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# On the pathwise uniqueness for a class of SPDEs driven by Lévy noise in Hilbert spaces

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**Abstract**

This paper seeks to prove the pathwise uniqueness of an abstract stochastic partial differential equation in Hilbert spaces driven by both Poisson random measure and the Wiener process with Hölder continuous drift. The main idea is based on the corresponding infinite-dimensional Kolmogorov equation. In addition, the main result is further supported by the help of an example.

Keywords: Poisson Random Measure; Pathwise Uniqueness; Infinite Dimensional Kolmogorov Equations; Lévy Noise
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## 1 Introduction

The theory of stochastic differential equations driven by Lévy noise has been a central issue for different studies. Understanding how the noise can improve the uniqueness for SPDEs, while the corresponding PDEs do not have this property, is a topic of great interest. For instance, consider the following deterministic differential equation:

$$dX_t = f(X_t)dt, \qquad X_0 = x_0. \tag{1.1}$$

It is well-known that if $f$ is a Lipschitz continuous function, the equation is well-posed, instead if $f$ is only Hölder continuous, it is not well-posed. For example, let $f(x) = |x|^\gamma$ with $0 < \gamma < 1$ and $X_0 = 0$, then equation (1.1) has two solutions: $X_t \equiv 0$ and $X_t = (1-\gamma)t^{\frac{1}{1-\gamma}}$. Nonetheless, if equation (1.1) is considered by a strong enough noise, the equation becomes well-posed for many singular $f$'s. To put it simply, consider the following SDE on $\mathbb{R}^m$:

$$dX_t = b(X_t)dt + dZ_t, \qquad X_0 = x_0. \tag{1.2}$$

In the case that $Z_t$ is an $m$-dimensional standard Wiener noise and $b$ is a bounded measurable function, Veretennikov [18] first proved the existence of a unique strong solution for SDE (1.2). In the finite dimensional case, this result has been extended in various directions, see for example [7], [9], [8], [10], [12], [13]. Also Veretennikov's result has been extended to stochastic evolution equations in Hilbert and Banach spaces, see [2], [3], [4], [5]. Studying the pathwise

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uniqueness of SDE (1.2), when $Z_t$ is a symmetric $\alpha$-stable process or Poisson random measure, is faced with more difficulties. Tanaka et al. [17] analyzed the SDE (1.2) with a symmetric $\alpha$-stable noise and proved the existence of a unique strong solution for this equation in the case $m = 1$ and $\alpha \geq 1$ with bounded continuous drift $b$. Also they substantiated that when $m = 1$ and $\alpha + \beta < 1$, even a bounded and $\beta$-Hölder continuous function b is not enough for the pathwise uniqueness . On the other hand, when the parameters $\alpha, \beta$ satisfy $\beta > 1 - \frac{\alpha}{2}$ with $\alpha \in [1, 2)$, Priola et al. [15] demonstrated the pathwise uniqueness for SDE (1.2) when the function $b$ is bounded and $\beta$-Hölder continuous.

In the case of infinite dimensional case, Desheng Yang [19] proved the pathwise uniqueness of stochastic evolution equations in Hilbert spaces with $\alpha$-stable noise and bounded $\beta$-Hölder continuous drift term. Also, Sun et al. [16] proved the pathwise uniqueness of stochastic evolution equations in Hilbert spaces driven by cylindrical $\alpha$-stable process and $\beta$-Hölder continuous drift term.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space and $N(dt, dx)$ be a Poisson random measure on the Borel $\sigma$-algebra $\mathbb{B}(\mathbb{R}^+ \times \mathcal{H})$ with intensity measure $\nu(dx)dt$. This paper is designed to prove the pathwise uniqueness of the mild solution of the following stochastic differential equation,

$$dX_t = AX_t dt + B(t, X_t)dt + F(t, X_t)dt + dW_t + \int_{\mathcal{H}} x\overline{N}(dt, dx), \qquad X_0 = x \in \mathcal{H}, \tag{1.3}$$

where

- $(\mathcal{H}, \langle ., . \rangle, |.|)$ is a separable Hilbert space.
- $A : D(A) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal generator of a $C_0$-semigroup $(e^{tA})_{t\geq 0}$ of linear operators in Hilbert space $\mathcal{H}$.
- $B : [0, T] \times \mathcal{H} \longrightarrow \mathcal{H}$ is Hölder continuous in $x$ uniformly in $t$ and $F : [0, T] \times \mathcal{H} \longrightarrow \mathcal{H}$ is locally Lipschitz continuous.
- $W_t$ is a Wiener process in Hilbert space $\mathcal{H}$ with covariance operator $Q$ and $\overline{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$ denotes the compensated Poisson random measure corresponding to $N(dt, dx)$ and it is independent of $W_t$. For more details about Poisson random measure and Wiener process in Hilbert spaces and definition and properties of integral with respect to $W_t$ and $\overline{N}(dt, dx)$, consult [14], [6].

As far as we know, little attention has been paid to the pathwise uniqueness for stochastic evolution equations in Hilbert spaces driven by both Wiener process and Poisson random measure with Hölder continuous drift. The main technical ingredient of this study is a regularity result for a nonhomogeneous infinite dimensional Kolmogorov equations.

This paper is organized as follows. The second section provides an explanation for some notations, assumptions and results used throughout this paper and in Section 3, we go through the regularity results of the Ornstein-Ulenbeck semigroup and the associated Kolmogorov equation. Also we focus on the proof of the main theorem and corroborate our main claim by an example.

## 2 Preliminaries

We are given a separable Hilbert space $\mathcal{H}$ (equipped with inner product $\langle ., . \rangle$ and norm $|.|$). In the sequel, the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ will be denoted by $L(\mathcal{H}, \mathcal{H})$, also $\|.\|_{L(\mathcal{H},\mathcal{H})}$ and $\|.\|_{L(\mathcal{H},L(\mathcal{H},\mathcal{H}))}$ always denote the usual operator norm of a linear bounded operator from $\mathcal{H}$ into $\mathcal{H}$ and $L(\mathcal{H}, \mathcal{H})$, respectively. On the other hand, $\|.\|_{HS}$ indicates the Hilbert–Schmidt norm (cf. [6], chapter 4).

For $\alpha, T > 0$, the space $C\big([0, T]; C_b^\alpha(\mathcal{H}, \mathcal{H})\big)$ stands for all continuous bounded functions $f : [0, T] \times \mathcal{H} \longrightarrow \mathcal{H}$ for which there exists $C > 0$ such that:

$$|f(t, x) - f(t, y)| \leq C|x - y|^\alpha, \qquad x, y \in \mathcal{H}, t \in [0, T],$$

equipped with the following norm

$$\|f\|_\alpha = \sup_{t\in[0,T],x\in\mathcal{H}} |f(t, x)| + \sup_{t\in[0,T]} \sup_{x\neq y\in\mathcal{H}} \frac{|f(t, x) - f(t, y)|}{|x - y|^\alpha},$$

and $Lip(\mathcal{H},\mathcal{H})$ stands for the usual space of Lipschitz continuous functions in $\mathcal{H}$. Throughout this paper, we fix a complete orthonormal system $\{e_n\}_{n\geq 1}$ for Hilbert space $\mathcal{H}$. Also, we shall denote the components of any $\mathcal{H}$-valued function $\varphi$ with respect to $\{e_n\}_{n\geq 1}$ by $\varphi_n$, i.e., $\varphi_n(x) = \langle \varphi(x), e_n\rangle$.

Now we list the assumptions used throughout this paper

$\mathbb{H}1$. $A$ is self adjoint and $Ae_n = -\alpha_n e_n$ with $\alpha_n > 0$, $\alpha_n \uparrow \infty$,

$\mathbb{H}2$. $\int_{\mathcal{H}} |x|^2 \nu(dx) < \infty$,

$\mathbb{H}3$. $\sum_{n=1}^{\infty} \frac{\|B_n\|_\alpha^2}{\alpha_n} < \infty$,

$\mathbb{H}4$. the operator $Q_t = \int_0^t e^{sA} Q e^{sA^\star} ds$ is trace class,

$\mathbb{H}5$. $e^{tA}(\mathcal{H}) \subseteq Q_t^{\frac{1}{2}}(\mathcal{H})$ for all $t > 0$,

$\mathbb{H}6$. the bounded operator $\Lambda_t = Q_t^{\frac{-1}{2}} e^{tA}$ satisfies $\int_0^T \|\Lambda_t\|_{L(\mathcal{H},\mathcal{H})}^{1+\theta} dt < \infty$ for some $\theta \geq \max\,(\alpha, 1-\alpha)$, where by $Q_t^{\frac{-1}{2}}$ we denote the pseudo-inverse of $Q_t^{\frac{1}{2}}$,

$\mathbb{H}7$. $B \in C\big([0,T]; C_b^\alpha(\mathcal{H},\mathcal{H})\big)$ and $F \in C\big([0,T]; Lip(\mathcal{H},\mathcal{H})\big)$ for some $\alpha, T > 0$.

In this part, we introduce Ornstein-Ulenbeck process and provide some results which play an important role in the proof of our main results. Firstly, let us define some of the spaces we use throughout this paper

(i) $B_b(\mathcal{H})$ (resp. $B_b(\mathcal{H},\mathcal{H})$) denotes the space of all bounded real-valued (resp. $\mathcal{H}$-valued) functions on $\mathcal{H}$.

(ii) $UC_b(\mathcal{H},\mathcal{H})$ stands for the space of all uniformly continuous and bounded functions from $\mathcal{H}$ into $\mathcal{H}$.

(iii) $UC_b^2(\mathcal{H},\mathcal{H})$ is the space of all functions $f : \mathcal{H} \longrightarrow \mathcal{H}$ which are twice Fréchet differentiable on $\mathcal{H}$ with a uniformly continuous and bounded second derivative $D^2 f$.

We consider the following equation:

$$dZ_t^x = AZ_t^x dt + dW_t + \int_{\mathcal{H}} y\overline{N}(dt,dy), \qquad Z_0^x = x \in \mathcal{H}. \tag{2.1}$$

This equation has a unique mild solution for any initial value $x \in \mathcal{H}$ as follows

$$Z_t^x = e^{tA}x + \int_0^t e^{(t-s)A} dW_s + \int_0^t \int_{\mathcal{H}} e^{(t-s)A} y\overline{N}(ds,dy).$$

The solution $Z_t^x$ is called the Ornstein-Ulenbeck process. The process $W_A(t) = e^{tA}x + \int_0^t e^{(t-s)A} dW_s$ is Gaussian with mean $e^{tA}x$ and covariance operator $Q_t = \int_0^t e^{sA} Q e^{sA^\star} ds$, see [6] for more details . We denote the law of $W_A(t)$ by $N_{e^{tA}x,Q_t}$. Suppose that $R_t : B_b(\mathcal{H}) \longrightarrow B_b(\mathcal{H})$ is the corresponding semigroup defined as $R_t\varphi(x) = E[\varphi(Z_t^x)]$, then we have

$$R_t\varphi(x) = E[\varphi(Z_t^x)] = \int_{\mathcal{H}} \int_{\mathcal{H}} \varphi(y+z) N_{e^{tA}x,Q_t}(dy) \mu_t^x(dz),$$

where $\mu_t^x$ is the law of

$$Y_t^x = \int_0^t \int_{\mathcal{H}} e^{(t-s)A} y\overline{N}(ds,dy).$$

Let us introduce the analogous semigroup on $\mathcal{H}$-valued functions as follows:

$$\overline{R}_t\varphi(x) = E[\varphi(Z_t^x)], \qquad \varphi \in B_b(\mathcal{H},\mathcal{H}).$$

Clearly, we have

$$\langle \overline{R}_t\varphi(x), g\rangle = R_t\varphi_g(x), \qquad \varphi_g(x) = \langle \varphi(x), g\rangle,\ g \in \mathcal{H}.$$

The following theorem is argued in [3].

**Theorem 2.1.** Let $R_t^0$ be the semigroup defined on $B_b(\mathcal{H},\mathcal{H})$ as $R_t^0\varphi(x) = \int_{\mathcal{H}} \varphi(e^{tA}x+y)N_{Q_t}(dy)$, then by considering the assumption $\mathbb{H}5$, we get

$$\varphi \in UC_b(\mathcal{H},\mathcal{H}) \Longrightarrow R_t^0\varphi \in UC_b^2(\mathcal{H},\mathcal{H}),$$

for every $t > 0$.

Furthermore, the differential $DR_t^0\varphi(x) \in L(\mathcal{H},\mathcal{H})$ at each fixed $x \in \mathcal{H}$ is the linear operator given by

$$DR_t^0\varphi(x)g = \int_{\mathcal{H}} \langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle \varphi(e^{tA}x+y)N_{Q_t}(dy), \qquad g \in \mathcal{H},$$

and the second order derivative $D^2R_t^0\varphi(x) \in L\big(\mathcal{H}, L(\mathcal{H},\mathcal{H})\big)$ is as follows

$$[D^2R_t^0\varphi(x)g]k = \int_{\mathcal{H}} [\langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle\langle \Lambda_t k, Q_t^{\frac{-1}{2}} y\rangle - \langle \Lambda_t g, \Lambda_t k\rangle]\varphi(e^{tA}x+y)N_{Q_t}(dy).$$

Finally,

$$\begin{aligned} &\|DR_t^0\varphi(x)\|_{L(\mathcal{H},\mathcal{H})} \leq \|\Lambda_t\|_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty, \\ &\|D^2R_t^0\varphi(x)\|_{L(\mathcal{H},L(\mathcal{H},\mathcal{H}))} \leq \sqrt{2}\|\Lambda_t\|^2_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty, \end{aligned}$$

where $\|\varphi\|_\infty = \sup_{x\in\mathcal{H}} |\varphi(x)|$.

As a consequence of this theorem, we prove the following lemma.

**Lemma 2.2.** Let assumption $\mathbb{H}5$ hold. Then for every $t > 0$ and $\varphi \in UC_b(\mathcal{H},\mathcal{H})$, we have $\overline{R}_t\varphi \in UC_b^2(\mathcal{H},\mathcal{H})$, moreover for each $t > 0$ and $g,k \in \mathcal{H}$, we get

$$D\overline{R_t}\varphi(x)g = \int_{\mathcal{H}}\int_{\mathcal{H}} \langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle \varphi(e^{tA}x+y+z)N_{Q_t}(dy)\mu_t(dz),$$

and

$$[D^2\overline{R_t}\varphi(x)g]k = \int_{\mathcal{H}}\int_{\mathcal{H}} [\langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle\langle \Lambda_t k, Q_t^{\frac{-1}{2}} y\rangle - \langle \Lambda_t g, \Lambda_t k\rangle]\varphi(e^{tA}x+y+z)N_{Q_t}(dy)\mu_t(dz).$$

Finally, we have

$$\begin{aligned} &\|D\overline{R_t}\varphi(x)\|_{L(\mathcal{H},\mathcal{H})} \leq \|\Lambda_t\|_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty, \\ &\|D^2\overline{R_t}\varphi(x)\|_{L(\mathcal{H},L(\mathcal{H},\mathcal{H}))} \leq \sqrt{2}\|\Lambda_t\|^2_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty. \end{aligned}$$

**Proof .** The validity of part 1 and 2 is obtained from the preceding theorem, so it suffices to prove the last part. We have

$$\begin{aligned} |D\overline{R_t}\varphi(x)g| &\leq \int_{\mathcal{H}} \left| \int_{\mathcal{H}} \langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle \varphi(e^{tA}x+y+z)N_{Q_t}(dy)\right| \mu_t(dz) \\ &\leq \int_{\mathcal{H}} \|\Lambda_t\|_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty |g|\mu_t(dz) = \|\Lambda_t\|_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty|g|, \end{aligned}$$

and

$$\begin{aligned} \big|[D^2\overline{R_t}\varphi(x)g]k\big| &\leq \int_{\mathcal{H}} \left| \int_{\mathcal{H}} [\langle \Lambda_t g, Q_t^{\frac{-1}{2}} y\rangle\langle \Lambda_t k, Q_t^{\frac{-1}{2}} y\rangle - \langle \Lambda_t g, \Lambda_t k\rangle]\varphi(e^{tA}x+y+z)N_{Q_t}(dy)\right| \mu_t(dz) \\ &\leq \sqrt{2}\int_{\mathcal{H}} \|\Lambda_t\|^2_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty|g||k|\mu_t(dz) = \sqrt{2}\|\Lambda_t\|^2_{L(\mathcal{H},\mathcal{H})}\|\varphi\|_\infty|g||k|. \end{aligned}$$

□

## 3 Main Results

At first, we consider the sequence of the following backward equations of Kolmogorov type on $[0,T]$ with values in $\mathcal{H}$:

$$\begin{aligned}\frac{\partial V_n}{\partial t}(t,x) + \langle Ax, DV_n(t,x)\rangle + \langle B(t,x), DV_n(t,x)\rangle + \frac{1}{2}Tr\big(QD^2V_n(t,x)\big)\\ + \int_{\mathcal{H}} \big(V_n(t,x+y) - V_n(t,x) - \langle DV_n(t,x), y\rangle\big)\nu(dy) = B_n(t,x),\\ V_n(T,x) = 0,\end{aligned} \tag{3.1}$$

where $B_n$ denotes the $n$th component of $B$. In the following lemma, we prove that equation (3.1) has a unique regular solution for each $n \geq 1$ and we also analyze the properties of the $\mathcal{H}$-valued function $V(t,x) = \sum_{n=1}^{\infty} V_n(t,x)e_n$.

**Lemma 3.1.** Let the assumptions $\mathbb{H}1$–$\mathbb{H}7$ hold. Then for small enough $T$, equation (3.1) has a unique solution for each $n \geq 1$ such that $V \in C\big([0,T]; UC_b^2(\mathcal{H},\mathcal{H})\big)$. If we set $K_T = \|DV\|_\infty$, then $\lim_{T\to 0} K_T = 0$. Furthermore, for some constant $C_T > 0$, we have $\|D^2V_n\|_\infty \leq C_T\|B_n\|_\alpha$ for every $n \in \mathbb{N}$.

**Proof .** For the sake of simplicity, we use the following forward notations for the PDE. The final result will apply to the backward PDE (3.1).

$$\begin{aligned}\frac{\partial V_n}{\partial t}(t,x) = \langle Ax, DV_n(t,x)\rangle + \frac{1}{2}Tr\big(QD^2V_n(t,x)\big) +\\ \int_{\mathcal{H}} \big(V_n(t,x+y) - V_n(t,x) - \langle DV_n(t,x), y\rangle\big)\nu(dy) + \langle B(t,x), DV_n(t,x)\rangle + B_n(t,x),\\ V_n(0,x) = 0.\end{aligned} \tag{3.2}$$

For any $\varphi$ of class $C^2$, applying the Itô's formula to the process $\big(\varphi(Z_t^x)\big)_{t\geq 0}$, we have

$$\begin{aligned} d\,\varphi(Z_t^x) &= \langle D\,\varphi(Z_{t^-}^x), d\,Z_t^x\rangle + \frac{1}{2}Tr\left[Q\,D^2\,\varphi(Z_{t^-}^x)\right]dt\\ &\quad + \int_{\mathcal{H}} \big(\,\varphi(Z_{t^-}^x + y) - \varphi(Z_{t^-}^x) - \langle D\,\varphi(Z_{t^-}^x), y\rangle\,\big)N(dt,dy)\\ &= \langle D\,\varphi(Z_{t^-}^x), A\,Z_t^x\rangle\,dt + \langle D\,\varphi(Z_{t^-}^x), dW_t\rangle\\ &\quad + \langle D\,\varphi(Z_{t^-}^x), \int_{\mathcal{H}} y\overline{N}(dt,dy)\rangle + \frac{1}{2}Tr\left[Q\,D^2\,\varphi(Z_{t^-}^x)\right]dt\\ &\quad + \int_{\mathcal{H}} \big(\,\varphi(Z_{t^-}^x + y) - \varphi(Z_{t^-}^x) - \langle D\,\varphi(Z_{t^-}^x), y\rangle\,\big)N(dt,dy).\end{aligned}$$

Consequently

$$\begin{aligned} R_t\varphi(x) = E\left[\varphi(Z_t^x)\right] &= \varphi(x) + E\int_0^t \langle D\varphi(Z_{r^-}^x), A\,Z_r^x\rangle\,dr\\ &\quad + \frac{1}{2}E\int_0^t Tr\left[Q\,D^2\,\varphi(Z_{r^-}^x)\right]dr\\ &\quad + E\int_0^t\int_{\mathcal{H}} \big(\varphi(Z_{r^-}^x + y) - \varphi(Z_{r^-}^x) - \langle\varphi(Z_{r^-}^x), y\rangle\,\big)\nu(dy)dr.\end{aligned}$$

By the dominated convergence theorem, we have

$$\begin{aligned}\Lambda\varphi(x) = \lim_{t\downarrow 0}\frac{R_t\varphi(x) - \varphi(x)}{t} &= \langle Ax, D\varphi(x)\rangle + \frac{1}{2}Tr\left[Q\,D^2\,\varphi(x)\right]\\ &\quad + \int_{\mathcal{H}} \big(\varphi(x+y) - \varphi(x) - \langle D\varphi(x), y\rangle\,\big)\nu(dy),\end{aligned}$$

which $\Lambda$ denotes the infinitesimal generator of semigroup $R_t$. Therefore, equation (3.2) is equivalent to the following equation for each fixed $t \geq 0$.

$$\frac{\partial V_n}{\partial t}(t,x) = \Lambda V_n(t,x) + \langle B(t,x), DV_n(t,x)\rangle + B_n(t,x),$$
$$V_n(0,x) = 0. \tag{3.3}$$

By writing the PDE (3.3) in convolution form, we get

$$V_n(t,x) = \int_0^t R_{t-s}\big(\langle B(s), DV_n(s)\rangle + B_n(s)\big)(x)ds,$$

where we have used $B(s)$ instead of $B(s,.)$ for simplicity and so on. Notice that for a fixed $t \geq 0$, the Fréchet derivative of $V(t,.)$ at each $x \in \mathcal{H}$ is a Linear operator on $\mathcal{H}$ and for every $p \in \mathcal{H}$,

$$\langle DV(t,x)p, e_n\rangle = \langle DV_n(t,x), p\rangle.$$

Therefore, we have

$$V_n(t,x) = \int_0^t R_{t-s}\big(\langle DV(s)B(s) + B(s), e_n\rangle\big)(x)ds,$$

and finally we get the following $\mathcal{H}$-valued equation

$$V(t,x) = \int_0^t \overline{R}_{t-s}\big(\langle B(s), D\rangle V(s) + B(s)\big)(x)ds, \tag{3.4}$$

where $V(t,x) = \sum_{n=1}^{\infty} V_n(t,x)e_n$ and we have denoted $\sum_{n=1}^{\infty} \langle B(s), DV_n(s)\rangle\, e_n$ by $\langle B(s), D\rangle V(s)$. Finally, according to Lemma 2.2, the semigroups $R_t^0$ and $\overline{R}_t$ have the same regularity properties, so by considering Theorem 5 in [3], equation (3.4) has a unique solution with the mentioned properties in the theorem and this completes the proof. □

Now, let $V$ be the solution of the equation (3.4), under assumptions $\mathbb{H}1$–$\mathbb{H}7$ the following Zvonkin's transformation holds.

**Theorem 3.2.** $X_t$ is a mild solution of equation (1.3) if and only if it satisfies the following equation

$$\begin{aligned} X_t = {} & e^{tA}\big(x - V(0,x)\big) + V(t,X_t) + \int_0^t Ae^{(t-s)A}V(s,X_s)ds \\ & - \int_0^t e^{(t-s)A}DV(s,X_s)F(s,X_s)ds + \int_0^t e^{(t-s)A}F(s,X_s)ds \\ & - \int_0^t e^{(t-s)A}DV(s,X_{s^-})dW_s + \int_0^t e^{(t-s)A}dW_s \\ & + \int_0^t\int_{\mathcal{H}} e^{(t-s)A}\big(V(s,X_{s^-}+y) - V(s,X_{s^-})\big)\overline{N}(ds,dy) \\ & + \int_0^t\int_{\mathcal{H}} e^{(t-s)A}y\overline{N}(ds,dy). \end{aligned} \tag{3.5}$$

**Proof .** According to Lemma 3.1, $V_n$ is smooth enough for each $n \in \mathbb{N}$, so by using Itô's formula, we have

$$\begin{aligned} dV_n(t,X_t) = {} & \frac{\partial V_n}{\partial t}(t,X_t)dt + \langle DV_n(t,X_{t^-}), dX_t\rangle + \frac{1}{2}Tr\big(QD^2V_n(t,X_t)\big)dt \\ & + \int_{\mathcal{H}}\big(V_n(t,X_{t^-}+y) - V_n(t,X_{t^-}) - \langle DV_n(t,X_{t^-}), y\rangle\big)N(dt,dy). \end{aligned} \tag{3.6}$$

By substituting $X_{t^-}$ with $x$ in equation (3.1) and then substituting the value of $\frac{\partial V_n}{\partial t}(t,x)$ in equation (3.6), we arrive at

$$
\begin{aligned}
&dV_n(t,X_t) = B_n(t,X_t)dt - \langle AX_t, DV_n(t,X_t)\rangle dt - \langle B(t,X_t), DV_n(t,X_t)\rangle dt \\
&- \frac{1}{2}Tr\big(QD^2V_n(t,X_t)\big)dt - \int_{\mathcal{H}} \big(V_n(t,X_{t^-}+y) - V_n(t,X_{t^-}) - \langle DV_n(t,X_{t^-}),y\rangle\big)\nu(dy)dt \\
&+ \langle DV_n(t,X_t), AX_t\rangle dt + \langle DV_n(t,X_t), B(t,X_t)\rangle dt + \langle DV_n(t,X_t), F(t,X_t)\rangle dt \\
&+ \langle DV_n(t,X_{t^-}), dW_t\rangle + \int_{\mathcal{H}} \langle DV_n(t,X_{t^-}),y\rangle \overline{N}(dt,dy) \\
&+ \frac{1}{2}Tr\big(QD^2V_n(t,X_t)\big)dt + \int_{\mathcal{H}} \big(V_n(t,X_{t^-}+y) - V_n(t,X_{t^-}) - \langle DV_n(t,X_{t^-}),y\rangle\big)N(dt,dy).
\end{aligned}
$$

After some simplification, we get

$$
\begin{aligned}
dV_n(t,X_t) &= B_n(t,X_t)dt + \langle DV_n(t,X_t), F(t,X_t)\rangle dt + \langle DV_n(t,X_{t^-}), dW_t\rangle \\
&\quad + \int_{\mathcal{H}} \big(V_n(t,X_{t^-}+y) - V_n(t,X_{t^-})\big)\overline{N}(dt,dy) \\
&= \big\langle B(t,X_t)dt + DV(t,X_t)F(t,X_t)dt + DV(t,X_{t^-})dW_t \\
&\quad + \int_{\mathcal{H}} \big(V(t,X_{t^-}+y) - V(t,X_{t^-})\big)\overline{N}(dt,dy), e_n\big\rangle.
\end{aligned}
$$

Namely

$$
\begin{aligned}
dV(t,X_t) &= B(t,X_t)dt + DV(t,X_t)F(t,X_t)dt + DV(t,X_{t^-})dW_t \\
&\quad + \int_{\mathcal{H}} \big(V(t,X_{t^-}+y) - V(t,X_{t^-})\big)\overline{N}(dt,dy),
\end{aligned}
\tag{3.7}
$$

where $V(t,x) = \sum_{n=1}^{\infty} V_n(t,x)e_n$. By substituting the value of $B(t,X_t)dt$ from equation (3.7) in equation (1.3), we get

$$
\begin{aligned}
dX_t &= AX_tdt + dV(t,X_t) - DV(t,X_t)F(t,X_t)dt + F(t,X_t)dt \\
&\quad - \int_{\mathcal{H}} \big(V(t,X_{t^-}+y) - V(t,X_{t^-})\big)\overline{N}(dt,dy) \\
&\quad - DV(t,X_{t^-})dW_t + dW_t + \int_{\mathcal{H}} y\overline{N}(dt,dy).
\end{aligned}
$$

From the usual variation of constant method, we get

$$
\begin{aligned}
X_t &= e^{tA}x + \int_0^t e^{(t-s)A}dV(s,X_s) - \int_0^t e^{(t-s)A}DV(s,X_s)F(s,X_s)ds \\
&\quad + \int_0^t e^{(t-s)A}F(s,X_s)ds - \int_0^t e^{(t-s)A}DV(s,X_{s^-})dW_s + \int_0^t e^{(t-s)A}dW_s \\
&\quad + \int_0^t \int_{\mathcal{H}} e^{(t-s)A}\big(V(s,X_{s^-}+y) - V(s,X_{s^-})\big)\overline{N}(ds,dy) \\
&\quad + \int_0^t \int_{\mathcal{H}} e^{(t-s)A}y\overline{N}(ds,dy).
\end{aligned}
\tag{3.8}
$$

Using integration by parts in the first integral of the equation above, we have

$$
\int_0^t e^{(t-s)A}dV(s,X_s) = e^{tA}V(0,x) + V(t,X_t) + \int_0^t Ae^{(t-s)A}V(s,X_s)ds.
$$

We put this formula in equation (3.8) and finally get

$$
\begin{aligned}
X_t = e^{tA}\big(x - V(0,x)\big) + V(t,X_t) + \int_0^t Ae^{(t-s)A}V(s,X_s)ds \\
- \int_0^t e^{(t-s)A}DV(s,X_s)F(s,X_s)ds + \int_0^t e^{(t-s)A}F(s,X_s)ds \\
- \int_0^t e^{(t-s)A}DV(s,X_{s^-})dW_s + \int_0^t e^{(t-s)A}dW_s \\
+ \int_0^t \int_{\mathcal{H}} e^{(t-s)A}\big(V(s,X_{s^-}+y) - V(s,X_{s^-})\big)\overline{N}(ds,dy) \\
+ \int_0^t \int_{\mathcal{H}} e^{(t-s)A}y\overline{N}(ds,dy).
\end{aligned}
\tag{3.9}
$$

□

We are now in a position to prove our main theorem.

**Theorem 3.3.** Assume that assumptions $\mathbb{H}1$–$\mathbb{H}7$ hold, then for sufficiently small $T$, pathwise uniqueness holds for mild solution of equation (1.3) on $[0,T]$.

**Proof .** Let $X_t^1$ and $X_t^2$ be two solutions of equation (1.3), both starting at $x \in \mathcal{H}$. By Theorem 3.2, the difference $Y_t = X_t^1 - X_t^2$ satisfies

$$
\begin{aligned}
Y_t = V(t,X_t^1) - V(t,X_t^2) + \int_0^t Ae^{(t-s)A}[V(s,X_s^1) - V(s,X_s^2)]ds \\
- \int_0^t e^{(t-s)A}[DV(s,X_s^1)F(s,X_s^1) - DV(s,X_s^2)F(s,X_s^2)]ds \\
+ \int_0^t e^{(t-s)A}[F(s,X_s^1) - F(s,X_s^2)]ds - \int_0^t e^{(t-s)A}[DV(s,X_{s^-}^1) - DV(s,X_{s^-}^2)]dW_s \\
+ \int_0^t \int_{\mathcal{H}} e^{(t-s)A}\big[V(s,X_{s^-}^1+y) - V(s,X_{s^-}^2+y) - V(s,X_{s^-}^1) + V(s,X_{s^-}^2)\big]\overline{N}(ds,dy),
\end{aligned}
$$

we claim that $E(\int_0^T |Y_t|^2 dt) = 0$. Let us fix $T$ and choose its value at the end. By Lemma 3.1,

$$
\begin{aligned}
|V(t,X_t^1) - V(t,X_t^2)| \le K_T|X_t^1 - X_t^2|, \qquad t \in [0,T], \\
\|DV(t,X_t^1) - DV(t,X_t^2)\|_{L(\mathcal{H},\mathcal{H})} \le C_T|X_t^1 - X_t^2|, \qquad t \in [0,T].
\end{aligned}
\tag{3.10}
$$

By maximal inequality, we have

$$
\|\int_0^{\cdot} Ae^{(\cdot-s)A}f(s)ds\|^2_{L^2([0,T];\mathcal{H})} \le M_T\|f\|^2_{L^2([0,T];\mathcal{H})},
\tag{3.11}
$$

in which $C_T$ is independent of $f$. Note that $C_T$ does not go to zero as $T \longrightarrow 0$. Applying the following inequality

$$
(x_1 + ... + x_n)^2 \le n(x_1^2 + ... + x_n^2),
$$

we get the following inequality

$$\int_0^T |Y_t|^2 dt \le 6\int_0^T |V(t,X_t^1)-V(t,X_t^2)|^2 dt + 6\int_0^T \left|\int_0^t Ae^{(t-s)A}[V(s,X_s^1)-V(s,X_s^2)]ds\right|^2 dt$$
$$-6\int_0^T \left|\int_0^t e^{(t-s)A}[DV(s,X_s^1).F(s,X_s^1)-DV(s,X_s^2).F(s,X_s^2)]ds\right|^2 dt$$
$$+6\int_0^T \left|\int_0^t e^{(t-s)A}[F(s,X_s^1)-F(s,X_s^2)]ds\right|^2 dt$$
$$+6\int_0^T \left|\int_0^t e^{(t-s)A}[DV(s,X_{s^-}^1)-DV(s,X_{s^-}^2)]dW_s\right|^2 dt$$
$$+6\int_0^T \left|\int_0^t \int_{\mathcal{H}} e^{(t-s)A}[V(s,X_{s^-}^1+y)-V(s,X_{s^-}^2+y)-V(s,X_{s^-}^1)+V(s,X_{s^-}^2)]\overline{N}(ds,dy)\right|^2 dt,$$

by inequalities (3.10) and (3.11), we get

$$\int_0^T |Y_t|^2 dt \le 6K_T^2\int_0^T |Y_t|^2 dt + 6M_TK_T\int_0^T |Y_t|^2 dt + 6TC_T'\int_0^T |Y_t|^2 dt$$
$$+6TK_T\int_0^T |Y_t|^2 dt + 6\int_0^T \left|\int_0^t e^{(t-s)A}[DV(s,X_{s^-}^1)-DV(s,X_{s^-}^2)]dW_s\right|^2 dt$$
$$+6\int_0^T \left|\int_0^t \int_{\mathcal{H}} e^{(t-s)A}[V(s,X_{s^-}^1+y)-V(s,X_{s^-}^2+y)-V(s,X_{s^-}^1)+V(s,X_{s^-}^2)]\overline{N}(ds,dy)\right|^2 dt,$$

since by Lemma 3.1, $\lim_{T\to 0} K_T = 0$, for small enough $T$, we have

$$6K_T^2\int_0^T |Y_t|^2 dt + 6M_TK_T\int_0^T |Y_t|^2 dt + 6TC_T'\int_0^T |Y_t|^2 dt + 6TK_T\int_0^T |Y_t|^2 dt \le \frac{1}{2}\int_0^T |Y_t|^2 dt,$$

thus we get

$$\int_0^T |Y_t|^2 dt \le 12\int_0^T \left|\int_0^t e^{(t-s)A}[DV(s,X_{s^-}^1)-DV(s,X_{s^-}^2)]dW_s\right|^2 dt$$
$$+12\int_0^T \left|\int_0^t \int_{\mathcal{H}} e^{(t-s)A}[V(s,X_{s^-}^1+y)-V(s,X_{s^-}^2+y)-V(s,X_{s^-}^1)+V(s,X_{s^-}^2)]\overline{N}(ds,dy)\right|^2 dt,$$

now taking expectation from both sides yields

$$\int_0^T E|Y_t|^2 dt \le 12\int_0^T E\left|\int_0^t e^{(t-s)A}[DV(s,X_{s^-}^1)-DV(s,X_{s^-}^2)]dW_s\right|^2 dt \tag{3.12}$$
$$+12\int_0^T E\left|\int_0^t \int_{\mathcal{H}} e^{(t-s)A}[V(s,X_{s^-}^1+y)-V(s,X_{s^-}^2+y)-V(s,X_{s^-}^1)+V(s,X_{s^-}^2)]\overline{N}(ds,dy)\right|^2 dt,$$

also we have

$$\begin{aligned}
&\left\|e^{(t-s)A}[DV(s,X^1_{s^-})-DV(s,X^2_{s^-})]\sqrt{Q}\right\|^2_{HS}\\
&=\sum_{n,m=1}^{\infty}\left\langle e^{(t-s)A}\left(DV(s,X^1_{s^-})-DV(s,X^2_{s^-})\right)\sqrt{Q}\,e_m,e_n\right\rangle^2\\
&=\sum_{n,m=1}^{\infty}e^{-2\,\alpha_n(t-s)}\left\langle\left(DV(s,X^1_{s^-})-DV(s,X^2_{s^-})\right)\sqrt{Q}\,e_m,e_n\right\rangle^2\\
&=\sum_{n,m=1}^{\infty}e^{-2\,\alpha_n(t-s)}\left\langle\left(DV_n(s,X^1_{s^-})-DV_n(s,X^2_{s^-})\right),\sqrt{Q}e_m\right\rangle^2\\
&=\sum_{n=1}^{\infty}e^{-2\,\alpha_n(t-s)}\sum_{m=1}^{\infty}\left\langle\sqrt{Q}\big(DV_n(s,X^1_{s^-})-DV_n(s,X^2_{s^-})\big),e_m\right\rangle^2\\
&\le\|Q\|_{L(\mathcal{H},\mathcal{H})}\sum_{n=1}^{\infty}e^{-2\,\alpha_n(t-s)}\left|DV_n(s,X^1_{s^-})-DV_n(s,X^2_{s^-})\right|^2\\
&\le\|Q\|_{L(\mathcal{H},\mathcal{H})}\sum_{n=1}^{\infty}e^{-2\,\alpha_n(t-s)}\left\|D^2V_n\right\|^2_{\infty}\left|X^1_{s^-}-X^2_{s^-}\right|^2.
\end{aligned}$$

And by Lemma 3.1, we know that

$$\|D^2V_n\|_\infty\le C_T\|B_n\|_\alpha,$$

hence

$$\begin{aligned}
&\int_0^T E\left|\int_0^t e^{(t-s)A}[DV(s,X^1_{s^-})-DV(s,X^2_{s^-})]dW_s\right|^2dt\\
&\qquad\le C_T^2\|Q\|_{L(\mathcal{H},\mathcal{H})}\big(\int_0^T\sum_{n=1}^{\infty}e^{-2\alpha_n t}\|B_n\|^2_\alpha dt\big)\int_0^T|Y_s|^2ds,
\end{aligned}$$

by assumption $\mathbb{H}3$, $\int_0^T\sum_{n=1}^\infty e^{-2\alpha_n t}\|B_n\|^2_\alpha dt$ is finite and goes to zero when $T\to 0$. Hence the inequality (3.12) for small $T$ becomes

$$\int_0^T E|Y_t|^2dt\le 24\int_0^T E\left|\int_0^t\int_{\mathcal{H}}e^{(t-s)A}[V(s,X^1_{s^-}+y)-V(s,X^2_{s^-}+y)-V(s,X^1_{s^-})+V(s,X^2_{s^-})]\overline{N}(ds,dy)\right|^2dt. \quad (3.13)$$

By the following inequality

$$\begin{aligned}
&\left|V(s,X^1_{s^-}+y)-V(s,X^2_{s^-}+y)-V(s,X^1_{s^-})-V(s,X^2_{s^-})\right|\\
&\qquad=\left|\int_0^1\big(DV(s,X^1_{s^-}+zy)-DV(s,X^2_{s^-}+zy)\big)dz\right|\\
&\qquad\le\int_0^1 K_T|y||X^1_{s^-}-X^2_{s^-}|dz=K_T|y||X^1_{s^-}-X^2_{s^-}|,
\end{aligned}$$

and inequality (3.13), we get

$$\begin{aligned}
&\int_0^T E|Y_t|^2dt\le\\
&\quad 24\int_0^T\big(\int_0^t\int_{\mathcal{H}}E\left|e^{(t-s)A}[V(s,X^1_{s^-}+y)-V(s,X^2_{s^-}+y)-V(s,X^1_{s^-})+V(s,X^2_{s^-})]\right|^2\nu(dy)ds\big)dt\\
&\le 24K_T^2\int_0^T\big(\int_0^t\int_{\mathcal{H}}E|X^1_{s^-}-X^2_{s^-}|^2|y|^2\nu(dy)ds\big)dt\\
&\le 24TK_T^2\int_{\mathcal{H}}|y|^2\nu(dy)\int_0^T E|Y_t|^2dt,
\end{aligned}$$

by the assumption $\mathbb{H}2$, $\int_{\mathcal{H}}|y|^2\nu(dy)$ is finite. Hence, $24TK_T^2\int_{\mathcal{H}}|y|^2\nu(dy)\to 0$ as $T\to 0$ and this gives $E(\int_0^T|Y_t|^2dt)=0$. This implies $X_t^1=X_t^2$. The proof is completed. □

## 4 An illustrated example

In this section, motivated by [3] and [11], we provide an example of equation (1.3) to illustrate an application of our main theorem.

Let $\mathcal{H} = [L^2([0,2\pi]^d)]^m$, endowed with the usual norm $\|.\|$ on this space and let $(b_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Consider the following vector-valued equation for the unknown $X = (X_1, ..., X_m)$,

$$dX(t,\eta) = \big((\Delta^\beta + I)X(t,\eta) + B\big(X(t,.)(\eta)\big)\big)dt + \Delta^{\frac{-\gamma}{2}} dW(t,\eta) + \Delta^{\frac{-\rho}{2}} \int_{\mathbb{R}^m} \sigma\tilde{\pi}(ds, d\eta, d\sigma), \tag{4.1}$$

for $t \geq 0$, $\eta \in [0,2\pi]^d$ and $\sigma \in \mathbb{R}^m$, with periodic boundary conditions, where for a fixed $R > 0$, the function $B$ is defined as follows:

$$B(f)(\eta) = g(\eta) \int_{[0,2\pi]^d} \sqrt[3]{|f(x)| \wedge R}\,dx, \tag{4.2}$$

for all $f = (f_1, ..., f_m) \in \mathcal{H}$ and given $g \in [L^\infty([0,2\pi]^d)]^m$.

Here $W = (W_1, ..., W_m)$ is a space-time white noise with values in $\mathbb{R}^m$, $0 \leq \gamma < \frac{1}{3}$, $\rho > 2d$ and $\beta > \frac{d(1+\gamma)}{2}$. Moreover, $\Delta^\beta$, $\Delta^{\frac{-\gamma}{2}}$ and $\Delta^{\frac{-\rho}{2}}$ are pseudodifferential operators, acting componentwise [$\Delta^\beta X = (\Delta^\beta X_1, ..., \Delta^\beta X_m)$ and so on]. $\tilde{\pi}$ is a compensated Poisson random measure on $(0,\infty] \times [0,2\pi]^d \times \mathbb{R}^m$ with intensity measure $\lambda^1 \otimes \lambda^d \otimes \mu$ where $\mu$ is a Lévy measure on $\mathbb{R}^m$ satisfying $\int_{\mathbb{R}^m} \sigma^2 \mu(d\sigma) < \infty$ and $\lambda^d$ stands for d-dimensional Lebesgue measure.

Also, $Z(t, d\eta) = \int_0^t \int_{\mathbb{R}^m} \sigma\tilde{\pi}(ds, d\eta, d\sigma)$ is an informal representation of an impulsive cylindrical process $(Z(t))_{t\geq 0}$ on $\mathcal{H}$ with jump size intensity $\nu$ in the sense of Definition 7.23 in [14].

For $\rho > 2d$, the natural embedding of $U_0 = \Delta^{\frac{-\rho}{2}}(\mathcal{H}) \subseteq \mathcal{H}$ into $\mathcal{H}$ is Hilbert–Schmidt, therefore by Example 2.5 in [11] the following series

$$\Delta^{\frac{-\rho}{2}} Z(t) := \sum_{n=1}^{\infty} \int_0^t \int_{[0,2\pi]^d} \int_{\mathbb{R}^m} \sigma b_n(\eta)\tilde{\pi}(ds, d\eta, d\sigma)\Delta^{\frac{-\rho}{2}} b_n, \qquad t \geq 0, \tag{4.3}$$

converges for each $T \geq 0$ in $M_T^2(\mathcal{H})$ (the space of all càdlàg square integrable $\mathcal{H}$-valued $(\mathcal{F}_t)$-martingales equipped with norm $\|X\|_{M_T^2(\mathcal{H})} = (E\|X(T)\|_{\mathcal{H}}^2))^{\frac{1}{2}}$ and defines a Lévy process that satisfies the assumption $\mathbb{H}2$ with covariance operator $Q = \int_{\mathbb{R}^m} \sigma^2\mu(d\sigma)\Delta^{-\rho}$ and intensity measure $\nu = (\lambda^d \otimes \mu) \circ \varphi^{-1}$ where $\varphi(x,y) = \sum_{n=1}^{\infty} y b_n(x)\Delta^{\frac{-\rho}{2}} b_n$ (convergence in $L^2([0,2\pi]^d \times \mathbb{R}^m, \lambda^d \otimes \mu; \mathcal{H})$).

Notice that $\Delta^{\frac{-\rho}{2}} Z(t)$ is an $\mathcal{H}$-valued pure jump Lévy process which belongs to $M_T^2(\mathcal{H})$, therefore by Lévy-Itô decomposition theorem, we have $\Delta^{\frac{-\rho}{2}} Z(t) = \int_0^t \int_{\mathcal{H}} y\overline{N}(dt, dy)$ where $\overline{N}(dt, dy)$ is the corresponding Poisson random measure on $[0,t] \times \mathcal{H}$. Hence, the last term of equation (4.1) can be written as

$$\Delta^{\frac{-\rho}{2}} \int_{\mathbb{R}^m} \sigma\tilde{\pi}(ds, d\eta, d\sigma) = \int_{\mathcal{H}} y\overline{N}(dt, dy).$$

Setting $A = \Delta^\beta + I$ with $D(A) = [\mathcal{H}^{2\beta}([0,2\pi]^d)_{per}]^m$ where $\mathcal{H}^{2\beta}([0,2\pi]^d)_{per}$ is the classical Sobolev space with periodic boundary conditions. The equation (4.1) is rewritten in the form of equation (1.3) on $\mathcal{H}$.

Let us check the assumptions stated in section 2. Consider the following function

$$k(y) = \sqrt[3]{|y| \wedge R}, \qquad y \in \mathbb{R}^m,$$

for some constant $C_R > 0$, we have

$$|k(x) - k(y)| \leq C_R \sqrt[3]{|x - y|},$$

consequently,

$$\begin{aligned}
\|Bf - Bf'\|^2 &\leq \int_{[0,2\pi]^d} \left| g(x) \int_{[0,2\pi]^d} [k(f(y)) - k(f'(y))]dy \right|^2 dx \\
&\leq C\|g\|_\infty^2 \left| \int_{[0,2\pi]^d} [k(f(y)) - k(f'(y))]dy \right|^2 \\
&\leq C'\|g\|_\infty^2 C_R^{12} \Big( \int_{[0,2\pi]^d} |f(y) - f'(y)|^2 dy \Big)^{\frac{1}{3}} \\
&= C'\|g\|_\infty^2 C_R^{12} \|f - f'\|^{\frac{1}{6}}.
\end{aligned}$$

Therefore, the function $B$ is $\frac{1}{12}$-Hölder and it is clear that $B$ is bounded, so the assumption $\mathbb{H}7$ is satisfied. Now, for

$$B_n f = \langle g, b_n \rangle \int_{[0,2\pi]^d} k\big(f(y)\big) dy,$$

we have

$$\begin{aligned}|B_n f - B_n f^{'}|^2 &= |\langle g, b_n \rangle|^2 \left| \int_{[0,2\pi]^d} [k(f(y)) - k(f^{'}(y))] dy \right|^2 \\ &\leq C |\langle g, b_n \rangle|^2 \|f - f^{'}\|^{\frac{1}{6}},\end{aligned}$$

therefore

$$\frac{|B_n f - B_n f^{'}|}{\|f - f'\|^{\frac{1}{12}}} \leq \sqrt{C} |\langle g, b_n \rangle|,$$

then we get $\|B_n\|^2_{\frac{1}{12}} \leq \sqrt{C} |\langle g, b_n \rangle|^2$ and

$$\sum_{n=1}^{\infty} \frac{\|B_n\|^2_{\frac{1}{12}}}{\alpha_n} \leq \sqrt{C} \sum_{n=1}^{\infty} \frac{|\langle g, b_n \rangle|^2}{\alpha_n} \leq \frac{\sqrt{C}}{\alpha_1} \sum_{n=1}^{\infty} |\langle g, b_n \rangle|^2 = \frac{\sqrt{C}}{\alpha_1} \|g\|^2 < \infty,$$

hence the assumption $\mathbb{H}3$ is satisfied.

Notice that in this example $Q = A^{-\gamma}$ and it can be easily seen that $\Lambda_t = A^{\frac{1+\gamma}{2}} (I - e^{2tA})^{-\frac{1}{2}} e^{tA}$ and $Q_t = \frac{1}{2} A^{-1-\gamma} (I - e^{2tA})$. Finally, since $0 \leq \gamma < \frac{1}{3}$ and $\beta > \frac{d(1+\gamma)}{2}$, according to Lemma 9 in [3], the pair $Q_t$ and $\Lambda_t$ satisfy assumptions $\mathbb{H}4$–$\mathbb{H}6$ with $\theta = \frac{1}{2}$.

## 5 Conclusions and future work

In conclusion, we proved pathwise uniqueness for stochastic evolution equations in Hilbert spaces driven by both Poisson random measure and Wiener process with Hölder continuous drift. Thus, we extended the work done by Flandoli et al. [3] which generalized Veretennikov's fundamental result to infinite dimensions for Wiener noise. The point of the trick was removing the non-regular drift $B$ and replacing it with some new terms by means of the corresponding infinite dimensional Kolmogorov equation, which was proved to have good Lipschitz properties. As we saw the corresponding infinite dimensional Kolmogorov equation was a powerful tool and it could provide a strong foundation for future works in this area. Finally, some problems are offered that will help expand and strengthen the results of our work.

- Can a similar result be obtained by considering equation (1.3) with multiplicative noise under Lipschitz noise coefficients for the next step?
- Does the equation (1.3) still have the pathwise uniqueness property by considering equation (1.3) with values in Banach spaces? For more details about Poisson random measure and Wiener process in Banach spaces and definition and properties of integral with respect to $W_t$ and $\overline{N}(dt, dx)$, consult [1].
- Does the same idea work to prove the weak uniqueness of equation (1.3)?

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