Int. J. Nonlinear Anal. Appl. In Press, 1–9
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29727.4245

Directional returns to scale in data envelopment analysis

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(Communicated by Haydar Akca)

Abstract

In evaluating the DMUs, two factors, namely the efficiency and the production size, are considered. When the production size of a unit is not optimal, its RTS determines that changing the resources in which direction, would enhance its productivity. In most past research, RTS is considered to be increasing or decreasing, and frontier analysis is used to determine it. In this paper, a method based on MPSS is developed that in addition to determining the RTS for each unit in a directional manner, the shortest changes in resources for achieving the suitable production size are also obtained. In this approach, the computational complexity and ambiguity in the unit's RTS are absent.

Keywords: envelopment analysis, returns to scale, MPSS
2020 MSC: 90C08

1 Introduction

The efficiency of an organization is affected by two factors:

- *the internal factors*, which are the ability of the organization in order to achieve the maximum productivity of the organization, and they can be examined via efficiency evaluation methods. Having the production function in a specific production area, one can determine weather the examined organization is working under optimal conditions or some of it's factors shows any weakness. The production function is unknown in most cases and different science researches from variety of research fields study to estimate it. **D**ata **E**nvelopment **A**nalysis(DEA) is a non-parametric technique that unlike the parametric approaches, instead of estimating the constraints of the function, calculates the set of feasible production actions and uses it's maximal and dominant boundary as a suitable estimation for the production function.
- *The external factors* which are the external environmental conditions forced upon the organization and affects the efficiency of the organization. One of these conditions could be production resources that is in the disposal of the organization and is called the "size" of the organization. Human resources, budget and the equipment in dispose are among the resources which determine the size of production. Organization with different sizes, presents different results. Having the bigger production size does not always yields the better results, but it's often the case that small production size, would prevent the better results. Note that different known and unknowns environmental conditions could affect the productivity of an organization which are not necessarily dependent to each other.

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Received: July 2022 *Accepted:* December 2022

If the basis for performance evaluations is set to be the Homogeneous observed productive activities within a specific time window, the examination would be relative and the evaluation would be the frontier production function which could be estimated from various parametric approaches, or a Production Possibility Set (PPS) which is based on a set of accepted facts, which is determined in the data envelopment analysis.

2 Productivity

The return of a production activity is calculated with the comparison of the obtained results and used resources. This basic notion of performance measurement, is dependent to the production technology. With changes in the size of the production activity, the maximum output and hence the maximum return could be changed. With regards to the life cycle of a technology which includes these four sections:

The Initial stage: for most companies, this stage is the most expensive period for the product. On one hand, sales are increasing with a very low rate, on the other hand the expense for research and development and marketing is very high. Crossing this period is extremely hard, this phase is called start-up in companies. As it was mentioned before, in this stage companies desperately need investments while their return time would not occur soon. In other words, the use of production resources is too high while the achieved results are negligible and hence the efficiency of the activity is low. In this stage, the organization is not yet formed and maybe because of this reason they should not be compared to the homogeneous organizations.

The growth stage: this stage is usually accompanied with a powerful sales and benefits. The company could use the efficiency relative to the increasing returns to scale. Because of that the profit margin and the ultimate benefit is continuously increasing. It is for this reason that most companies would attempt to expand their organization which would lead in the increasing the consumption of production resources in order to maximize their growth potential. In fact, the progress of the companies within this phase is it's size getting bigger and this goal would be worthy if it is accompanied with maintaining up-to date technology. the end of this phase, have the most valuable productive situation which is called the most productivity scale size in MPSS. Along with efficiency, being close to this phase would result in increasing the productivity of an organization.

The maturity stage: in this session, the product is well introduced and the goal of companies is to achieve larger market share. This session is the most competitive phase in the company's active period and they should have smart investigations in their marketing. It is also possible that the companies attempt to make adjustments to their products in order to achieve better competitive advantages in the market. Thus more investments or enlargements of the company does not lead into profits but it is R&D which would cause in better outputs and more profits. The returns to scale for this session is decreasing. In the beginning of this session, the organization is in the MPSS position and does not have the tendency to get away from it. Keeping up the efficiency along with staying near to MPSS are the performance measurement of the units.

The Decline stage: the capacity of market for one product is limited and as the market gets full, companies would present their products in lower cost markets and hence achieving less profits which would decrease the activity return. The occurrence of this phase is almost inevitable for most products. The beginning of this phase is the gateway to the concept of congestion for organizations, and the organizations within this situation would have to decrease the size of their input based on the current capacity of the market in order to get back to the competitive phase.

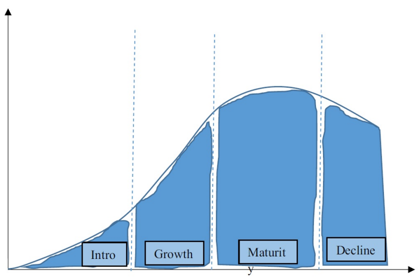

Figure 1: Organizational Life Cycle

Based on what we said, the return of a productive activity, is affected by the efficiency and the size of it's action. In the following we would present the mathematical models for these concepts in the data envelopment analysis.

Suppose that n production organizations which in the data envelopment literature are referred to as **D**ecision **M**aking **U**nits (DMU), are supposed to be compared to each other productivity wise in an specific period of time. The jth unit uses m resources with the quantitative values $X_j = (x_{1j}, \cdots, x_{mj})^{\mathsf{T}}$ in order to produce s desirable results with quantitative values $Y_j = (y_{1j}, \cdots, y_{sj})^{\mathsf{T}}$. In related works, the vectors X_j and Y_j are called the input and output vectors of the DMU_j and it's assumed that their values are additive. (this assumption is valid since it could easily be achieved via normalization). Also it's assumed that their values are non-zero and non-negative. With assigning values to the input and output factors, a measurement for the return R could be obtained via:

$$R = \frac{v_1x_1 + \cdots v_mx_m}{u_1y_1 + \cdots + u_sy_s} = \frac{\sum_{i=1}^{m} v_ix_i}{\sum_{r=1}^{s} u_ry_r} \tag{2.1}$$

In which v_i is the quantitative value of input $x_i, (i = 1, \cdots m)$ and u_r is the quantitative value of output $y_r, (r = 1, \cdots s)$. Hence the efficiency of input and output vectors (X, Y) in the hyperplane satisfying the equation $R\sum_{r=1}^{s} u_ry_r - \sum_{i=1}^{m} v_ix_i = 0$ which crosses the point (X, Y) and the origin. Based on the relative comparison, the **M**ost **P**roductivity **S**cale **S**ize (MPSS) set, include DMUs which have the highest return among other DMUs, in other words:

$$MPSS = \{DMU_p | R_p = \max_j \{R_j\}\} \tag{2.2}$$

in which $R_j = \frac{\sum_{i=1}^{m} v_ix_{ij}}{\sum_{r=1}^{s} u_ry_{rj}}$ is the efficiency of $DMU_j, (j = 1 \cdots n)$.

In most cases, the coefficient vectors $V = (v_1, \cdots, v_m)^{\mathsf{T}}$ and $U = (u_1, \cdots, u_s)^{\mathsf{T}}$ are unknown and in DEA their values are adjusted relative to the unit under evaluation DMU_p, in a way that the best return among other units is achieved

$$(V^*, U^*) = \arg \max_{(U,V)>(0,0)} \left\{ \frac{R_p}{\max_j R_j} \right\}. \tag{2.3}$$

Thus if a unit is not in the MPSS situation relative to it's corresponding coefficient vector, it would not be in this situation with any other coefficient vectors. The weights obtained from the above model are equivilant to the weights achieved via the CCR model in DEA which proposed in 1978 by Charnes-Cooper-Rhoeds, and thus we have:

$$\begin{aligned} e_p^{CCR} = \max & \frac{\sum_{r=1}^{s} u_ry_{rp}}{\sum_{i=1}^{m} v_ix_{ip}} \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_ry_{rj}}{\sum_{i=1}^{m} v_ix_{ij}} \leq 1 \qquad j = 1, \cdots, n \\ & v_i \geq \epsilon, \qquad i = 1, \cdots, m \\ & u_r \geq \epsilon, \qquad r = 1, \cdots, s \end{aligned} \tag{2.4}$$

in which ϵ is a non-Archimedean infinitesimal epsilon. in the model (2.4), e_p^{CCR} is the normalized return of the unit under evaluation DMU_p which in the DEA literature is usually referred to as the efficiency score of CCR. it's obvious that if $e_p^{CCR} = 1$, the DMU_p with it's coefficient vector in equation (2.4) is located in MPSS. Thus in situations in which the coefficient vector is unknown, the MPSS set is assumed to have the form:

$$MPSS = \{DMU_p | e_p^{CCR} = 1\}$$

regardless of the observed points, here we present the MPSS situation:

Definition 2.1. Relative to the set of units under evaluation, the set of input vectors which are included in the convex combination of the input vectors of MPSS points, is called an MPSS region. Assuming that J_{MPSS} is the set including the indices of the MPSS units, we define:

$$X_{MPSS} = \underset{j \in J_{MPSS}}{convex} \{X_j\}$$

it's obvious that for any answer (V^*, U^*) achieved in model (**??**), the hyperplane

$$\sum_{r=1}^{s} u_r^* y_r - \sum_{i=1}^{m} v_i^* x_i = 0 \tag{2.5}$$

includes all points (X, Y) which represent the highest return if $X \in X_{MPSS}$. Every hyperplane in (2.5) is called and MPSS hyperplane. The convex combination of every two MPSS point which is located in the MPSS hyperplane, if observed, is an MPSS point, but if these points are located in different hyperplanes this statement is not necessarily true. But this statement is different for MPSS points, meaning that the convex combination of every two MPSS points are in MPSS region.

3 Technical efficiency and scale

If a unit is not MPSS, it does not necessarily mean that it does not posess the maximum productivity or it's inefficient. Unsuitable production situation regarding the usage of resources and or the production size, prevent the inclusion of DMU in the MPSS set, but it does not prevent the maximum production and efficiency. Efficiency is the optimal usage of resources in order to reach the maximum productivity and is completely dependent on the technology in that production area.

In 1984, Banker-Charnes-Cooper developed the basic DEA model for evaluating the efficiency named BCC which is based on this axiom that the production function is a concave, continuous and envelopment function. Instead of estimating the production function, they estimated the **P**roduction **P**ossibility **S**et (PPS), with **V**ariable **R**eturns to **S**cale (VRS) based on Including observations, free disposal in the input and output and convexity. The envelopment form of the BCC model for calculating the efficiency score of the unit under evaluation DMU_p is presented in the following way:

$$\begin{aligned} e_p^{BCC} =& \max \varphi \\ \text{s.t.} \quad & \sum_{i=1}^{m} \lambda_j x_{ij} + s_i^- \qquad i = 1, \cdots m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \varphi y_{rp}, \qquad r = 1, \cdots, s \\ & \sum_{i=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \qquad j = 1, \cdots, n \end{aligned} \tag{3.1}$$

e_p^{BCC} is the efficiency score of the unit under evaluation. This unit is technically efficient if $e_p^{BCC} = 1$. Of course if for all optimal answers of model (**??**) we have $s_i^- = s_r^+ = 0, (r = 1, \cdots s, i = 1, \cdots, m)$ then it's efficient in pareto koopmans context. the efficiency score achieved from (3.1) is at least 1 ($1 \leq e_p^{BCC}$). model (3.1) determines if unit DMU_p, with using all the resources in it's disposal have the e_p^{BCC} percentage of production power relative to the similar technologies and hence have the technical inefficiency of $e_p^{BCC} - 1$ which in term is called Inefficient. Although after appropriate evaluation of the outputs, the values s_i^- and s_r^+ provides the opportunity of using less resources respectively and hence producing more results separately which is a complete evaluation criterion. The dual of (3.1) represent the support hyperplane on PPS in the point DMU_p which is:

$$\begin{aligned} & \min \sum_{i=1}^{m} v_i x_{ip} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r u_{rp} + u_0 = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0, \qquad j = 1, \cdots, n \\ & v_i \geq 0, \qquad i = 1, \cdots, m \\ & u_r \geq 0, \qquad r = 1, \cdots, s \end{aligned} \tag{3.2}$$

for each answer like (V', U', u_0') for model (3.2), the hyperplane

$$\sum_{r=1}^{s} u_r' y_r - \sum_{i=1}^{m} v_i' x_i + u_0' = 0 \tag{3.3}$$

is a supporting hyperplane in the coordinates of DMU_p on the PPS. the hyperplane (3.3) have some important tips in the evaluating the efficient unit DMU_p:

- if $DMU_p \notin MPSS$, then in each optimal solution of (3.2) we have $u'_0 \neq 0$. Although the vice versa does not always hold.
- Since the set of solutions of a linear programming is convex, if there exists two different solutions of (3.2) with positive and negative u'_0, then in the optimal solution it's zero and hence $DMU_p \in MPSS$.
- if $u'_0 > 0$ then $\sum_{r=1}^{s} u'_r y_{rp} - \sum_{i=1}^{m} v'_i x_{ip} < 0$ meaning that if the hyperplane (3.3) is transformed parallel so that it crosses the origin, then the coordinates of DMU_p would be above it, that is the production average (current efficiency) of the unit is more that the production rate.
- if $u'_0 < 0$ then $\sum_{r=1}^{s} u'_r y_{rp} - \sum_{i=1}^{m} v'_i x_{ip} > 0$ meaning that if the hyperplane (3.3) is transformed parallel so that it crosses the origin, then the coordinates of DMU_p would be below it, that is the production average (current efficiency) of the unit is less that the production rate.

these tips will be used in section 5.

it's obvious that $e_p^{CCR} \leq e_p^{BCC}$ and hence if $e_p^{CCR} = 1$ then DMU_p is technically efficient. Having this in mind, the scale efficiency score could be achieved via:

$$e_p^{SE} = \frac{e_p^{CCR}}{e_p^{BCC}} \tag{3.4}$$

this score combines both the performance evaluation and the optimal activity size. Having $e_p^{SE} = 1$ means that the unit under evaluation have the most productivity scale size, and it's called scale efficient. Otherwise in case $e_p^{scale} < 1$, the unit is scale inefficient and is not in the optimal production situation. If the unit is not in the optimal production situation, the production size relative to the optimal production situation can not be determined from the scale efficiency score. In other words, is the production resources higher or lower than the optimal value?

4 Returns to scale

the location of a Unit located on the frontiers of production, relative to the optimal production region in MPSS, is called Returns to Scale. If a unit reaches the MPSS region with increasing it's resources, it's said that the unit shows an increasing returns to scale. In other words, increase in the resources lead into the increase in return. If decreasing the resources pushes the unit into MPSS, it's said that the unit shows decreasing returns to scale and decreasing the resources would lead into higher return. In other cases, the returns to scale is considered to be constant which may ignore the changes that does not affect the input and output simultaneously.

One way to determine the RTS of an efficient technical point located on the frontier of productivity, is to compare the marginal product and average product which is referred to as the Scale Elasticity. $e_p^{\text{scale Elastity}}$ which in the case of singe input-output can be calculated via:

$$e_p^{\text{scale Elastity}} = \frac{\frac{dy}{dx}}{\frac{y}{x}}\Big|_{(x,y)} = (x_p, y_p) \tag{4.1}$$

if the production rate is higher(lower) than the average production rate, increasing(decreasing) the resources would cause in higher return. If they are equivalent, changes in the resources would not affect the return. Hence:

- if $e_p^{\text{scale elastity}} > 1$ increasing returns to scale for the DMU_p is detected.
- if $e_p^{\text{scale elastity}} < 1$ decreasing returns to scale for the DMU_p is detected.
- if $e_p^{\text{scale elastity}} = 1$ returns to scale for the DMU_p remains constant.

This factor can not be calculated in the case of multiple input-output unless the appropriate changes relative to the input-output is used in order to transform it to a single input-output form. Suppose that:

$$\beta(\alpha) = \max\{\beta| \, ((\alpha+1)X_p, (\beta+1)Y_p) \in PPS\} \tag{4.2}$$

such that the multiple input-output vector (X, Y) is turned into the single input-output vector $(\alpha, \beta(\alpha))$. Note that the corresponding vector to the unit under evaluation DMU_p is $(\alpha, \beta(\alpha)) = (0, 0)$. In this case the scale elasticity could be calculated via:

$$e_p^{\text{scale elastity}} = \frac{d\beta(\alpha)}{d\alpha}|_{\alpha=0} \tag{4.3}$$

Among all possible changes, equation (4.3) only considers the increase or decrease relative to the input and output and ignores other changes in the resources, while it's possible that changes in other directions may increase the return.

Using support hyperplanes on PPS, could be a solution to the mentioned problem. for every optimal solution, the hyperplanes (3.3) are estimations of production rate relative to the small changes in the resources. As we seen in the section 3, the sign of u_0' is the relation between production rate and production average, but we should have in mind that:

- if for some optimal solutions of (3.2) we have $u_0' > 0$ it means that DMU_p shows decreasing RTS.
- if for all optimal solutions of (3.2) we have $u_0' < 0$ it means that DMU_p shows increasing RTS.
- if for all optimal solutions of (3.2) we have $u_0' = 0$ it means that the unit under evaluation DMU_p is in MPSS and it's RTS and changes in resources is not necessary.

Using the scale efficiency, one could determine if the activity size of a unit is optimal or not. And if it's not optimal, using the supporting hyperplanes in (3.2) could determine whether increasing or decreasing the production size would lead into increasing the return. The question that we are after in this research is that: Is it possible that changes in resources in non radial and inappropriate directions would lead into the increasing efficiency? if so, what directions?

In other words, in this paper we are looking for a change in the resources which is not necessarily appropriate, and may lead into increasing, decreasing and or leaving some resources unchanged but in that direction the return and productivity could be increased.

5 Proposed method

As it was discussed before, in the production possibility set with variable returns to scale, the maximum efficiency is achieved in X_{MPSS}. Based on definition 2.1, this set have the form:

$$X_{MPSS} = \{X | X = \sum_{j \in J_{MPSS}} \mu_j X_j, \sum_{j \in J_{MPSS}} \mu_j = 1, \mu_j \geq 0 (j \in J_{MPSS})\}. \tag{5.1}$$

The resources in PPS which have shorter distance to the X_{MPSS} set, if working efficient, have higher return. Hence for determining the RTS of a unit under evaluation which is outside the MPSS position, it's enough to determine the shortest production position distance to the X_{MPSS} using:

$$\mathtt{d_x} = (d_1, \cdots, d_m)^{\mathtt{T}} = \frac{1}{\sum_{i=1}^m |\alpha_i^*|} (\alpha_1^*, \cdots, \alpha_m^*)^{\mathtt{T}} \tag{5.2}$$

in which $\alpha^* = (\alpha_1^*, \cdots, \alpha_m^*)^{\mathtt{T}}$ is one of the optimal solutions for the problem:

$$\begin{aligned} &\min \sum_{i=1}^m |\alpha_i| \\ \text{s.t.} \quad &\sum_{j \in J_{MPSS}} \lambda_j x_{ij} = x_{ip} + \alpha_i, \qquad i = 1, \cdots, m \\ &\sum_{j \in J_{MPSS}} \lambda_j y_{rj} = y_{rp} + \beta_r, \qquad r = 1, \cdots, s \\ &\sum_{j \in J_{MPSS}} \lambda_j = 1, \qquad \lambda_j \geq 0, \end{aligned} \tag{5.3}$$

the point $X_p - \alpha^*$ is in MPSS position and has the shortest distance to the X_p position. The production technology from point X_p and in the direction of $\mathtt{d}$ shows increasing efficiency. Model (5.3) could easily become a linear model:

$$\begin{aligned} \min \sum_{i=1}^{m} \alpha_i' + \alpha_i'' & \\ \text{s.t.} \quad \sum_{j \in J_{MPSS}} \lambda_j x_{ij} = x_{ip} + \alpha_i' - \alpha_i'', & \qquad i = 1, \cdots, m \\ \sum_{j \in J_{MPSS}} \lambda_j y_{rj} = y_{rp} + \beta_r' - \beta_r'', & \qquad r = 1, \cdots, s \\ \sum_{j \in J_{MPSS}} \lambda_j = 1, & \\ \alpha_i', \alpha_i'' \geq 0, \qquad i = 1, \cdots, m & \\ \beta_r', \beta_r'' \geq 0, \qquad r = 1, \cdots, m & \end{aligned} \tag{5.4}$$

in which:

$$\begin{aligned} \alpha_i = \alpha_i' - \alpha_i'', \qquad \alpha_i'\alpha_i'' = 0 \qquad i = 1, \cdots, m \\ \beta_r = \beta_r' - \beta_r'', \qquad \beta_r'\beta_r'' = 0 \qquad r = 1, \cdots, s \end{aligned}$$

Assuming that α^*, $\beta^* = (\beta_1^*, \cdots, \beta_s^*)^{\mathtt{T}}$ and $(\mu_1^*, \cdots, \mu_s^*)^{\mathtt{T}}$ is an optimal solution for (5.4), it's obvious that the projected point

$$\left(X_p', Y_p'\right) = (X_p + \alpha^*, Y_p + \beta^*) \tag{5.5}$$

is in the MPSS position and is Pareto-Koopmans efficient. And the unit under evaluation could reach the MPSS of the projection point $\left(X_p', Y_p'\right)$ in (5.5) with changing it's input resources from X_p in the direction $\mathtt{d_x}$ with magnitude $\sigma = \sum_{i=1}^{m} |\alpha_i^*|$. Since DMU_p is out of the MPSS position, vector $\alpha^* \neq \mathtt{0}$, but it's not necessarily non-positive or non-negative , the same also holds for the changing output results vector obtained from this transformation, β^*. Hence the concept of increasing and or decreasing returns to scale is irrelevant here and only the changing and or constant returns to scale could be relevant. All units which their resource situation is in X_{MPSS}, have a constant returns to scale, otherwise they have a variable returns to scale. Anyway, in this transformation the return would increase regarding to (2.1). Althowgh it's possible that $\beta^* = 0$. The transformation direction would be:

$$\mathtt{d} = \frac{1}{\sum_{i=1}^{m} |\alpha_i^*| + \sum_{r=1}^{s} |\beta_r^*|} \left((\alpha_1^*, \cdots, \alpha_r^*)^{\mathtt{T}}, (\beta_1^*, \cdots, \beta_s^*)^{\mathtt{T}}\right) \tag{5.6}$$

using (5.2) we would have:

$$\mathtt{d} = \frac{1}{\sum_{i=1}^{m} |\alpha_i^*| + \sum_{r=1}^{s} |\beta_r^*|} \left(\sum_{i=1}^{m} |\alpha_i^*| \mathtt{d_x}, (\beta_1^*, \cdots, \beta_s^*)^{\mathtt{T}}\right) = \frac{\sum_{i=1}^{m} |\alpha_i^*|}{\sum_{i=1}^{m} |\alpha_i^*| + \sum_{r=1}^{s} |\beta_r^*|} \left(\mathtt{d_x}, \frac{1}{\sum_{i=1}^{m} |\alpha_i^*|} (\beta_1^*, \cdots, \beta_s^*)^{\mathtt{T}}\right).$$

So, the rate of change in the output corresponding to one unit change in the direction $\mathtt{d_x}$ is equivalent to $Rate(\mathtt{d_x}) = \frac{1}{\sum_{i=1}^{m} |\alpha_i^*| + \sum_{r=1}^{s} |\beta_r^*|}$ in the direction:

$$\mathtt{d_y} = (d_{m+1}, \cdots, d_{m+s})^{\mathtt{T}} = \frac{1}{\sum_{r=1}^{s} |\beta_r^*|} (\beta_1^*, \cdots, \beta_s^*)^{\mathtt{T}}$$

only of $\beta^* \neq 0$, otherwise $Rate(\mathtt{d_x}) = 0$.

Note 1: in the above method, the shortest path to the MPSS set is used as a measurement to determine the RTS situation, while all directions that lead to the MPSS could be used too. Hence the RTS situation of a frontier point is also dependent on the chosen direction. As it was mentioned, in most methods available, determining the appropriate RTS direction (equivalently here $\alpha^* > 0$ or $\alpha^* < 0$) is used, thus the comparison results of that methods with the proposed method could differ.

Note 2: The returns to scale concept is one of the features of the production function. The efficient points are used as estimated points of this function and is used in RTS studies. Although the projection of the inefficient points on the frontier of PPS:

$$\left(\bar{X}_p, \bar{Y}_p\right) = (X_p, \varphi^* Y_p) \tag{5.7}$$

which in φ^* is an optimal solution for (3.1), could be used to determine the RTS of an estimated point from the production function with the same resources. In most studies, the result of RTS, (5.7) is assigned to the inefficient unit (X_p, Y_p).

Example 5.1. for better understanding, assume that we have 9 decision making units marked with $\boxed{1}$ to $\boxed{9}$, which uses two input resources I_1 and I_2 in order to produce a single output O_1. The data for these units are presented in table 1. After using model (2.4), we can determine that units $\boxed{2}$, $\boxed{3}$ and $\boxed{9}$ are among the MPSS units and the set X_{MPSS}, in the input area could be calculated as the convex combination of these three points. Figure 2 shows the input area in which the above set is represented in it. After calculating the scale and technical efficiency, with model (3.1) and (3.4) respectively, it could be understood that although unit $\boxed{8}$ is not MPSS, but it is in this region, and hence this unit have the appropriate production size but as it can be determined from it's efficiency score, it's inefficient. The units within MPSS region shows constant RTS region and the rest have varying RTS region. After applying model (5.4) on the projection of the units on the PPS, we obtained the shortest paths to the X_{MPSS} set with the values α_1^* and α_2^* which the change direction $\mathtt{d_x}$, the change rate σ and the change rate of the outputs in the direction of $Rate(\mathtt{d_s})$ is calculable based on it. As it can be seen, the change directions could be different from the direction of appropriate increasing and or decreasing the inputs.

Table 1: the input-output for the DMUs and their performance results

DMUs	I_1	I_2	O_1	e_p^{CCR}	e_p^{BCC}	e_p^{SE}	X_{MPSS}	RTS	Projected Points $(\bar{x}_{1p}, \bar{x}_{2p}, \bar{y}_{1p})$	$(\alpha_1^*, \alpha_2^*, \beta_1^*)$	Direction $\mathbf{d}_x$	σ	Rate $(\mathbf{d}_x)$
[1]	2	2	1	2.00	1.00	2.00	Out	V	$(2, 2, 1)$	$(2, 0, 2)$	$(1, 0)$	2	0.25
[3]	4	2	3	1.00	1.00	1.00	In	C	$(4, 2, 3)$	–	–	–	–
[2]	3	5	4	1.00	1.00	1.00	In	C	$(3, 5, 4)$	–	–	–	–
[5]	6	4	4	1.25	1.00	1.25	Out	V	$(6, 4, 4)$	$(-2.67, 0, -0.67)$	$(-1, 0)$	2.67	0.33
[6]	4	4	2	2.00	1.88	1.06	Out	V	$(4, 4, 3.76)$	$(-0.67, 0, -0.47)$	$(-1, 0)$	0.67	0.91
[7]	6	6	4	1.50	1.00	1.50	Out	V	$(6, 6, 4)$	$(-3, -1, 0)$	$(-0.75, -0.25)$	4	0.00
[8]	2	7	2	1.33	1.00	1.33	Out	V	$(2, 7, 2)$	$(1, -2, 2)$	$(0.33, -0.67)$	3	0.2
	3.5	3.5	2	1.75	1.75	1.00	In	C	$(3.5, 3.5, 3.5)$	–	–	–	–
[4]	3	3	3	1.00	1.00	1.00	In	C	$(3, 3, 3)$	–	–	–	–

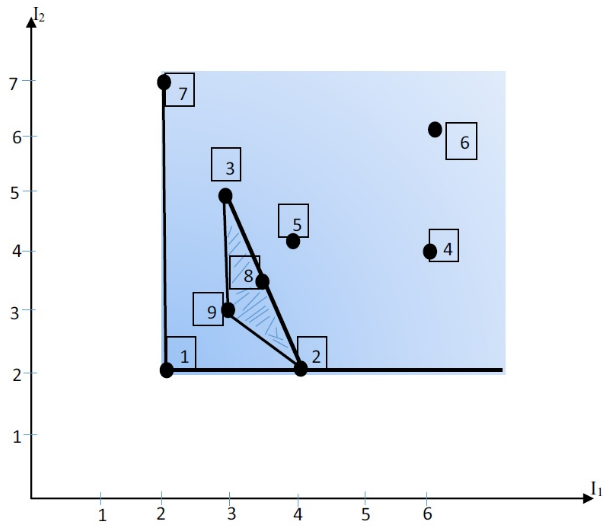

Figure 2: Input Region and X_{MPSS} set

6 Results

Based on the developed models for determining the optimal production size, we have shown that the RTS for a unit could be inappropriate, and be determined in different and inappropriate ways. Among all directions for increasing the efficiency of the production size, the changes towards the X_{MPSS} set in the shortest path could be an straightforward ground for determining the RTS situation of a point.

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