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Fixed point uniqueness of generalized (ψ, φ)-weak contractions in partially ordered metric spaces under suitable constraints

Gholamreza Heidary Joonaghany

Department of Mathematics, Arak Branch, Islamic Azad University, Arak, Iran

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Abstract

In this paper, by providing an example, I show that the condition which produced by Radenović and Kadelburg in [Generalized weak contractions in partially ordered metric spaces, Comput. Math. Appl. 60 (2010) pp. 1776-1783] is not sufficient for uniqueness of the fixed point. Furthermore, a new sufficient condition is introduced for the uniqueness of the fixed point. Some suitable examples are furnished to demonstrate the validity of the hypotheses of my results.

Keywords: Fixed Point Uniqueness, Generalized (ψ, φ)-Weak Contraction, Ordered Metric Spaces
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1 Introduction and Preliminaries

The study of fixed point theory in the setting of a partially ordered metric space was first started in 2004 by Ran and Reurings [11] and then by Nieto and López (see [6, 7]). Many authors obtained several interesting results in ordered metric spaces, for example, see [3, 4, 5, 8, 10].

The following definitions and notations will be used in this paper.

Definition 1.1. [9] Let $(X, \preceq)$ be a partially ordered set and let T and S be two self-maps on X. Then

1) the elements $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds and we denote it by $x \preceq\succeq y$.
2) a subset A of X is said to be well ordered if any two elements of A are comparable .
3) the ordered metric space $(X, \preceq, d)$ is called regular whenever if a nondecreasing sequence $\{x_n\}$ in $(X, \preceq)$ converges to $x \in X$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.
4) T is called nondecreasing with respect to $\preceq$ if $x \preceq y$ implies $Tx \preceq Ty$.
5) the mappings T and S are called weakly increasing if $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in X$. A mapping T is called weakly increasing If T and T are weakly increasing maps which means that $Tx \preceq T^2x$ for each $x \in X$.

The control functions were introduced by Đorić [1] as follows:

Email address: ghheidari@iau-arak.ac.ir (Gholamreza Heidary Joonaghany)

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Definition 1.2. A pair (ψ, φ) of self-maps on $[0, \infty)$ is called a pair of control functions, if the following conditions are satisfied:

1. ψ is a continuous nondecreasing function and $\psi(t) = 0$ if and only if $t = 0$.
2. φ is lower semi-continuous with $\varphi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on control functions (see, e.g. [1, 2, 9, 12]).

Consider the following notations:

- $m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}$,
- $n(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)+d(y,Sy)}{2}, \frac{d(x,Sy)+d(y,Tx)}{2}\right\}$,
- $m_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$,
- $n_T(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$,

for all $x, y \in X$, where T and S are two self-maps on the metric space (X, d).

In 2010, Radenović and Kadelburg [9], studied generalized weak contractions in partially ordered metric spaces and extended result of Đorić [1], Harjani and Sadarangani [2], as well as Zhang and Song [12]. They obtained the following results:

Theorem 1.3. [9, Theorem 3.1] Let $(X, \preceq, d)$ be an ordered complete metric space and (T, S) be a pair of weakly increasing maps on X such that for any two comparable elements $x, y \in X$

$$\psi(d(Tx, Sy)) \leq \psi(m(x, y)) - \phi(m(x, y)), \tag{1.1}$$

where (ψ, φ) is a pair of control functions.
Then T and S have a common fixed point, provided by at least one of the following cases holds:

(i) T or S is continuous, or

(ii) X is regular.

Corollary 1.4. [9, Theorem 3.3] Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$ for some $x_0 \in X$ and for every comparable elements $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \psi(m_T(x, y)) - \phi(m_T(x, y)), \tag{1.2}$$

where (ψ, φ) is a pair of control functions. Then, in each of the following two cases, T has a fixed point.

(i) T is continuous, or

(ii) X is regular.

Theorem 1.5. [9, Theorem 3.4] Let $(X, \preceq, d)$ be an ordered complete metric space and (T, S) be a weakly increasing par of self-maps on X. Suppose that there exist a pair of control functions (ψ, φ) such that for any two comparable elements $x, y \in X$,

$$\psi(d(Tx, Sy)) \leq \psi(n(x, y)) - \phi(n(x, y)).$$

Then in each of the following two cases the mappings T and S have at least one common fixed point:

(i) T either S is continuous, or

(ii) X is regular.

Theorem 1.6. [9, Theorem 4.3] Let all the conditions of Corollary 1.4 be fulfilled and let the following condition hold:
(a) for arbitrary two points $x, y \in X$ there exists $z \in X$ which is comparable with both x and y.
Then the fixed point of T is unique.

In the first part of this paper, with presenting an example, it is shown that Theorem 1.6 is not true as it stands. Then, it is shown that, by adding a suitable condition, the uniqueness of common fixed point can be proved.

2 Main Results

The following example indicates that Theorem 1.6 may fail.

Example 2.1. Let $X = \{O = (0,0), A = (2,2), B = (0,2), C = (2,0)\} \subseteq \mathbb{R}^2$ be endowed with the metric d defined by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$

Suppose that relation $\preceq$ is defined on X as follows:

$$\preceq \; = \{(O,O), (A,A), (B,B), (C,C), (B,O), (B,A), (C,O), (C,A)\}.$$

Then $(X, \preceq, d)$ is a regular ordered complete metric space. Suppose that $T : X \to X$ is defined as follows:

$$T(O) = O, \; T(A) = A, \; T(B) = C, \; T(C) = B.$$

Then, T is nondecreasing with respect to $\preceq$ and $O \preceq O = TO$. Choosing $\psi(t) = t$ and $\varphi(t) = \frac{t}{a}$ for any $t \geq 0$, where $a \geq 2 + \sqrt{2}$ is a real number, one can verify that, all conditions of Theorem 1.6 are satisfied. Indeed, for the points A and B, we have $A \preceq\succeq B$ and

$$\psi(d(TA, TB)) = \psi(d(A, C)) = 2.$$

On the other hand

$$\begin{aligned} m_T(A, B) &= \max\left\{d(A, B), d(A, A), d(B, C), \tfrac{d(A,C)+d(B,A)}{2}\right\} \\ &= \max\{2, 0, 2\sqrt{2}, 2\} \\ &= 2\sqrt{2}. \end{aligned}$$

Thus, one has

$$\begin{aligned} \psi(m_T(A, B)) - \varphi(m_T(A, B)) &= \psi(2\sqrt{2}) - \varphi(2\sqrt{2}) \\ &= 2\sqrt{2} - \tfrac{2\sqrt{2}}{a} \\ &\geq 2 \;\; (\textit{because } a \geq 2 + \sqrt{2}). \end{aligned}$$

Consequently

$$\psi(d(TA, TB)) \leq \psi(m_T(A, B)) - \varphi(m_T(A, B)).$$

The case of points A and C is treated similarly since it is really the same as the case of points A and B. Again, for the points O and C, we have $O \preceq\succeq C$ and

$$\psi(d(TO, TC)) = \psi(d(O, B)) = 2.$$

On the other hand we have

$$\begin{aligned} m_T(O, C) &= \max\left\{d(O, C), d(O, O), d(C, B), \tfrac{d(O,B)+d(C,O)}{2}\right\} \\ &= \max\{2, 0, 2\sqrt{2}, 2\} \\ &= 2\sqrt{2}. \end{aligned}$$

Thus, we get

$$\begin{array}{rcl}\psi(m_T(O,C))-\varphi(m_T(O,C)) & = & \psi(2\sqrt{2})-\varphi(2\sqrt{2}) \\ & = & 2\sqrt{2}-\frac{2\sqrt{2}}{a} \\ & \geq & 2 \ \ (\textit{because } a\geq 2+\sqrt{2}).\end{array}$$

Consequently, we have

$$\psi(d(TO,TC))\leq\psi(m_T(O,C))-\varphi(m_T(O,C)).$$

The case of points O and B is treated similarly since it is really the same as the case of points O and C. So, the relation (1.1) is established for any two comparable elements of X. Also, it is clear that the condition (a) is reliable. Thus, all the conditions of Theorem 1.6 be fulfilled. However, O and A are distinct fixed points of T.

Note that these two fixed points of T are incomparable; there are two points (B and C) which are comparable to both of them, but none of them satisfies the condition $Z\preceq TZ$.

The preceding example shows that the condition (a) in Theorem 1.6 can not grantee the uniqueness of the fixed point and the theorem contains a gap. In the next result, by assuming an additional condition, this result can be repaired.

Theorem 2.2. Let $(X,\preceq,d)$ be an ordered complete metric space and $T:X\to X$ be a nondecreasing map such that, for every comparable elements $x,y\in X$,

$$\psi(d(Tx,Ty))\leq\psi(m_T(x,y))-\phi(m_T(x,y)), \tag{2.1}$$

where (ψ,φ) is a pair of control functions. Also suppose that the following condition holds:
(b) for arbitrary non-comparable elements $x,y\in X$ there exists $z\in X$ which is comparable with x and y and $z\preceq Tz$.
Then, in each of the following two cases, T has a unique fixed point.

(i) T is continuous, or

(ii) X is regular.

Proof . We first claim that the fixed points set of T is nonempty. Indeed, if X is a singleton set, say $X=\{x_0\}$, then $x_0=T(x_0)$ and x_0 is the unique fixed point of T. So, one can suppose that X has at least two distinct elements. One can consider following two cases:

(I) any two elements of X are comparable. In this case, one of the following two state occurs:

- (I_1) $x_0\preceq T(x_0)$, for some $x_0\in X$.
  In this case, it follows from Corollary 1.4, that T has a fixed point and the claim is proved.

- (I_2) $T(x)\preceq x$, for any $x\in X$. (Notice that any two elements of X are comparable)

  We define a new order $\ll$ on X as follows:

  $$x\ll y\Leftrightarrow y\preceq x,$$

  for any $x,y\in X$. Now, for any $x\in X$, we have $x\ll Tx$. It is easy to verify that, all the conditions of Corollary 1.4 are satisfied for $(X,\ll,d)$, except (maybe) the regularity of $(X,\ll,d)$ which we do not need it here (Because, any two elements of X are comparable). So by Corollary 1.4, T has a fixed point z in $(X,\ll,d)$. It is clear that z is a fixed point of T in $(X,\preceq,d)$.

(II) there exist at least two incomparable points in X. With this assumption, our claim is proved by combining the condition (b)and Corollary 1.4.

Now let u and v be two fixed points of T. One of the following two cases can occurs:

(1) $u \preceq\succeq v$. In this case, we have

$$\begin{array}{rcl}\psi(d(u,v)) & = & \psi(d(Tu,Tv) \\ & \leq & \psi(m_T(u,v)) - \phi(m_T(u,v)),\end{array}$$

where

$$\begin{array}{rcl}m_T(u,v) & = & \max\left\{d(u,v),\ d(u,Tu),\ d(v,Tv),\ \frac{d(u,Tv)+d(v,Tu)}{2}\right\} \\ & = & \max\left\{d(u,v), 0, 0,\ \frac{d(u,v)+d(v,u)}{2}\right\} \\ & = & d(u,v).\end{array}$$

Thus, it follows that

$$\psi(d(u,v)) \leq \psi(d(u,v)) - \phi(d(u,v)),$$

which is a contradiction unless $d(u,v) = 0$, i.e., $u = v$.

(2) u and v are not comparable. In this case by the hypothesis (b), there exists $z \in X$ such that $z \preceq\succeq u$ and $z \preceq\succeq v$ and $z \preceq Tz$. Put $y_n := T^n y$, for any $y \in X$ and $n \geq 0$. Since T is nondecreasing, we obtain that $u = u_n \preceq\succeq z_n$, $v = v_n \preceq\succeq z_n$ for each $n \geq 0$. If there exists $n_0 \geq 0$ such that, $z_{n_0} = u$ then $v \preceq\succeq z_{n_0} = u$ and so from item (1), $u = v$. Thus, we can assume that $z_n \neq u$, $\forall n \geq 0$. Since $z \preceq Tz$,, proceeding as in the proof of [9, Theorem 3.1], one can conclude that $\{z_n\}$ is a convergent sequence and

$$\lim_{n\to\infty} d(z_{n-1}, z_n) = 0. \tag{2.2}$$

Now, we claim that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

Indeed, for any $n \geq 1$ we have

$$u = u_n \preceq\succeq z_n.$$

Hence,

$$\begin{array}{rcl}\psi(d(u,z_n)) & = & \psi(d(Tu_{n-1}, Tz_{n-1}) \\ & \leq & \psi(m_T(u_{n-1}, z_{n-1})) - \phi(m_T(u_{n-1}, z_{n-1})) \\ & \leq & \psi(m_T(u, z_{n-1})) - \phi(m_T(u, z_{n-1})) \\ & < & \psi(m_T(u, z_{n-1}))(because m_T(u, z_{n-1}) \neq 0)\end{array}$$

where

$$\begin{array}{rcl}m_T(u, z_{n-1}) & = & \max\left\{d(u, z_{n-1}),\ d(u,Tu),\ d(z_{n-1}, Tz_{n-1}),\ \frac{d(u,Tz_{n-1})+d(z_{n-1},Tu)}{2}\right\} \\ & = & \max\left\{d(u, z_{n-1}),\ d(z_{n-1}, z_n),\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\right\}.\end{array}$$

Using the fact that ψ is a nondecreasing function, it follows that

$$d(u, z_n) < \max\left\{d(u, z_{n-1}),\ d(z_{n-1}, z_n),\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\right\}. \tag{2.3}$$

Now, one can consider the following two cases:

(i_1) there exists a sequence $\{n_k\}_{k\geq 0}$ of distinct positive integers that

$$d(u, z_{n_k-1}) \leq d(z_{n_k-1}, z_{n_k}).$$

In this case, (2.2) implies that

$$\lim_{k\to\infty} d(u, z_{n_k-1}) = 0,$$

and using the fact that $\{z_n\}$ is a convergent sequence, one can conclude that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

(i_2) there exists $n_0 \geq 1$ such that

$$d(u, z_n) > d(z_{n-1}, z_n),$$

for any $n \geq n_0$. In this case, (2.2) implies that, for any $n \geq n_0$

$$\begin{aligned} d(u, z_n) &< \max\left\{d(u, z_{n-1}), d(u, z_n), \frac{d(u,z_n)+d(z_{n-1},u)}{2}\right\} \\ &= \max\{d(u, z_{n-1}), d(u, z_n)\}. \end{aligned}$$

So, one can conclude that

$$d(u, z_n) < d(u, z_{n-1}),$$

for any $n \geq n_0$. Thus, for any , the sequence $\{d(u, z_n)\}_{n \geq n_0}$ is non-increasing and bounded below. So, it has a limit $l \geq 0$. In addition, we have:

$$\lim_{n\to\infty} m_T(u, z_{n-1}) = l.$$

Passing to (upper)limit in the relation

$$\psi(d(u, z_n)) \leq \psi(m_T(u, z_{n-1})) - \phi(m_T(u, z_{n-1})),$$

one concludes that

$$\psi(l)) \leq \psi(l) - \phi(l).$$

Which is a contradiction unless $l = 0$. So, in any case, we proved that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

In the same way, one can show that

$$\lim_{n\to\infty} d(v, z_n) = 0.$$

Finally, for any $n \geq 0$, we have

$$0 \leq d(u, v) \leq d(u, z_n) + d(v, z_n).$$

Letting $n \to \infty$, we obtain that $d(u, v) = 0$, i.e., $u = v$.

Hence, in any case, the fixed point of T is unique. □

Theorem 2.3. Assume that all the conditions of Corollary 1.4 are satisfied. Also suppose that the following condition holds:
(c) for arbitrary non-comparable two fixed points $x, y \in X$ there exists $z \in X$ which is comparable with x and y, and also $z \preceq Tz$.

Then T has a unique fixed point.

Proof . By Corollary 1.4, T has at least a fixed point. Remainder of proof is similar to Theorem 2.2. □

Theorem 2.4. Assume that all the conditions of Corollary 1.4 are satisfied. Then T has a unique fixed point if and only if the set of all fixed points of T is well ordered.

Proof . By Corollary 1.4, T has at least a fixed point. Now, if the fixed point of T is unique then the set of all fixed points of T is a singleton and so is well ordered.

Conversely, suppose that the set of all fixed points of T is well ordered, and u and v are two distinct fixed point of T. Then $u \preceq\succeq v$. Similarly as in the proof of item (1) in Theorem 2.2, it can be shown that $u = v$. □

Remark 2.5. If $m_T(x, y)$ is replaced by $n_T(x, y)$ in the Theorem 2.3, then we can replace the condition (b) by the condition (a), i.e., we have the following theorem:

Theorem 2.6. Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$, for some $x_0 \in X$ and for any two comparable elements $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \psi(n_T(x, y)) - \phi(n_T(x, y)), \tag{2.4}$$

where (ψ, φ) is a pair of control functions. Furthermore, let the condition (a) of Theorem 1.6 hold. If T is continuous or X is regular, then T has a unique fixed point.

Proof . Firstly, similar to the proof of Corollary 1.4, One can conclude that T has a fixed point.
Now, let u and v be two fixed points of T. One can consider the following two cases:

(1) $u \preceq\succeq v$.
In this case, similarly as in the proof of item (1) in Theorem 2.2, it can be shown that $u = v$.

(2) u and v are not comparable.
In this case by the hypothesis (a) of Theorem 1.6, there exist $z \in X$ such that $z \preceq\succeq u$ and $z \preceq\succeq v$.
By using the employed notations in the proof of Theorem 2.2, (without lose of generality) one can suppose that $z_n \neq u$, for all $n \geq 0$ and prove that $u = u_n \preceq\succeq z_n$ and $v = v_n \preceq\succeq z_n$ for each $n \geq 0$. Now, for any $n \geq 1$, $u = u_n \preceq\succeq z_n$, hence,

$$\begin{array}{rcl} \psi(d(u, z_n)) & = & \psi(d(Tu_{n-1}, Tz_{n-1}) \\ & \leq & \psi(n_T(u_{n-1}, z_{n-1})) - \phi(n_T(u_{n-1}, z_{n-1})) \\ & \leq & \psi(n_T(u, z_{n-1})) - \phi(n_T(u, z_{n-1})), \end{array}$$

where

$$\begin{array}{rcl} n_T(u, z_{n-1}) & = & \max\left\{d(u, z_{n-1}),\ \frac{d(u,Tu)+d(z_{n-1},Tz_{n-1})}{2},\ \frac{d(u,Tz_{n-1})+d(z_{n-1},Tu)}{2}\right\} \\ & = & \max\left\{d(u, z_{n-1}),\ \frac{d(z_{n-1},z_n)}{2},\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\right\} \\ & \leq & \max\left\{d(u, z_{n-1}),\ \frac{d(z_{n-1},u)+d(u,z_n)}{2},\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\right\} \\ & = & \max\left\{d(u, z_{n-1}),\ \frac{d(z_{n-1},u)+d(u,z_n)}{2}\right\} \\ & \leq & \max\{d(u, z_{n-1}), d(u, z_n)\}. \end{array}$$

Similar to the proof of Theorem 2.2,one can complete the proof. □

Example 2.7. As we saw, in the Example 2.4, the condition (a) is hold, but the mapping T has two distinct fixed points. Now, notice that, the condition (b) of the Theorem 2.2 is not satisfied. In fact, for the points A and O there is no $Z \in X$ which is comparable with A and O and $Z \preceq TZ$. Also, notice that, the mapping T is not satisfied in the Theorem 2.6. Indeed,, for the points A and B we have $A \preceq\succeq B$ and

$$\begin{array}{rcl} n_T(A, B) & = & \max\left\{d(A, B), \frac{d(A,A)+d(B,C)}{2}, \frac{d(A,C)+d(B,A)}{2}\right\} \\ & = & \max\{2, \sqrt{2}, 2\} \\ & = & 2. \end{array}$$

So, for any pair (ψ, φ) of control functions we have:

$$\psi(d(TA, TB)) = \psi(d(A, C)) = \psi(2).$$

Also,

$$\begin{array}{rcl} \psi(n_T(A, B)) - \varphi(n_T(A, B)) & = & \psi(2) - \varphi(2) \\ & < & \psi(2) \quad (\textit{because } \varphi(2) > 0) \\ & = & \psi(d(TA, TB)). \end{array}$$

Hence the conditions (2.4) of Theorem 2.4 is not satisfy. Finally, note that all conditions of Theorem 2.3 are satisfied, except condition (c) which is not established. In fact, O and A are two non comparable fixed points of T and there is no $Z \in X$ which is comparable with O, A and TZ.

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