{n_k} = z$ for some $z \in \mathcal{P}$, then T have a fixed point. Because

$$\|x_{n_k} - Tz\| \preceq \beta\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|,$$

so by taking the limit from both sides of above inequality when k tends to the infinity, we have $\lim_{k\to\infty} x_{n_k} = Tz$, and hence $z \in F(T)$.

Theorem 4.12. Let T be a self-mapping on a locally weakly compact convex subset Y of a Dedekind complete algebraic cone normed space X with cone $\mathcal{P}$. Assume that $T : Y \to Y$ satisfies condition (C_α) as defined by Definition (4.3) for some $\alpha = \gamma e_{\mathcal{A}}$, where $\gamma \in (0, 1)$. Let X satisfies the Opial condition, and T has a fixed point. Define a sequence $\{x_n\}$ in Y by $x_1 \in Y$ and

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, (n \in \mathbb{N})$$

for $\lambda \in [\gamma, 1)$. Then $\{x_n\}$ weakly converges to a fixed point of T .

Proof . Let z be a fixed point of T and $0 \neq c \in \mathcal{P}$. Then, the set

$$\{x \in X : \|x - z\|_{\mathcal{P}} \preceq c\}$$

is weakly compact, convex and T-invariant. Therefore, without loss of generality, we may assume that Y is weakly compact. By Theorem 4.9, $\{x_n\}$ is an a.f.p.s. for T. Now same as the proof of [5, Theorem 6] we can consider the following two cases:

Case I) $\{x_n\}$ has a cluster point. In this case, let $y \in Y$ be a cluster point of $\{x_n\}$. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging by cone norm $\|.\|_{\mathcal{P}}$ to y. If $Ty \neq y$, then by the Opial condition of X,

$$\liminf_{n\to\infty} \|x_{n_j} - Ty\|_{\mathcal{P}} \prec \liminf_{n\to\infty} \|x_{n_j} - y\|_{\mathcal{P}}.$$

On the other hand, if $c = \frac{1}{2} \liminf_{n\to\infty} \|x_{n_j} - Ty\|_{\mathcal{P}} \succ 0$, then

$$\lambda\|x_{n_j} - Tx_{n_j}\|_{\mathcal{P}} \preceq \|x_{n_j} - Tx_{n_j}\|_{\mathcal{P}} \prec c \prec \|x_{n_j} - y\|_{\mathcal{P}}$$

for sufficiently large $n \in \mathbb{N}$. Since T satisfies C_γ, we have

$$\|Tx_{n_j} - Ty\|_{\mathcal{P}} \preceq \|x_{n_j} - y\|_{\mathcal{P}}.$$

So,

$$\begin{aligned} \|x_{n_j} - Ty\|_{\mathcal{P}} &\preceq \|x_{n_j} - Tx_{n_j}\|_{\mathcal{P}} + \|Tx_{n_j} - Ty\|_{\mathcal{P}} \\ &\preceq \|x_{n_j} - Tx_{n_j}\|_{\mathcal{P}} + \|x_{n_j} - y\|_{\mathcal{P}}. \end{aligned}$$

Taking $n \to \infty$, we obtain

$$\liminf_{n\to\infty} \|x_{n_j} - Ty\|_{\mathcal{P}} \preceq \liminf_{n\to\infty} \|x_{n_j} - y\|_{\mathcal{P}},$$

a contradiction. This shows that y is a fixed point for T. Since T satisfies condition $C_{\gamma e_{\mathcal{A}}}$ and $\gamma\|x_n - Ty\|_{\mathcal{P}} = \gamma\|x_n - y\|_{\mathcal{P}} \preceq \|x_n - Ty\|_{\mathcal{P}}$, we have

$$\begin{aligned} \|x_{n+1} - y\|_{\mathcal{P}} &= \|(1-\lambda)x_n + \lambda Tx_n - (1-\lambda+\lambda)y\|_{\mathcal{P}} \\ &\preceq (1-\lambda)\|x_n - y\|_{\mathcal{P}} + \lambda\|Tx_n - Ty\|_{\mathcal{P}} \\ &\preceq (1-\lambda)\|x_n - y\|_{\mathcal{P}} + \lambda\|x_n - y\|_{\mathcal{P}} = \|x_n - y\|_{\mathcal{P}}. \end{aligned}$$

Therefore, $\{\|x_n - y\|_{\mathcal{P}}\}$ is a decreasing sequence. This fact and the relation $\lim_{j\to\infty} \|x_{n_j} - y\|_{\mathcal{P}} = 0$ lead to $\lim_{n\to\infty} x_n = y$.

Case II) $\{x_n\}$ has no cluster point. In this case, arguing by contradiction, we assume that $\{x_n\}$ does not converge weakly. Since Y is weakly compact, we can choose sub-sequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to distinct points $y_1, y_2 \in Y$, respectively. Since X satisfies the Opial condition, by an argument similar to case I, y_1 and y_2 are fixed points of T and also $\{\|x_n - y_1\|_{\mathcal{P}}\}$ and $\{\|x_n - y_2\|_{\mathcal{P}}\}$ are decreasing. Using the Opial condition again, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-y_1\|_{\mathcal{P}} &= \lim_{j\to\infty}\|x_{n_j}-y_1\|_{\mathcal{P}}\\ &\prec \lim_{j\to\infty}\|x_{n_j}-y_2\|_{\mathcal{P}}\\ &= \lim_{n\to\infty}\|x_n-y_2\|_{\mathcal{P}}\\ &= \lim_{k\to\infty}\|x_{n_k}-y_2\|_{\mathcal{P}}\\ &\prec \lim_{k\to\infty}\|x_{n_k}-y_1\|_{\mathcal{P}}\\ &= \lim_{n\to\infty}\|x_n-y_1\|_{\mathcal{P}},\end{aligned}$$

a contradiction. Therefore, we obtain the desired result. □

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