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Twin positive solutions to the RL-type nonlinear FBVPs with p-Laplacian operator

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Abstract

In this work, we establish the existence of at least two positive solutions for a coupled system of p-Laplacian fractional-order boundary value problems. Establishing the existence of positive solutions to the problem is challenging for a variety of reasons, the most important of which is a lack of compatibility with the kernel. To address these issues, we have included the necessary conditions for overcoming certain methodological hurdles on the kernel as well as adapting to the problem's nature of positivity. The method is based on the AH functional fixed point theorem.

Keywords: Fractional derivative, boundary value problem, p-Laplacian, Integral equation, Kernel, positive solution
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1 Introduction

Several researchers have been fascinated by Fractional differential equations (FDEqs) since they are extensively used in the applied mathematics of a range of scientific and technical situations. Fractional derivatives have the benefit of enabling more degrees of freedom in models, making them more effective for simulating real-world phenomena. The ability to grab hereditary features of various tasks due to the operator's nonlocality and to simulate substances of intermediate characteristics are benefits of fractional derivatives over classical derivatives like viscoelasticity of substances intermediate between particles [17, 15, 11, 16].

In effect, the classical p-Laplacian is a collection of differential equations ruled by its most often utilized operators, which keeps appearing and generates considerable interest in fields like biophysics, glaciology, non-Newtonian fluid flow, physics, turbulent filtration in porous media, and so on [9]. In fact, differential equations including many functions or dependent variables, as well as multiple relationships between them, are frequently encountered. As a result of all of this, the system of differential equations is established. For some latest results on the existence of positive solutions for FDEqs and systems of FDEqs dealing with different boundary conditions, we suggest the reader to the monograph [10], as well as the papers [1, 5, 6, 4, 18, 13].

Large sectors including telecommunications equipment and communication systems, synthetic chemical, automobile, medical, and pharmaceutical depend on boundary value problems (BVPs) to simulate complex phenomena at

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different stages for both the design and construction of high-tech goods, and positive solutions seem to be useful in such applied settings. Positive solutions to p-Laplacian BVPs have been extensively studied; see [12, 3, 23]. Recently, authors have aimed to establish the theories on p-Laplacian FBVPs [7, 8, 19, 20]. In [22], the eigenvalue intervals in which at least one positive solution to the FBVPs exists have been discussed.

In this paper, inspired and motivated by recent developments on the existence of positive solutions for FBVPs, we study a system of nonlinear p-Laplacian FBVP for $\mathbf{z} \in [\mathtt{a}, \mathtt{b}]$:

$$\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{n}_1}\left[\phi_p\left(\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{k}_1}\varpi_1(\mathbf{z})\right)\right] + \mathtt{g}_1\big(\mathbf{z}, \varpi_1(\mathbf{z}), \varpi_2(\mathbf{z})\big) = 0, \tag{1.1}$$

$$\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{s}_1}\left[\phi_p\left(\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{r}_1}\varpi_2(\mathbf{z})\right)\right] + \mathtt{g}_2\big(\mathbf{z}, \varpi_1(\mathbf{z}), \varpi_2(\mathbf{z})\big) = 0, \tag{1.2}$$

$$\left.\begin{aligned} \psi_1\varpi_1(\mathtt{a}) - \psi_2\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{k}_2}\varpi_1(\mathtt{a}) &= 0,\\ \psi_3\varpi_1(\mathtt{b}) + \psi_4\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{k}_3}\varpi_1(\mathtt{b}) &= 0,\\ \phi_p\left(\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{k}_1}\varpi_1(\mathtt{a})\right) &= 0, \end{aligned}\right\} \tag{1.3}$$

$$\left.\begin{aligned} \beta_1\varpi_2(\mathtt{a}) - \beta_2\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{r}_2}\varpi_2(\mathtt{a}) &= 0,\\ \beta_3\varpi_2(\mathtt{b}) + \beta_4\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{r}_3}\varpi_2(\mathtt{b}) &= 0,\\ \phi_p\left(\mathfrak{D}_{\mathtt{a}^+}^{\mathbf{r}_1}\varpi_2(\mathtt{a})\right) &= 0, \end{aligned}\right\} \tag{1.4}$$

where $\mathfrak{D}_{\mathtt{a}^+}^{\bullet}$ is the Riemann–Liouville derivative, $\phi_p(\varrho) = |\varrho|^{p-2}\varrho$, $p \in (1, \infty)$, $\phi_p^{-1} = \phi_q$, $q = \dfrac{p}{p-1}$, $\mathbf{k}_1, \mathbf{r}_1 \in (1, 2]$, $\mathbf{n}_1, \mathbf{s}_1, \mathbf{k}_2, \mathbf{r}_2, \mathbf{k}_3, \mathbf{r}_3 \in (0, 1]$, $\mathtt{b} > \mathtt{a} \geq 0$, $\psi_k \in \mathbb{R}, k = \overline{1, 4}$ are constants s.t. either $\psi_1^2 + \psi_2^2 > 0$ or $\psi_3^2 + \psi_4^2 > 0$ and $\beta_k \in \mathbb{R}, k = \overline{1, 4}$ are constants s.t. either $\beta_1^2 + \beta_2^2 > 0$ or $\beta_3^2 + \beta_4^2 > 0$.

We consider the following assumptions throughout the paper:

($\mathtt{K}_1$) $\mathbf{k}_1 > \mathbf{k}_j + 1,\ j = \overline{2, 3}$,

($\mathtt{K}_2$) $\dfrac{\mathtt{b}^{\mathbf{k}_1 - 1}}{\Gamma(\mathbf{k}_1 - \mathbf{k}_2)} > \dfrac{\psi_1 \mathtt{a}^{\mathbf{k}_2}}{\Gamma(\mathbf{k}_1)\psi_2}$,

($\mathtt{K}_3$) $\dfrac{\psi_1 \mathtt{a}^{\mathbf{k}_2}}{\psi_2} > \dfrac{(\mathbf{k}_2 - 2)!\big[{\mathbf{k}_1}^2 + \mathbf{k}_2(\mathbf{k}_2 + 1) - \mathbf{k}_1(2\mathbf{k}_2 + 1)\big]}{(\mathbf{k}_1 - \mathbf{k}_2)!}$,

($\mathtt{K}_4$) $\mathbf{r}_1 > \mathbf{r}_j + 1,\ j = \overline{2, 3}$,

($\mathtt{K}_5$) $\dfrac{\beta_1 \mathtt{a}^{\mathbf{r}_2}}{\Gamma(\mathbf{r}_1)\beta_2} < \dfrac{\mathtt{b}^{\mathbf{r}_1 - 1}}{\Gamma(\mathbf{r}_1 - \mathbf{r}_2)}$,

($\mathtt{K}_6$) $\dfrac{\big[\mathbf{r}_1^2 + \mathbf{r}_2(\mathbf{r}_2 + 1) - \mathbf{r}_1(2\mathbf{r}_2 + 1)\big](\mathbf{r}_2 - 2)!}{(\mathbf{r}_1 - \mathbf{r}_2)!} < \dfrac{\beta_1 \mathtt{a}^{\mathbf{r}_2}}{\beta_2}$,

($\mathtt{K}_7$) $\mathtt{g}_i : [\mathtt{a}, \mathtt{b}] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous, for $i = \overline{1, 2}$.

To use the AH fixed point theorem, we derive the criteria for the existence of at least two positive solutions to the system of p-Laplacian FBVP (1.1)–(1.4). Our approach to achieving the required outcomes is standard, but its application in the framework of the current problem is new. By a positive solution of the FBVP (1.1)–(1.4), we mean $\big(\varpi_1(\mathbf{z}), \varpi_2(\mathbf{z})\big) \in \big(\mathtt{C}[\mathtt{a}, \mathtt{b}]\big)^2$ satisfying (1.1)–(1.4) with $\varpi_i(\mathbf{z}) > 0, i = \overline{1, 2}, \forall\ \mathbf{z} \in [\mathtt{a}, \mathtt{b}]$.

The content of this article is structured in the following manner. The second section contains some preliminary findings. In Section 3, we present the main theorems, and in Section 4, we use an example to demonstrate our conclusions.

2 Preliminaries

Denote

$$\left.\begin{aligned} \wp_1 &= \psi_1 - \frac{\psi_2\Gamma(\mathbf{k}_1-1)\mathtt{a}^{-\mathbf{k}_2}}{\Gamma(\mathbf{k}_1-\mathbf{k}_2-1)},\\ \wp_2 &= \frac{\psi_2\Gamma(\mathbf{k}_1)\mathtt{a}^{-\mathbf{k}_2+1}}{\Gamma(\mathbf{k}_1-\mathbf{k}_2)} - \mathtt{a}\psi_1,\\ \widetilde{\wp}_1 &= \psi_3\mathtt{b}^{\mathbf{k}_1-1} + \frac{\psi_4\Gamma(\mathbf{k}_1)\mathtt{b}^{\mathbf{k}_1-\mathbf{k}_3-1}}{\Gamma(\mathbf{k}_1-\mathbf{k}_3)},\\ \widetilde{\wp}_2 &= \psi_3\mathtt{b}^{\mathbf{k}_1-2} + \frac{\psi_4\Gamma(\mathbf{k}_1-1)\mathtt{b}^{\mathbf{k}_1-\mathbf{k}_3-2}}{\Gamma(\mathbf{k}_1-\mathbf{k}_3-1)}. \end{aligned}\right\} \tag{2.1}$$

Lemma 2.1. Let $\Delta_1 = \wp_1\widetilde{\wp}_1 + \wp_2\widetilde{\wp}_2 \neq 0$. For every $\mathfrak{h}(\mathbf{z}) \in \mathtt{C}[\mathtt{a},\mathtt{b}]$, the FBVP

$$\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathbf{z}) + \mathfrak{h}(\mathbf{z}) = 0,\ \mathbf{z}\in[\mathtt{a},\mathtt{b}], \tag{2.2}$$

$$\left.\begin{aligned} \psi_1\varpi_1(\mathtt{a}) - \psi_2\mathfrak{D}^{\mathbf{k}_2}_{\mathtt{a}^+}\varpi_1(\mathtt{a}) = 0,\\ \psi_3\varpi_1(\mathtt{b}) + \psi_4\mathfrak{D}^{\mathbf{k}_3}_{\mathtt{a}^+}\varpi_1(\mathtt{b}) = 0, \end{aligned}\right\} \tag{2.3}$$

has a unique solution $\varpi_1(\mathbf{z}) = \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\mathbf{z},\varrho)\mathfrak{h}(\varrho)d\varrho$, where $\mathcal{H}_1(\mathbf{z},\varrho)$ is given by

$$\mathcal{H}_1(\mathbf{z},\varrho) = \begin{cases} \mathcal{H}_{11}(\mathbf{z},\varrho), & \mathbf{z}\le\varrho,\\ \mathcal{H}_{12}(\mathbf{z},\varrho), & \varrho\le\mathbf{z}, \end{cases} \tag{2.4}$$

$$\begin{aligned} \mathcal{H}_{11}(\mathbf{z},\varrho) =& \frac{1}{\Delta_1}\left[\wp_1\mathbf{z}^{\mathbf{k}_1-1} + \wp_2\mathbf{z}^{\mathbf{k}_1-2}\right]\left[\frac{\psi_3}{\Gamma(\mathbf{k}_1)} + \frac{\psi_4(\mathtt{b}-\varrho)^{-\mathbf{k}_3}}{\Gamma(\mathbf{k}_1-\mathbf{k}_3)}\right](\mathtt{b}-\varrho)^{\mathbf{k}_1-1},\\ \mathcal{H}_{12}(\mathbf{z},\varrho) =& \mathcal{H}_{11}(\mathbf{z},\varrho) - \frac{(\mathbf{z}-\varrho)^{\mathbf{k}_1-1}}{\Gamma(\mathbf{k}_1)},\ \text{also } \wp_1,\widetilde{\wp}_1,\wp_2 \text{ and } \widetilde{\wp}_2 \text{ are given in (2.1).} \end{aligned}$$

Proof . Let $\varpi_1(\mathbf{z}) \in \mathtt{C}[\mathtt{a},\mathtt{b}]$ be the solution of FBVP (2.2)–(2.3). Then (2.2) can be uniquely expressed as $\varpi_1(\mathbf{z}) = \sum_{j=1}^{2} c_j\mathbf{z}^{\mathbf{k}_1-j} - \int_{\mathtt{a}}^{t}\frac{(\mathbf{z}-\varrho)^{\mathbf{k}_1-1}}{\Gamma(\mathbf{k}_1)}\mathfrak{h}(\varrho)d\varrho$. By using (2.3) it yields

$$c_j = \frac{\wp_j}{\Delta_1}\int_{\mathtt{a}}^{\mathtt{b}}\left[\frac{\psi_3}{\Gamma(\mathbf{k}_1)} + \frac{\psi_4(\mathtt{b}-\varrho)^{-\mathbf{k}_3}}{\Gamma(\mathbf{k}_1-\mathbf{k}_3)}\right](\mathtt{b}-\varrho)^{\mathbf{k}_1-1}\mathfrak{h}(\varrho)d\varrho,$$

where $\wp_j, j = \overline{1,2}$ are given in (2.1). Hence the unique solution of (2.2)–(2.3) is $\varpi_1(\mathbf{z}) = \int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\mathfrak{h}(\varrho)d\varrho$, where $\mathcal{H}_1(\mathbf{z},\varrho)$ is given in (2.4). □

Lemma 2.2. Let $\mathfrak{h}_1(\mathbf{z}) \in \mathtt{C}[\mathtt{a},\mathtt{b}]$. Then the FDEq

$$\mathfrak{D}^{\mathbf{n}_1}_{\mathtt{a}^+}\left[\phi_p\left(\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathbf{z})\right)\right] + \mathfrak{h}_1(\mathbf{z}) = 0,\quad \mathbf{z}\in[\mathtt{a},\mathtt{b}], \tag{2.5}$$

with (1.3) has a unique solution

$$\varpi_1(\mathbf{z}) = \int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathfrak{h}_1(\tau)d\tau\right)d\varrho, \tag{2.6}$$

where $\mathcal{H}_1(\mathbf{z},\varrho)$ is given in (2.4).

Proof . An equivalent integral equation for (2.5) is:

$$\phi_p\left(\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathbf{z})\right) = \widetilde{c}_1\mathbf{z}^{\mathbf{n}_1-1} - \int_{\mathtt{a}}^{t}\frac{(\mathbf{z}-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathfrak{h}_1(\tau)d\tau.$$

Using $\phi_p\Big(\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathtt{a})\Big)=0$, one can get $\widetilde{c_1}=0$. Therefore

$$\phi_p\Big(\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathbf{z})\Big)+\int_{\mathtt{a}}^{t}\frac{(\mathbf{z}-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathfrak{h}_1(\tau)d\tau=0.$$

So, the FBVP (2.5), (1.3) is equivalent to the problem:

$$\begin{cases}\mathfrak{D}^{\mathbf{k}_1}_{\mathtt{a}^+}\varpi_1(\mathbf{z})+\phi_q\left(\displaystyle\int_{\mathtt{a}}^{t}\frac{(\mathbf{z}-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathfrak{h}_1(\tau)d\tau\right)=0, \quad \mathbf{z}\in[\mathtt{a},\mathtt{b}]\\ \psi_1\varpi_1(\mathtt{a})-\psi_2\mathfrak{D}^{\mathbf{k}_2}_{\mathtt{a}^+}\varpi_1(\mathtt{a})=0,\\ \psi_3\varpi_1(\mathtt{b})+\psi_4\mathfrak{D}^{\mathbf{k}_3}_{\mathtt{a}^+}\varpi_1(\mathtt{b})=0.\end{cases}$$

Lemma 2.1 implies that the FBVP (2.5), (1.3) has a unique solution:

$$\varpi_1(\mathbf{z})=\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathfrak{h}_1(\tau)d\tau\right)d\varrho.$$

□

Lemma 2.3. [14] Suppose $(\mathtt{K}_1)$-$(\mathtt{K}_3)$ hold. Then $\mathcal{H}_1(\mathbf{z},\varrho)$ has the properties:

$$\begin{aligned}(i)&\ \ \mathcal{H}_1(\mathbf{z},\varrho)\geq 0,\ \forall\ \mathbf{z},\varrho\in[\mathtt{a},\mathtt{b}],\\ (ii)&\ \ \mathcal{H}_1(\mathbf{z},\varrho)\leq\mathcal{H}_1(\varrho,\varrho),\ \forall\ \mathbf{z},\varrho\in[\mathtt{a},\mathtt{b}],\\ (iii)&\ \ \mathcal{H}_1(\mathbf{z},\varrho)\geq\Psi_1\mathcal{H}_1(\varrho,\varrho),\ \forall\ \mathbf{z}\in I,\ \ \varrho\in[\mathtt{a},\mathtt{b}],\end{aligned}$$

where $I=\left[\frac{3\mathtt{a}+\mathtt{b}}{4},\frac{\mathtt{a}+3\mathtt{b}}{4}\right]$ and $\Psi_1=\left(\frac{3\mathtt{a}+\mathtt{b}}{4\mathtt{b}}\right)^{\mathbf{k}_1-1}$.

Denote

$$\left.\begin{aligned}\aleph_1&=\beta_1-\frac{\beta_2\Gamma(\mathbf{r}_1-1)\mathtt{a}^{-\mathbf{r}_2}}{\Gamma(\mathbf{r}_1-\mathbf{r}_2-1)},\\ \aleph_2&=\frac{\beta_2\Gamma(\mathbf{r}_1)\mathtt{a}^{-\mathbf{r}_2+1}}{\Gamma(\mathbf{r}_1-\mathbf{r}_2)}-\mathtt{a}\beta_1,\\ \widetilde{\aleph}_1&=\frac{\beta_4\Gamma(\mathbf{r}_1)\mathtt{b}^{\mathbf{r}_1-\mathbf{r}_3-1}}{\Gamma(\mathbf{r}_1-\mathbf{r}_3)}+\beta_3\mathtt{b}^{\mathbf{r}_1-1},\\ \widetilde{\aleph}_2&=\frac{\beta_4\Gamma(\mathbf{r}_1-1)\mathtt{b}^{\mathbf{r}_1-\mathbf{r}_3-2}}{\Gamma(\mathbf{r}_1-\mathbf{r}_3-1)}+\beta_3\mathtt{b}^{\mathbf{r}_1-2}.\end{aligned}\right\}\tag{2.7}$$

Lemma 2.4. Let $\Delta_2=\aleph_1\widetilde{\aleph}_1+\aleph_2\widetilde{\aleph}_2\neq 0$. For every $\mathtt{j}(\mathbf{z})\in\mathtt{C}[\mathtt{a},\mathtt{b}]$, the FBVP

$$\mathfrak{D}^{\mathbf{r}_1}_{\mathtt{a}^+}\varpi_2(\mathbf{z})+\mathtt{j}(\mathbf{z})=0,\ \mathbf{z}\in[\mathtt{a},\mathtt{b}],\tag{2.8}$$

$$\left.\begin{aligned}\beta_1\varpi_2(\mathtt{a})-\beta_2\mathfrak{D}^{\mathbf{r}_2}_{\mathtt{a}^+}\varpi_2(\mathtt{a})=0,\\ \beta_3\varpi_2(\mathtt{b})+\beta_4\mathfrak{D}^{\mathbf{r}_3}_{\mathtt{a}^+}\varpi_2(\mathtt{b})=0,\end{aligned}\right\}\tag{2.9}$$

has a unique solution $\varpi_2(\mathbf{z})=\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\mathbf{z},\varrho)\mathtt{j}(\varrho)d\varrho$, where $\mathcal{H}_2(\mathbf{z},\varrho)$ is given by

$$\mathcal{H}_2(\mathbf{z},\varrho)=\begin{cases}\mathcal{H}_{21}(\mathbf{z},\varrho), & \mathbf{z}\leq\varrho,\\ \mathcal{H}_{22}(\mathbf{z},\varrho), & \varrho\leq\mathbf{z},\end{cases}\tag{2.10}$$

$$\begin{aligned}\mathcal{H}_{21}(\mathbf{z},\varrho)=&\frac{1}{\Delta_2}\Big[\aleph_1\mathbf{z}^{\mathbf{r}_1-1}+\aleph_2\mathbf{z}^{\mathbf{r}_1-2}\Big]\left[\frac{\beta_3}{\Gamma(\mathbf{r}_1)}+\frac{\beta_4(\mathtt{b}-\varrho)^{-\mathbf{r}_3}}{\Gamma(\mathbf{r}_1-\mathbf{r}_3)}\right](\mathtt{b}-\varrho)^{\mathbf{r}_1-1},\\ \mathcal{H}_{22}(\mathbf{z},\varrho)=&\mathcal{H}_{21}(\mathbf{z},\varrho)-\frac{(\mathbf{z}-\varrho)^{\mathbf{r}_1-1}}{\Gamma(\mathbf{r}_1)},\ \text{ also } \aleph_1,\widetilde{\aleph}_1,\aleph_2 \text{ and } \widetilde{\aleph}_2 \text{ are given in (2.7).}\end{aligned}$$

Lemma 2.5. Let $\mathfrak{h}_2(\mathbf{z}) \in \mathtt{C}[\mathtt{a},\mathtt{b}]$. Then the FDEq

$$\mathfrak{D}^{\mathbf{s}_1}_{\mathtt{a}^+}\left[\phi_p\left(\mathfrak{D}^{\mathbf{r}_1}_{\mathtt{a}^+}\varpi_2(\mathbf{z})\right)\right] + \mathfrak{h}_2(\mathbf{z}) = 0, \quad \mathbf{z} \in [\mathtt{a},\mathtt{b}], \tag{2.11}$$

with (1.4) has a unique solution

$$\varpi_2(\mathbf{z}) = \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_2(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\mathfrak{h}_2(\tau)d\tau\right)d\varrho, \tag{2.12}$$

where $\mathcal{H}_2(\mathbf{z},\varrho)$ is given in (2.10).

Lemma 2.6. [14] Suppose $(\mathtt{K}_4)$-$(\mathtt{K}_6)$ hold. Then $\mathcal{H}_2(\mathbf{z},\varrho)$ has the properties:

$$\begin{aligned} &(i)\ \ \mathcal{H}_2(\mathbf{z},\varrho) \geq 0,\ \forall\ \mathbf{z},\varrho \in [\mathtt{a},\mathtt{b}],\\ &(ii)\ \ \mathcal{H}_2(\mathbf{z},\varrho) \leq \mathcal{H}_2(\varrho,\varrho),\ \forall\ \mathbf{z},\varrho \in [\mathtt{a},\mathtt{b}],\\ &(iii)\ \ \mathcal{H}_2(\mathbf{z},\varrho) \geq \Psi_2\mathcal{H}_2(\varrho,\varrho),\ \forall\ \mathbf{z} \in I,\ \ \varrho \in [\mathtt{a},\mathtt{b}], \end{aligned}$$

where $I = \left[\frac{3\mathtt{a}+\mathtt{b}}{4}, \frac{\mathtt{a}+3\mathtt{b}}{4}\right]$ and $\Psi_2 = \left(\frac{3\mathtt{a}+\mathtt{b}}{4\mathtt{b}}\right)^{\mathbf{r}_1-1}$.

Remark 2.7. $\mathcal{H}_1(\mathbf{z},\varrho) \geq \widetilde{\Psi}\mathcal{H}_1(\varrho,\varrho)$ and $\mathcal{H}_2(\mathbf{z},\varrho) \geq \widetilde{\Psi}\mathcal{H}_2(\varrho,\varrho)$, for all $(\mathbf{z},\varrho) \in I \times [\mathtt{a},\mathtt{b}]$, where $\widetilde{\Psi} = \min\left\{\Psi_1,\Psi_2\right\}$.

Theorem 2.8. [2] Let $\mathtt{K}$ be a cone in the real Banach space $\mathtt{B}$. Suppose α and γ are increasing, nonnegative continuous functionals on $\mathtt{K}$ and θ is nonnegative continuous functional on $\mathtt{K}$ with $\theta(0) = 0$ s.t., for some positive numbers Ξ_3 and Ξ_4, $\gamma(\varpi) \leq \theta(\varpi) \leq \alpha(\varpi)$ and $\|\varpi\| \leq \Xi_4\gamma(\varpi)$, for all $\varpi \in \overline{\mathtt{K}(\gamma,\Xi_3)}$. Suppose that there exist positive numbers Ξ_1 and Ξ_2 with $\Xi_1 < \Xi_2 < \Xi_3$ s.t. $\theta(\chi\varpi) \leq \chi\theta(\varpi)$, for all $0 \leq \chi \leq 1$ and $\varpi \in \partial\mathtt{K}(\theta,\Xi_2)$. Further, let $\mathtt{T} : \overline{\mathtt{K}(\gamma,\Xi_3)} \to \mathtt{K}$ be a completely continuous operator s.t.

(*i*) $\gamma(\mathtt{T}\varpi) > \Xi_3$, for all $\varpi \in \partial\mathtt{K}(\gamma,\Xi_3)$,

(*ii*) $\theta(\mathtt{T}\varpi) < \Xi_2$, for all $\in \partial\mathtt{K}(\theta,\Xi_2)$,

(*iii*) $\mathtt{K}(\alpha,\Xi_1) \neq \emptyset$ and $\alpha(\mathtt{T}\varpi) > \Xi_1$ for all $\varpi \in \partial\mathtt{K}(\alpha,\Xi_1)$.

Then, $\mathtt{T}$ has at least two fixed points $\varpi_1,\varpi_2 \in \overline{\mathtt{K}(\gamma,\Xi_3)}$ s.t. $\Xi_1 < \alpha(\varpi_1)$ with $\theta(\varpi_1) < \Xi_2$ and $\Xi_2 < \theta(\varpi_2)$ with $\gamma(\varpi_2) < \Xi_3$.

3 Main Results

We introduce the following notations for computational ease:

$$\mathtt{M}_1 = \max\left\{\begin{array}{l} \widetilde{\Psi}\phi_q\left(\dfrac{(\mathtt{b}-\mathtt{a})^{\mathbf{n}_1}}{4^{\mathbf{n}_1}\Gamma(\mathbf{n}_1+1)}\right)\displaystyle\int_{\varrho\in I}\mathcal{H}_1(\varrho,\varrho)d\varrho,\\ \\ \widetilde{\Psi}\phi_q\left(\dfrac{(\mathtt{b}-\mathtt{a})^{\mathbf{s}_1}}{4^{\mathbf{s}_1}\Gamma(\mathbf{s}_1+1)}\right)\displaystyle\int_{\varrho\in I}\mathcal{H}_2(\varrho,\varrho)d\varrho. \end{array}\right\} \tag{3.1}$$

$$\mathtt{M}_2 = \min\left\{\begin{array}{l} \phi_q\left(\dfrac{(\mathtt{b}-\mathtt{a})^{\mathbf{n}_1}}{4^{\mathbf{n}_1}\Gamma(\mathbf{n}_1+1)}\right)\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\varrho,\varrho)d\varrho,\\ \\ \phi_q\left(\dfrac{(\mathtt{b}-\mathtt{a})^{\mathbf{s}_1}}{4^{\mathbf{s}_1}\Gamma(\mathbf{s}_1+1)}\right)\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\varrho,\varrho)d\varrho. \end{array}\right\} \tag{3.2}$$

Consider the Banach space $\mathtt{B} = \mathtt{E} \times \mathtt{E}$, where $\mathtt{E} = \Big\{\varpi_1 : \varpi_1 \in \mathtt{C}[\mathtt{a}, \mathtt{b}]\Big\}$ be the Banach space equipped with the norm $\|(\varpi_1, \varpi_2)\| = \|\varpi_1\|_0 + \|\varpi_2\|_0$, for $(\varpi_1, \varpi_2) \in \mathtt{B}$ and the norm is specified by $\|\varpi_1\|_0 = \max_{i \in [\mathtt{a},\mathtt{b}]} |\varpi_1(\mathbf{z})|$. Define a cone $\mathtt{K} \subset \mathtt{B}$ by

$$\mathtt{K} = \Big\{(\varpi_1, \varpi_2) \in \mathtt{B} : \varpi_1(\mathbf{z}) \geq 0,\ \varpi_2(\mathbf{z}) \geq 0,\ \forall \mathbf{z} \in [\mathtt{a}, \mathtt{b}] \text{ and } \min_{\mathbf{z} \in I} [\varpi_1(\mathbf{z}) + \varpi_2(\mathbf{z})] \geq \widetilde{\Psi} \|(\varpi_1, \varpi_2)\|\Big\},$$

where $I = \left[\dfrac{3\mathtt{a} + \mathtt{b}}{4}, \dfrac{\mathtt{a} + 3\mathtt{b}}{4}\right]$ and $\widetilde{\Psi} = \min\Big\{\Psi_1, \Psi_2\Big\}$.

The system of FBVP (1.1)–(1.4) is well-known to be equivalent to:

$$\left.\begin{aligned} \varpi_1(\mathbf{z}) &= \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\mathbf{z}, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{n}_1 - 1}}{\Gamma(\mathbf{n}_1)} \mathtt{g}_1\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho, \\ \varpi_2(\mathbf{z}) &= \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_2(\mathbf{z}, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{s}_1 - 1}}{\Gamma(\mathbf{s}_1)} \mathtt{g}_2\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho. \end{aligned}\right\} \tag{3.3}$$

Let $\mathtt{T}_1, \mathtt{T}_2 : \mathtt{K} \to \mathtt{E}$ be the operators defined as

$$\begin{cases} \mathtt{T}_1(\varpi_1, \varpi_2)(\mathbf{z}) = \displaystyle\int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\mathbf{z}, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{n}_1 - 1}}{\Gamma(\mathbf{n}_1)} \mathtt{g}_1\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho, \\ \mathtt{T}_2(\varpi_1, \varpi_2)(\mathbf{z}) = \displaystyle\int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_2(\mathbf{z}, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{s}_1 - 1}}{\Gamma(\mathbf{s}_1)} \mathtt{g}_2\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho. \end{cases}$$

Consider the operator $\mathtt{T} : \mathtt{K} \to \mathtt{B}$, which is defined as

$$\mathtt{T}\big(\varpi_1, \varpi_2\big)(\mathbf{z}) = \Big(\mathtt{T}_1(\varpi_1, \varpi_2)(\mathbf{z}), \mathtt{T}_2(\varpi_1, \varpi_2)(\mathbf{z})\Big), \text{ for } (\varpi_1, \varpi_2) \in \mathtt{B}. \tag{3.4}$$

This is self-evident that the solution of the FBVP (1.1)–(1.4) is a fixed point of $\mathtt{T}$. Two fixed points of $\mathtt{T}$ are sought.

Lemma 3.1. If $(\mathtt{K}_1)$-$(\mathtt{K}_7)$ hold. Then $\mathtt{T} : \mathtt{K} \to \mathtt{K}$ is completely continuous.

Proof . One can clearly establish that the operator $\mathtt{T}$ is completely continuous with standard arguments, and we only need to verify $\mathtt{T}(\mathtt{K}) \subset \mathtt{K}$. Let $(\varpi_1, \varpi_2) \in \mathtt{K}$. Clearly, $\mathtt{T}_1(\varpi_1, \varpi_2)(\mathbf{z}) \geq 0$ and $\mathtt{T}_2(\varpi_1, \varpi_2)(\mathbf{z}) \geq 0$, for $\mathbf{z} \in [\mathtt{a}, \mathtt{b}]$. Also, for $(\varpi_1, \varpi_2) \in \mathtt{K}$,

$$\begin{aligned} \|\mathtt{T}_1(\varpi_1, \varpi_2)\|_0 &\leq \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\varrho, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{n}_1 - 1}}{\Gamma(\mathbf{n}_1)} \mathtt{g}_1\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho, \\ \|\mathtt{T}_2(\varpi_1, \varpi_2)\|_0 &\leq \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_2(\varrho, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{s}_1 - 1}}{\Gamma(\mathbf{s}_1)} \mathtt{g}_2\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho, \end{aligned}$$

and

$$\begin{aligned} \min_{\mathbf{z} \in I} \mathtt{T}_1(\varpi_1, \varpi_2)(\mathbf{z}) &= \min_{\mathbf{z} \in I} \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\mathbf{z}, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{n}_1 - 1}}{\Gamma(\mathbf{n}_1)} \mathtt{g}_1\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho \\ &\geq \widetilde{\Psi} \int_{\mathtt{a}}^{\mathtt{b}} \mathcal{H}_1(\varrho, \varrho) \phi_q\bigg(\int_{\mathtt{a}}^{\varrho} \frac{(\varrho - \tau)^{\mathbf{n}_1 - 1}}{\Gamma(\mathbf{n}_1)} \mathtt{g}_1\big(\tau, \varpi_1(\tau), \varpi_2(\tau)\big) d\tau\bigg) d\varrho \\ &\geq \widetilde{\Psi} \|\mathtt{T}_1(\varpi_1, \varpi_2)\|_0. \end{aligned}$$

Similarly, $\min_{\mathbf{z} \in I} \mathtt{T}_2(\varpi_1, \varpi_2)(\mathbf{z}) \geq \widetilde{\Psi} \|\mathtt{T}_2(\varpi_1, \varpi_2)\|_0$. Therefore,

$$\begin{aligned} \min_{\mathbf{z} \in I} \left[\sum_{i=1}^{2} \mathtt{T}_i\big(\varpi_1, \varpi_2\big)(\mathbf{z})\right] &\geq \widetilde{\Psi} \left[\sum_{i=1}^{2} \big\|\mathtt{T}_i(\varpi_1, \varpi_2)\big\|_0\right] \\ &= \widetilde{\Psi} \big\|(\mathtt{T}_1(\varpi_1, \varpi_2), \mathtt{T}_2(\varpi_1, \varpi_2))\big\| \\ &= \widetilde{\Psi} \big\|\mathtt{T}(\varpi_1, \varpi_2)\big\|. \end{aligned}$$

Thus, $\mathtt{T}(\mathtt{K}) \subset \mathtt{K}$. Hence $\mathtt{T}$ is a completely continuous operator, as per standard arguments relying on the Arzela–Ascoli theorem. □

Next let the non-negative, increasing, continuous functionals γ, θ and α on the cone $\mathtt{K}$ by

$$\left.\begin{aligned} \gamma(\varpi_1,\varpi_2) &= \min_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big],\\ \theta(\varpi_1,\varpi_2) &= \max_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big],\\ \alpha(\varpi_1,\varpi_2) &= \max_{\mathbf{z}\in[\mathtt{a},\mathtt{b}]}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big], \end{aligned}\right\} \tag{3.5}$$

and $\mathtt{K}(\psi,\Xi_3) = \{(\varpi_1,\varpi_2)\in\mathtt{K} : \psi(\varpi_1,\varpi_2) < \Xi_3\}$.

Theorem 3.2. Suppose that ($\mathtt{K}_1$)-($\mathtt{K}_7$) hold and there exist $0 < \Xi_1 < \Xi_2 < \Xi_3$ s.t. $\mathbf{g}_i$ for $i = \overline{1,2}$ satisfies:

$$(\mathtt{Q}_1)\ \mathbf{g}_i\big(\mathbf{z},\varpi_1(\mathbf{z}),\varpi_2(\mathbf{z})\big) > \phi_p\left(\frac{\Xi_1}{2\mathtt{M}_1}\right),\ \mathbf{z}\in I \textit{ and } (\varpi_1,\varpi_2)\in\Big[\widetilde{\Psi}\Xi_1,\Xi_1\Big],$$

$$(\mathtt{Q}_2)\ \mathbf{g}_i\big(\mathbf{z},\varpi_1(\mathbf{z}),\varpi_2(\mathbf{z})\big) < \phi_p\left(\frac{\Xi_2}{2\mathtt{M}_2}\right),\ \mathbf{z}\in [\mathtt{a},\mathtt{b}] \textit{ and } (\varpi_1,\varpi_2)\in\left[0,\frac{\Xi_2}{\widetilde{\Psi}}\right],$$

$$(\mathtt{Q}_3)\ \mathbf{g}_i\big(\mathbf{z},\varpi_1(\mathbf{z}),\varpi_2(\mathbf{z})\big) > \phi_p\left(\frac{\Xi_3}{2\mathtt{M}_1}\right),\ \mathbf{z}\in I \textit{ and } (\varpi_1,\varpi_2)\in\left[\Xi_3,\frac{\Xi_3}{\widetilde{\Psi}}\right].$$

Then the FBVP (1.1)–(1.4) has at least two positive solutions (ϖ_1,ϖ_2) and $(\widehat{\varpi}_1,\widehat{\varpi}_2)$ s.t.

$$\begin{cases} \Xi_1 < \max\limits_{\mathbf{z}\in[\mathtt{a},\mathtt{b}]}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big] \textit{ with } \max\limits_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big] < \Xi_2,\\ \Xi_2 < \max\limits_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\widehat{\varpi}_i(\mathbf{z})\Big] \textit{ with } \min\limits_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\widehat{\varpi}_i(\mathbf{z})\Big] < \Xi_3. \end{cases}$$

Proof . We seek two fixed points $(\varpi_1,\varpi_2),(\widehat{\varpi}_1,\widehat{\varpi}_2)\in\mathtt{K}$ of $\mathtt{T}$ defined by (3.4). From Lemmas 2.2, 2.3 and 3.1, we have $\mathtt{T}(\mathtt{K})\subset\mathtt{K}$. Also, $\mathtt{T}$ is completely continuous. For every $(\varpi_1,\varpi_2)\in\mathtt{K}$, by (3.5) one has

$$\gamma(\varpi_1,\varpi_2) \le \theta(\varpi_1,\varpi_2) \le \alpha(\varpi_1,\varpi_2), \tag{3.6}$$

$$\left.\begin{aligned} \big\|(\varpi_1,\varpi_2)\big\| &\le \left(\frac{1}{\widetilde{\Psi}}\right)\min_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big] = \left(\frac{1}{\widetilde{\Psi}}\right)\gamma(\varpi_1,\varpi_2)\\ &\le \left(\frac{1}{\widetilde{\Psi}}\right)\theta(\varpi_1,\varpi_2) \le \left(\frac{1}{\widetilde{\Psi}}\right)\alpha(\varpi_1,\varpi_2). \end{aligned}\right\} \tag{3.7}$$

For each $(\varpi_1,\varpi_2)\in\mathtt{K}$, (3.6), (3.7) imply $\gamma(\varpi_1,\varpi_2)\le\theta(\varpi_1,\varpi_2)\le\alpha(\varpi_1,\varpi_2)$ and $\big\|(\varpi_1,\varpi_2)\big\| \le \left(\frac{1}{\widetilde{\Psi}}\right)\alpha(\varpi_1,\varpi_2)$. For each $(\varpi_1,\varpi_2)\in\mathtt{K},\chi\in[0,1]$, we have

$$\begin{cases} \theta\big(\chi\varpi_1,\chi\varpi_2\big) = \max\limits_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\chi\varpi_i(\mathbf{z})\Big]\\ = \chi\max\limits_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big] = \chi\theta\big(\varpi_1,\varpi_2\big). \end{cases}$$

It's indeed evident that $\theta\big(0,0\big)=0$. Lets now verify that the essential conditions of Theorem 2.8 are met. Then, we will ensure that condition (*iii*) of Theorem 2.8 is obeyed. Because $\big(0,0\big)\in\mathtt{K}$ and $\mathtt{K}(\alpha,\Xi_1)\neq\emptyset,\Xi_1>0$. For

$\big(\varpi_1,\varpi_2\big)\in\partial\mathtt{K}(\alpha,\Xi_1)$, we have $\widetilde{\Psi}\Xi_1=\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big]\le\Big[\sum_{i=1}^{2}\big\|\varpi_i\big\|\Big]=\Xi_1,\ \mathbf{z}\in I$. Therefore,

$$
\begin{aligned}
\alpha\big(\mathtt{T}(\varpi_1,\varpi_2)(\mathbf{z})\big) &= \max_{\mathbf{z}\in[\mathtt{a},\mathtt{b}]}\mathtt{T}\big(\varpi_1,\varpi_2\big)(\mathbf{z})\\
&= \max_{\mathbf{z}\in[\mathtt{a},\mathtt{b}]}\left\{\begin{array}{l}\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathtt{g}_1\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\right)d\varrho+\\ \displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\mathtt{g}_2\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\right)d\varrho\end{array}\right.\\
&\ge \widetilde{\Psi}\left\{\begin{array}{l}\displaystyle\int_{\varrho\in I}\mathcal{H}_1(\varrho,\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\phi_p\left(\frac{\Xi_1}{2\mathtt{M}_1}\right)d\tau\right)d\varrho+\\ \displaystyle\int_{\varrho\in I}\mathcal{H}_2(\varrho,\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\phi_p\left(\frac{\Xi_1}{2\mathtt{M}_1}\right)d\tau\right)d\varrho\end{array}\right.\\
&\ge \left\{\begin{array}{l}\displaystyle\left(\frac{\Xi_1}{2\mathtt{M}_1}\right)\widetilde{\Psi}\phi_q\left(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{n}_1}}{4^{\mathbf{n}_1}\Gamma(\mathbf{n}_1+1)}\right)\int_{\varrho\in I}\mathcal{H}_1(\varrho,\varrho)d\varrho+\\ \displaystyle\left(\frac{\Xi_1}{2\mathtt{M}_1}\right)\widetilde{\Psi}\phi_q\left(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{s}_1}}{4^{\mathbf{s}_1}\Gamma(\mathbf{s}_1+1)}\right)\int_{\varrho\in I}\mathcal{H}_2(\varrho,\varrho)d\varrho\end{array}\right.\\
&= \frac{\Xi_1}{2}+\frac{\Xi_1}{2}=\Xi_1
\end{aligned}
$$

employing supposition $(\mathtt{Q}_1)$. We will check that condition (*ii*) of Theorem 2.8 is met. As $\big(\varpi_1,\varpi_2\big)\in\partial\mathtt{K}(\theta,\Xi_2)$, from (3.7) we have that

$$0\le\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big]\le\Big[\sum_{i=1}^{2}\big\|\varpi_i\big\|\Big]\le\left(\frac{\Xi_2}{\widetilde{\Psi}}\right),\ \mathbf{z}\in[\mathtt{a},\mathtt{b}].$$

Therefore,

$$
\begin{aligned}
\theta\big(\mathtt{T}(\varpi_1,\varpi_2)(\mathbf{z})\big) &= \max_{\mathbf{z}\in I}\mathtt{T}\big(\varpi_1,\varpi_2\big)(\mathbf{z})\\
&= \max_{\mathbf{z}\in I}\left\{\begin{array}{l}\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathtt{g}_1\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\right)d\varrho+\\ \displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\mathbf{z},\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\mathtt{g}_2\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\right)d\varrho\end{array}\right.\\
&\le \left\{\begin{array}{l}\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\varrho,\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\phi_p\left(\frac{\Xi_2}{2\mathtt{M}_2}\right)d\tau\right)d\varrho+\\ \displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\varrho,\varrho)\phi_q\left(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\phi_p\left(\frac{\Xi_2}{2\mathtt{M}_2}\right)d\tau\right)d\varrho\end{array}\right.\\
&\le \left\{\begin{array}{l}\displaystyle\left(\frac{\Xi_2}{2\mathtt{M}_2}\right)\phi_q\left(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{n}_1}}{4^{\mathbf{n}_1}\Gamma(\mathbf{n}_1+1)}\right)\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\varrho,\varrho)d\varrho+\\ \displaystyle\left(\frac{\Xi_2}{2\mathtt{M}_2}\right)\phi_q\left(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{s}_1}}{4^{\mathbf{s}_1}\Gamma(\mathbf{s}_1+1)}\right)\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\varrho,\varrho)d\varrho\end{array}\right.\\
&= \frac{\Xi_2}{2}+\frac{\Xi_2}{2}=\Xi_2
\end{aligned}
$$

using hypothesis $(\mathtt{Q}_2)$. Furthermore, we shall check the condition (*i*) of Theorem 2.8 is met via hypothesis $(\mathtt{Q}_3)$. Since

$\big(\varpi_1,\varpi_2\big)\in\partial\mathtt{K}(\theta,\Xi_3)$, from (3.7) one has $\min_{\mathbf{z}\in I}\Big[\sum_{i=1}^{2}\varpi_i(\mathbf{z})\Big]=\Xi_3$ and $\Xi_3\le\Big[\sum_{i=1}^{2}\big\|\varpi_i\big\|\Big]\le\Big(\frac{\Xi_3}{\widetilde{\Psi}}\Big)$. Then

$$
\begin{aligned}
\gamma\big(\mathtt{T}(\varpi_1,\varpi_2)(\mathbf{z})\big) &= \max_{\mathbf{z}\in I}\mathtt{T}\big(\varpi_1,\varpi_2\big)(\mathbf{z})\\
&= \max_{\mathbf{z}\in I}\left\{\begin{array}{l}\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\mathbf{z},\varrho)\phi_q\bigg(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\mathtt{g}_1\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\bigg)d\varrho+\\ \displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\mathbf{z},\varrho)\phi_q\bigg(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\mathtt{g}_2\big(\tau,\varpi_1(\tau),\varpi_2(\tau)\big)d\tau\bigg)d\varrho\end{array}\right.\\
&\ge \widetilde{\Psi}\left\{\begin{array}{l}\displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\varrho,\varrho)\phi_q\bigg(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{n}_1-1}}{\Gamma(\mathbf{n}_1)}\phi_p\Big(\frac{\Xi_3}{2\mathtt{M}_1}\Big)d\tau\bigg)d\varrho+\\ \displaystyle\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\varrho,\varrho)\phi_q\bigg(\int_{\mathtt{a}}^{\varrho}\frac{(\varrho-\tau)^{\mathbf{s}_1-1}}{\Gamma(\mathbf{s}_1)}\phi_p\Big(\frac{\Xi_3}{2\mathtt{M}_1}\Big)d\tau\bigg)d\varrho\end{array}\right.\\
&\le \left\{\begin{array}{l}\displaystyle\Big(\frac{\Xi_3}{2\mathtt{M}_1}\Big)\widetilde{\Psi}\phi_q\bigg(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{n}_1}}{4^{\mathbf{n}_1}\Gamma(\mathbf{n}_1+1)}\bigg)\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_1(\varrho,\varrho)d\varrho+\\ \displaystyle\Big(\frac{\Xi_3}{2\mathtt{M}_1}\Big)\widetilde{\Psi}\phi_q\bigg(\frac{(\mathtt{b}-\mathtt{a})^{\mathbf{s}_1}}{4^{\mathbf{s}_1}\Gamma(\mathbf{s}_1+1)}\bigg)\int_{\mathtt{a}}^{\mathtt{b}}\mathcal{H}_2(\varrho,\varrho)d\varrho\end{array}\right.\\
&= \frac{\Xi_3}{2}+\frac{\Xi_3}{2}=\Xi_3.
\end{aligned}
$$

Thus, all of the requirements of Theorem 2.8 are met. So, Theorem 3.2 is successfully employed to ensure that the system of p-Laplacian FBVP (1.1)–(1.4) has at least two positive solutions $(\varpi_1,\varpi_2),(\widehat{\varpi}_1,\widehat{\varpi}_2)$. □

4 Illustration

We provide a suitable example to support the use of Theorem 3.2. Let $p=2$, $\mathbf{k}_1=1.8$, $\mathbf{n}_1=0.6,\mathbf{k}_2=0.4,\mathbf{k}_3=0.5,\mathbf{r}_1=1.9,\mathbf{s}_1=0.7,\mathbf{r}_2=0.5,\mathbf{r}_3=0.6$, $\psi_1=\frac{9\pi^4}{175},\psi_2=1+\sqrt{5},\psi_3=\frac{7}{\pi}$, $\psi_4=\frac{6}{\log\pi}$, $\beta_1=\frac{103}{20},\beta_2=3e-5,\beta_3=\frac{43}{20},\beta_4=\frac{3\pi e}{5}$.

Consider a coupled system of p-Laplacian FBVP,

$$\mathfrak{D}_{1^+}^{0.6}\Big[\phi_p\Big(\mathfrak{D}_{1^+}^{1.8}\varpi_1(\mathbf{z})\Big)\Big]+\mathtt{g}_1\big(\mathbf{z},\varpi_1,\varpi_2\big)=0, \tag{4.1}$$

$$\mathfrak{D}_{1^+}^{0.7}\Big[\phi_p\Big(\mathfrak{D}_{1^+}^{1.9}\varpi_2(\mathbf{z})\Big)\Big]+\mathtt{g}_2\big(\mathbf{z},\varpi_1,\varpi_2\big)=0, \tag{4.2}$$

$$\left.\begin{aligned}\frac{9\pi^4}{175}\varpi_1(1)-\big(1+\sqrt{5}\big)\mathfrak{D}_{1^+}^{0.4}\varpi_1(1)&=0,\\ \frac{7}{\pi}\varpi_1(2)+\frac{6}{\log\pi}\mathfrak{D}_{1^+}^{0.5}\varpi_1(2)&=0,\\ \phi_p\Big(\mathfrak{D}_{1^+}^{1.8}\varpi_1(1)\Big)&=0,\end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned}\frac{103}{20}\varpi_2(1)-\big(3e-5\big)\mathfrak{D}_{1^+}^{0.5}\varpi_2(1)&=0,\\ \frac{43}{20}\varpi_2(2)+\frac{3\pi e}{5}\mathfrak{D}_{1^+}^{0.6}\varpi_2(2)&=0,\\ \phi_p\Big(\mathfrak{D}_{1^+}^{1.9}\varpi_2(1)\Big)&=0,\end{aligned}\right\} \tag{4.4}$$

where

$$\left\{\begin{aligned}\mathbf{g}_1(\mathbf{z},\varpi_1,\varpi_2)&=\frac{\sqrt{5}\Big[2\big(\varpi_1+\varpi_2\big)+15\pi\Big]^{\frac{21}{4}}}{6\pi^4\Big[5(\varpi_1+\varpi_2)^2+7215\Big]}-\sqrt[3]{110}e^{\varpi_1}\mathbf{z}^{\frac{17}{4}},\\ \mathbf{g}_2(\mathbf{z},\varpi_1,\varpi_2)&=\frac{37\pi\Big[\big(\varpi_1+\varpi_2\big)^2+4+\sqrt{17}\Big]^{\frac{41}{8}}}{(63+7\sqrt{78}\pi)[\big(\varpi_1+\varpi_2\big)^2+1067\pi]}-\frac{48}{\pi^2}e^{\varpi_2}\mathbf{z}^{\frac{21}{5}}.\end{aligned}\right.$$

By direct calculations,

$$\begin{cases} \wp_1 = \dfrac{9\pi^4}{175} - \dfrac{(1+\sqrt{5})\Gamma(1.8-1)(1)^{-0.4}}{\Gamma(1.8-0.4-1)} \approx 3.311118, \\ \wp_2 = \dfrac{(1+\sqrt{5})\Gamma(1.8)(1)^{-0.4+1}}{\Gamma(1.8-0.4)} - (1)\dfrac{9\pi^4}{175} \approx -1.612632, \\ \widetilde{\wp}_1 = \left(\dfrac{7}{\pi}\right)(2)^{1.8-1} + \dfrac{\left(\dfrac{6}{\log\pi}\right)\Gamma(1.8)(2)^{1.8-0.5-1}}{\Gamma(1.8-0.5)} \approx 10.576241, \\ \widetilde{\wp}_2 = \left(\dfrac{7}{\pi}\right)(2)^{1.8-2} + \dfrac{\left(\dfrac{6}{\log\pi}\right)\Gamma(1.8-1)(2)^{1.8-0.5-2}}{\Gamma(1.8-0.5-1)} \approx 3.195379. \end{cases}$$

$\Delta_1 \approx 29.866217$, $\Psi_1 = \left(\dfrac{3(1)+2}{4(2)}\right)^{1.8-1} \approx 0.6866003$,

$$\begin{cases} \aleph_1 = \left(\dfrac{103}{20}\right) - \dfrac{(3e-5)\Gamma(1.9-1)(1)^{-0.5}}{\Gamma(1.9-0.5-1)} \approx 5.149985, \\ \aleph_2 = \dfrac{(3e-5)\Gamma(1.9)(1)^{-0.5+1}}{\Gamma(1.9-0.5)} - (1)\left(\dfrac{103}{20}\right) \approx -5.149967, \\ \widetilde{\aleph}_1 = \dfrac{\left(\frac{3\pi e}{5}\right)\Gamma(1.9)(2)^{1.9-0.6-1}}{\Gamma(1.9-0.6)} + \left(\dfrac{43}{20}\right)(2)^{1.9-1} \approx 10.772151, \\ \widetilde{\aleph}_2 = \dfrac{\left(\frac{3\pi e}{5}\right)\Gamma(1.9-1)(2)^{1.9-0.6-2}}{\Gamma(1.9-0.6-1)} + \left(\dfrac{43}{20}\right)(2)^{1.9-2} \approx 3.132705, \end{cases}$$

$\Delta_2 \approx 39.343088$, $\Psi_2 = \left(\dfrac{3(1)+2}{4(2)}\right)^{1.9-1} \approx 0.655076$, $\widetilde{\Psi} = \min\Big\{\Psi_1, \Psi_2\Big\} = \min\Big\{0.6866003, 0.655076\Big\} = 0.655076$, $p = 2$, $I = \left[\dfrac{5}{4}, \dfrac{7}{4}\right]$,

$$\mathtt{M}_1 = \max\left\{ \begin{array}{l} \widetilde{\Psi}\phi_q\left(\dfrac{(2-1)^{0.6}}{4^{0.6}\Gamma(0.6+1)}\right)\displaystyle\int_{\varrho\in I} \mathcal{H}_1(\varrho,\varrho)d\varrho, \\ \widetilde{\Psi}\phi_q\left(\dfrac{(2-1)^{0.7}}{4^{0.7}\Gamma(0.7+1)}\right)\displaystyle\int_{\varrho\in I} \mathcal{H}_2(\varrho,\varrho)d\varrho \end{array} \right\} \approx 0.098518,$$

$$\mathtt{M}_2 = \min\left\{ \begin{array}{l} \phi_q\left(\dfrac{(2-1)^{0.6}}{4^{0.6}\Gamma(0.6+1)}\right)\displaystyle\int_1^2 \mathcal{H}_1(\varrho,\varrho)d\varrho, \\ \phi_q\left(\dfrac{(2-1)^{0.7}}{4^{0.7}\Gamma(0.7+1)}\right)\displaystyle\int_1^2 \mathcal{H}_2(\varrho,\varrho)d\varrho \end{array} \right\} \approx 0.120812.$$

Choosing $\Xi_1 = 3, \Xi_2 = 15$ and $\Xi_3 = 30$, then $0 < \Xi_1 < \Xi_2 < \Xi_3$ and $\mathbf{g}_i$ for $i = \overline{1,2}$ satisfies:

$$\begin{cases} \bullet\ \mathbf{g}_i\big(\mathbf{z}, \varpi_1, \varpi_2\big) > \phi_p\left(\dfrac{\Xi_1}{2\mathtt{M}_1}\right) \approx 15.225644,\ \mathbf{z} \in \left[\dfrac{5}{4}, \dfrac{7}{4}\right] \text{ and } (\varpi_1, \varpi_2) \in [1.96, 3], \\ \bullet\ \mathbf{g}_i\big(\mathbf{z}, \varpi_1, \varpi_2\big) < \phi_p\left(\dfrac{\Xi_2}{2\mathtt{M}_2}\right) \approx 62.079925,\ \mathbf{z} \in [1,2] \text{ and } (\varpi_1, \varpi_2) \in [0, 22.89], \\ \bullet\ \mathbf{g}_i\big(\mathbf{z}, \varpi_1, \varpi_2\big) > \phi_p\left(\dfrac{\Xi_3}{2\mathtt{M}_1}\right) \approx 152.256448,\ \mathbf{z} \in \left[\dfrac{5}{4}, \dfrac{7}{4}\right] \text{ and } (\varpi_1, \varpi_2) \in [30, 45.79]. \end{cases}$$

Then, all the conditions of Theorem 3.2 are satisfied. Therefore, it follows from Theorem 2.8, the p-Laplacian FBVP (4.1)-(4.4) has at least two positive solutions.

5 Conclusion

In this paper, the existence of at least two positive solutions for a coupled system of p-Laplacian FBVPs are explored by applying Avery–Henderson functional fixed point theorem. The results of this paper are essentially new in the sense that they are considered within the fractional calclus platform and in the context of p-Laplacian Operators. The fundamental requirements for overcoming the problem's positivity while also adapting to methodological obstacles on the kernel were discovered.

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