

The existence of a solution to more general proportional forms of fractional integrals via a measure of noncompactness

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Abstract

A fixed point theorem is proved using a newly constructed contraction operator in this article, and the solvability of a more general type of fractional integrals based here on the proportional derivative is analyzed. We also use suitable examples to illustrate our findings.

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1 Introduction

Fractional integral equations play a decisive role in real-world problems. The importance of fractional order integral equations has gained much research interest. The concept of an MNC is important in fixed point theory. Kuratowski [23] pioneered the idea of an MNC. Using the idea of an MNC, Darbo [12] established a result proving the presence of a fixed point for the so-called condensing operators in 1955. Fixed point theory and the MNC have numerous applications in analyzing various integral equations found in a wide range of real-world problems (see [3, 18, 14, 15, 17, 19, 20, 25, 13]). This theorem was highly valuable in establishing the solvability of several kinds of differential and integral equations ([6, 7, 8, 14, 16, 30], for example).

This article aims to generalize the fixed-point theorem of Darbo and apply this theorem in the control of the solvability of a fractional integral equation.

Let $(\mathfrak{J}, \|\cdot\|)$ be a real Banach space and $B(\theta, r) = \{z \in \mathfrak{J} : \|z - \theta\| \leq r\}$. If $\mathfrak{E} (\neq \emptyset) \subseteq \mathfrak{J}$. Also, $\bar{\mathfrak{E}}$ and $\text{Conv}\mathfrak{E}$ represent the closure and convex closure of \mathfrak{E} . Furthermore, let

- $\mathfrak{M}_{\mathfrak{J}} =$ The collection of all non-empty and bounded subsets of \mathfrak{J} ,
- $\mathfrak{N}_{\mathfrak{J}} =$ The collection of all relatively compact sets,
- $\mathbb{R} = (-\infty, \infty)$,
and

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- $\mathbb{R}_+ = [0, \infty)$.

The definition of an MNC is as follows: [9].

Definition 1.1. A function $\Omega : \mathfrak{M}_3 \rightarrow [0, \infty)$ is said to be an MNC in \mathfrak{J} if it fulfills axioms:

- (i) for all $\mathfrak{E} \in \mathfrak{M}_3$, $\Omega(\mathfrak{E}) = 0$ gives \mathfrak{E} is relatively compact.
- (ii) $\ker \Omega = \{\mathfrak{E} \in \mathfrak{M}_3 : \Omega(\mathfrak{E}) = 0\} \neq \emptyset$ and $\ker \Omega \subset \mathfrak{N}_3$.
- (iii) $\mathfrak{E} \subseteq \mathfrak{E}_1 \implies \Omega(\mathfrak{E}) \leq \Omega(\mathfrak{E}_1)$.
- (iv) $\Omega(\bar{\mathfrak{E}}) = \Omega(\mathfrak{E})$.
- (v) $\Omega(\text{Conv}\mathfrak{E}) = \Omega(\mathfrak{E})$.
- (vi) $\Omega(\chi\mathfrak{E} + (1-\chi)\mathfrak{E}_1) \leq \chi\Omega(\mathfrak{E}) + (1-\chi)\Omega(\mathfrak{E}_1)$ for $\chi \in [0, 1]$.
- (vii) if $\mathfrak{E}_c \in \mathfrak{M}_3$, $\mathfrak{E}_c = \bar{\mathfrak{E}}_c$, $\mathfrak{E}_{c+1} \subset \mathfrak{E}_c$ for $c = 1, 2, 3, \dots$ and $\lim_{c \rightarrow \infty} \Omega(\mathfrak{E}_c) = 0$ then $\bigcap_{c=1}^{\infty} \mathfrak{E}_c \neq \emptyset$.

The family $\ker\Omega$ is said to be the *kernel of measure* Ω . Since $\Omega(\mathfrak{E}_\infty) \leq \Omega(\mathfrak{E}_c)$, $\Omega(\mathfrak{E}_\infty) = 0$. So, $\mathfrak{E}_\infty = \bigcap_{c=1}^{\infty} \mathfrak{E}_c \in \ker\Omega$.

Some important theorems and definitions

The following are some fundamental theorems to recall:

Theorem 1.2. (Schauder [1]) Let \mathfrak{U} be a non-empty, closed and convex subset of a Banach Space \mathfrak{J} . Then every compact continuous map $\mathfrak{G} : \mathfrak{U} \rightarrow \mathfrak{U}$ has at least one fixed point.

Theorem 1.3. (Darbo[12]) Let \mathfrak{U} be a non-empty, bounded, closed and convex (NBCC) subset of a Banach Space \mathfrak{J} . Let $\mathfrak{G} : \mathfrak{U} \rightarrow \mathfrak{U}$ be a continuous mapping and there is a constant $\chi \in [0, 1)$ such that

$$\Omega(\mathfrak{G}\mathfrak{B}) \leq \chi\Omega(\mathfrak{B}), \quad \mathfrak{B} \subseteq \mathfrak{U}.$$

Then \mathfrak{G} has a fixed point.

The following related concepts are needed to establish an extension of Darbo's fixed point theorem:

Definition 1.4. ([26]) Let $\Lambda_1, \Lambda_2 : [0, \infty) \rightarrow \mathbb{R}$ be the two functions. Then the pair of maps (Λ_1, Λ_2) is called a pair of shifting distance functions, if it satisfies following conditions:

1. For $x, y \in [0, \infty)$ if $\Lambda_1(x) \leq \Lambda_2(y)$ then $x \leq y$.
2. For $x_n, y_n \in [0, \infty)$ such that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = z$, if $\Lambda_1(x_n) \leq \Lambda_2(y_n) \forall n$ then $z = 0$.

We denote by Λ a pair (Λ_1, Λ_2) of shifting distance functions.

As examples, we put $\Lambda_1(x) = x$, $\Lambda_2(x) = \epsilon x$, $x \geq 0$ and $\epsilon \in [0, 1)$. They are obviously a pair of shifting distance functions.

Definition 1.5. [2] A continuous function $g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is a function of \mathcal{C} - class if subsequent axioms hold true:

- (1) $g(m, n) \leq m$,
- (2) $g(m, n) = m$ implies that either $m = 0$ or $n = 0$. Also $g(0, 0) = 0$. A \mathcal{C} - class function is symbolized by \mathcal{C} .

For example,

- (1) $g(m, n) = m - n$,
- (2) $g(m, n) = am$, $0 < a < 1$.

Definition 1.6. [22] A function $\xi : [0, \infty) \rightarrow [0, \infty)$ is an alternating distance function if:

- (1) $\xi(x) = 0$ if and only if $x = 0$.
- (2) ξ is continuous and increasing.

We use Ξ to denote this class of functions. For example, $\xi(x) = (1 - b)x$, $0 \leq b < 1$.

Definition 1.7. [2] A continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ is an ultra altering distance function if $\phi(0) \geq 0$ and $\phi(t) > 0$, $t > 0$.

We use Φ to denote this class of functions.

Definition 1.8. A continuous function $h : [0, \infty) \rightarrow [0, \infty)$ is a function of \mathcal{A} class if $h(x) > x$, $x \in (0, \infty)$. Also $h(0) = 0$.

For example, $h(x) = \bar{m}x$, $\bar{m} > 1$.

Definition 1.9. Let $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous and non-decreasing mapping of \mathcal{B} class if $\gamma(t) = t$, $t \geq 0$.

2 Main Results

Theorem 2.1. Let \mathbb{U} be a NBCC subset of a Banach space \mathfrak{Z} . Also, let $\mathcal{T} : \mathbb{U} \rightarrow \mathbb{U}$ be continuous mapping with

$$\Lambda_1[h[\xi\{\mu(\mathcal{T}\Omega) + \gamma(\mu(\mathcal{T}\Omega))\}]] \leq \Lambda_2[g[\xi\{\mu(\Omega) + \gamma(\mu(\Omega))\}, \phi\{\mu(\Omega) + \gamma(\mu(\Omega))\}]] \quad (2.1)$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC and $(\Lambda_1, \Lambda_2) \in \Lambda$, $\phi \in \Phi$, $\xi \in \Xi$, $g \in \mathcal{C}$, $h \in \mathcal{A}$ and $\gamma \in \mathcal{B}$. Then \mathcal{T} has at least one fixed point in \mathbb{U} .

Proof . Let us create a sequence $\{\mathbb{U}_p\}_{p=1}^\infty$ with $\mathbb{U}_1 = \mathbb{U}$ and $\mathbb{U}_{p+1} = \text{Conv}(\mathcal{T}\mathbb{U}_p)$ for $p \in \mathbb{N}$. Also $\mathcal{T}\mathbb{U}_1 = \mathcal{T}\mathbb{U} \subseteq \mathbb{U} = \mathbb{U}_1$, $\mathbb{U}_2 = \text{Conv}(\mathcal{T}\mathbb{U}_1) \subseteq \mathbb{U} = \mathbb{U}_1$. Continuing in the similar manner gives $\mathbb{U}_1 \supseteq \mathbb{U}_2 \supseteq \mathbb{U}_3 \supseteq \dots \supseteq \mathbb{U}_p \supseteq \mathbb{U}_{p+1} \supseteq \dots$

If there exists $p_0 \in \mathbb{N}$ satisfying $\mu(\mathbb{U}_{p_0}) = 0$ then \mathbb{U}_{p_0} is a compact set. In this case Schauder's theorem implies \mathcal{T} has a FP in \mathbb{U} . Let $\mu(\mathbb{C}_p) > 0$, $p \in \mathbb{N}$. Now, for $p \in \mathbb{N}$, we have

$$\begin{aligned} \Lambda_1[h[\xi\{\mu(\mathbb{U}_{p+1}) + \gamma(\mu(\mathbb{U}_{p+1}))\}]] &= \Lambda_1[h[\xi\{\mu(\text{Conv}\mathcal{T}\mathbb{U}_p) + \gamma(\mu(\text{Conv}\mathcal{T}\mathbb{U}_p))\}]] \\ &= \Lambda_1[h[\xi\{\mu(\mathcal{T}\mathbb{U}_p) + \gamma(\mu(\mathcal{T}\mathbb{U}_p))\}]] \\ &\leq \Lambda_2[g[\xi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}, \phi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}]]. \end{aligned}$$

Using the condition (1) of definition 1.4, we get

$$\begin{aligned} h[\xi\{\mu(\mathbb{U}_{p+1}) + \gamma(\mu(\mathbb{U}_{p+1}))\}] &\leq g[\xi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}, \phi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}] \\ &\leq \xi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}. \end{aligned}$$

Clearly $\{\xi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}\}_{p=1}^\infty$ is a non-negative and non-increasing sequence hence there exists $a \geq 0$ such that

$$\lim_{p \rightarrow \infty} \xi\{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\} = a.$$

If possible let $a > 0$. As $p \rightarrow \infty$, we get

$$h(a) \leq a$$

which is a contradiction hence $a = 0$, i.e.,

$$\xi[\lim_{p \rightarrow \infty} \{\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))\}] = 0$$

i.e.,

$$\lim_{p \rightarrow \infty} [\mu(\mathbb{U}_p) + \gamma(\mu(\mathbb{U}_p))] = 0.$$

Using the definition 1.9, we get

$$\lim_{p \rightarrow \infty} \mu(\mathbb{U}_p) = 0.$$

Since $\mathbb{U}_p \supseteq \mathbb{U}_{p+1}$, by definition 1.1, we get $\mathbb{U}_\infty = \bigcap_{p=1}^\infty \mathbb{U}_p$ is a nonempty, closed and convex subset of \mathbb{U} and \mathbb{U}_∞ is \mathcal{T} invariant. Thus theorem 1.2 implies that \mathcal{T} has a fixed point in \mathbb{U} . This completes the proof. \square

Theorem 2.2. Let \mathbb{U} be a NBCC subset of a Banach space \mathfrak{Z} . Also $\mathcal{T} : \mathbb{U} \rightarrow \mathbb{U}$ is a continuous mapping with

$$h [\xi \{ \mu(\mathcal{T}\Omega) + \gamma(\mu(\mathcal{T}\Omega)) \}] \leq kg [\xi \{ \mu(\Omega) + \gamma(\mu(\Omega)) \}, \phi \{ \mu(\Omega) + \gamma(\mu(\Omega)) \}] \quad (2.2)$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC and $\phi \in \Phi, \xi \in \Xi, g \in \mathcal{C}, h \in \mathcal{A}$ and $\gamma \in \mathcal{B}$. Then \mathcal{T} has at least one fixed point in \mathbb{U} .

Proof . The result follows by taking $\Lambda_1(x) = x$ and $\Lambda_2(x) = kx$ in Theorem 2.1. \square

Theorem 2.3. Let \mathbb{U} be a NBCC subset of a Banach space \mathfrak{Z} . Also, let $\mathcal{T} : \mathbb{U} \rightarrow \mathbb{U}$ be a continuous mapping with

$$h [\xi \{ 2\mu(\mathcal{T}\Omega) \}] \leq kg [\xi \{ 2\mu(\Omega) \}, \phi \{ 2\mu(\Omega) \}] \quad (2.3)$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC and $\phi \in \Phi, \xi \in \Xi, g \in \mathcal{C}$ and $h \in \mathcal{A}$. Then \mathcal{T} has at least one fixed point in \mathbb{U} .

Proof . The result follows by taking $\gamma(x) = x$ in Theorem 2.2. \square

Theorem 2.4. Let \mathbb{U} be a NBCC subset of a Banach space \mathfrak{Z} . Also, let $\mathcal{T} : \mathbb{U} \rightarrow \mathbb{U}$ be a continuous mapping with

$$h [\xi \{ 2\mu(\mathcal{T}\Omega) \}] \leq k\xi \{ 2\mu(\Omega) \} \quad (2.4)$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC and $\xi \in \Xi$ and $h \in \mathcal{A}$. Then \mathcal{T} has at least one fixed point in \mathbb{U} .

Proof . Use $g(m, n) \leq m$ in Theorem 2.3. \square

Corollary 2.5. Let \mathbb{U} be a NBCC subset of a Banach space \mathfrak{Z} . Also, let $\mathcal{T} : \mathbb{U} \rightarrow \mathbb{U}$ be a continuous mapping with

$$\mu(\mathcal{T}\Omega) \leq \lambda\mu(\Omega), \quad \lambda = \frac{k}{\bar{k}} \in (0, 1). \quad (2.5)$$

where $\Omega \subset \mathbb{U}$ and μ is an arbitrary MNC. Then \mathcal{T} has at least one fixed point in \mathbb{U} .

Proof . Using $\xi(x) = x$ and $h(x) = \bar{k}x$ where $0 < k < 1, \bar{k} > 1$ in Theorem 2.4. we get DPFT. \square

3 Measure of noncompactness on $C([0, I])$

Consider the space $\mathfrak{Z} = C(U)$ which is the set of real continuous functions on U , where $U = [0, I]$. Then \mathfrak{Z} is a Banach space with the norm

$$\|A\| = \sup \{ |A(t)| : t \in U \}, \quad A \in \mathfrak{Z}.$$

Let $T(\neq \emptyset) \subseteq \mathfrak{Z}$ be bounded. For $A \in T$ and $\varepsilon > 0$, denote by $\mu(A, \varepsilon)$ the modulus of the continuity of A , i.e.,

$$\mu(A, \varepsilon) = \sup \{ |A(t_1) - A(t_2)| : t_1, t_2 \in U, |t_1 - t_2| \leq \varepsilon \}.$$

Moreover, we set

$$\mu(T, \varepsilon) = \sup \{ \mu(A, \varepsilon) : A \in T \}; \quad \mu_0(T) = \lim_{\varepsilon \rightarrow 0} \mu(T, \varepsilon).$$

It is well-known that the function μ_0 is a MNC in \mathfrak{Z} such that the Hausdorff MNC Γ is given by $\Gamma(T) = \frac{1}{2}\mu_0(T)$ (see [9]).

4 Solvability of a fractional integral equation

For $h \in (0, 1]$ and $\omega \in \mathbb{C}, Re(\omega) > 0$, we define the left fractional integral of w by [21]

$$({}_a U^{\omega, h, \sigma} w)(\varphi) = \frac{1}{h^\omega \Gamma(\omega)} \int_a^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} w(\vartheta) \sigma'(\vartheta) d\vartheta.$$

In this section, we will study the fractional integral equation shown below

$$\mathcal{H}(\varphi) = \Psi(\varphi, \mathcal{J}(\varphi, \mathcal{H}(\varphi)), ({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)), \quad (4.1)$$

where $\omega > 0$, $h \in (0, 1]$, $\varphi \in U = [0, I]$. Let

$$D_{e_0} = \{\mathcal{H} \in \mathfrak{Z} : \|\mathcal{H}\| \leq e_0\}.$$

Assume that

(A) $\Psi : U \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\mathcal{J} : U \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and there exists constants $\beta_1, \beta_2, \beta_3 \geq 0$ satisfying

$$|\Psi(\varphi, \mathcal{J}, U_1) - \Psi(\varphi, \bar{\mathcal{J}}, \bar{U}_1)| \leq \beta_1 |\mathcal{J} - \bar{\mathcal{J}}| + \beta_2 |U_1 - \bar{U}_1|, \quad \varphi \in U; \quad \mathcal{J}, U_1, \bar{\mathcal{J}}, \bar{U}_1 \in \mathbb{R}$$

and

$$|\mathcal{J}(\varphi, L_1) - \mathcal{J}(\varphi, L_2)| \leq \beta_3 |L_1 - L_2|, \quad L_1, L_2 \in \mathbb{R}.$$

(B) There exists $e_0 > 0$ satisfying

$$\bar{\Psi} = \sup \left\{ |\Psi(\varphi, \mathcal{J}, U_1)| : \varphi \in U, \mathcal{J} \in [-\hat{\mathcal{J}}, \hat{\mathcal{J}}], U_1 \in [-\hat{U}, \hat{U}] \right\} \leq e_0,$$

and

$$\beta_1 \beta_3 < 1,$$

where

$$\hat{\mathcal{J}} = \sup \{ |\mathcal{J}(\varphi, \mathcal{H}(\varphi))| : \varphi \in U, \mathcal{H}(\varphi) \in [-e_0, e_0] \}$$

and

$$\hat{U} = \sup \{ |({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)| : \varphi \in U, \mathcal{H}(\varphi) \in [-e_0, e_0] \}.$$

(C) Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function.

(D) $|\Psi(\varphi, 0, 0)| = 0$, $\mathcal{J}(\varphi, 0) = 0$.

(E) There exists a positive solution e_0 of the inequality

$$\beta_1 \beta_3 e_0 + \frac{\beta_2 e_0 I^{\omega-1}}{h^{\omega-1} (h-1) \Gamma(\omega)} \cdot e^{\frac{(h-1)I}{h}} \leq e_0.$$

Theorem 4.1. If conditions (A)-(E) hold, then the Eq.(4.1) has a solution in $\mathfrak{Z} = C(U)$.

Proof . Set the operator $\mathcal{S} : \mathfrak{Z} \rightarrow \mathfrak{Z}$ as follows:

$$(\mathcal{S}\mathcal{H})(\varphi) = \Psi(\varphi, \mathcal{J}(\varphi, \mathcal{H}(\varphi)), ({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)).$$

Step 1: We show that the function \mathcal{S} maps D_{e_0} into D_{e_0} . Let $\mathcal{H} \in D_{e_0}$. We have

$$\begin{aligned} |(\mathcal{S}\mathcal{H})(\varphi)| &\leq |\Psi(\varphi, \mathcal{J}(\varphi, \mathcal{H}(\varphi)), ({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)) - \Psi(\varphi, 0, 0)| + |\Psi(\varphi, 0, 0)| \\ &\leq \beta_1 |\mathcal{J}(\varphi, \mathcal{H}(\varphi)) - 0| + \beta_2 |({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi) - 0| \\ &\leq \beta_1 \beta_3 |\mathcal{H}(\varphi)| + \beta_2 |({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)|. \end{aligned}$$

Also,

$$\begin{aligned} |({}_0U^{\omega, h, \sigma} \mathcal{H})(\varphi)| &= \left| \frac{1}{h^\omega \Gamma(\omega)} \int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \right| \\ &\leq \frac{1}{h^\omega \Gamma(\omega)} \int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} \sigma'(\vartheta) |\mathcal{H}(\vartheta)| d\vartheta \\ &\leq \frac{e_0}{h^\omega \Gamma(\omega)} \int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} \sigma'(\vartheta) d\vartheta \\ &\leq \frac{e_0 I^{\omega-1} e^{\frac{(h-1)I}{h}}}{h^{\omega-1} (h-1) \Gamma(\omega)}. \end{aligned}$$

Hence, $\|\mathcal{S}\| < e_0$ gives

$$\|\mathcal{S}\| \leq \beta_1 \beta_3 e_0 + \frac{\beta_2 e_0 I^{\omega-1}}{h^{\omega-1}(h-1)\Gamma(\omega)} \cdot e^{\frac{(h-1)I}{h}} \leq e_0.$$

Due to the assumption (E), \mathcal{S} maps D_{e_0} into D_{e_0} .

Step 2: We show that \mathcal{S} is continuous on D_{e_0} . Let $\varepsilon > 0$ and $\mathcal{H}, \bar{\mathcal{H}} \in D_{e_0}$ such that $\|\mathcal{H} - \bar{\mathcal{H}}\| < \varepsilon$. We now have

$$\begin{aligned} |(\mathcal{S}\mathcal{H})(\varphi) - (\mathcal{S}\bar{\mathcal{H}})(\varphi)| &\leq |\Psi(\varphi, \mathcal{J}(\varphi, \mathcal{H}(\varphi)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi)) - \Psi(\varphi, \mathcal{J}(\varphi, \bar{\mathcal{H}}(\varphi)), ({}_0U^{\omega, h, \sigma}\bar{\mathcal{H}})(\varphi))| \\ &\leq \beta_1 |\mathcal{J}(\varphi, \mathcal{H}(\varphi)) - \mathcal{J}(\varphi, \bar{\mathcal{H}}(\varphi))| + \beta_2 |({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi) - ({}_0U^{\omega, h, \sigma}\bar{\mathcal{H}})(\varphi)|. \end{aligned}$$

Also,

$$\begin{aligned} |({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi) - ({}_0U^{\omega, h, \sigma}\bar{\mathcal{H}})(\varphi)| &= \left| \frac{1}{h^\omega \Gamma(\omega)} \int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} \sigma'(\vartheta) \{\mathcal{H}(\vartheta) - \bar{\mathcal{H}}(\vartheta)\} d\vartheta \right| \\ &\leq \frac{1}{h^\omega \Gamma(\omega)} \int_0^\varphi e^{\frac{(h-1)(\sigma(\varphi) - \sigma(\vartheta))}{h}} (\sigma(\varphi) - \sigma(\vartheta))^{\omega-1} \sigma'(\vartheta) |\mathcal{H}(\vartheta) - \bar{\mathcal{H}}(\vartheta)| d\vartheta \\ &< \frac{\varepsilon I^{\omega-1} e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)}. \end{aligned}$$

Hence, $\|\mathcal{H} - \bar{\mathcal{H}}\| < \varepsilon$ gives

$$|(\mathcal{S}\mathcal{H})(\varphi) - (\mathcal{S}\bar{\mathcal{H}})(\varphi)| < \beta_1 \beta_3 \varepsilon + \frac{\varepsilon \beta_2 I^{\omega-1} e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)}.$$

As $\varepsilon \rightarrow 0$, we get $|(\mathcal{S}\mathcal{H})(\varphi) - (\mathcal{S}\bar{\mathcal{H}})(\varphi)| \rightarrow 0$. This shows that \mathcal{S} is continuous on D_{e_0} .

Step 3: An estimate of \mathcal{S} with respect to μ_0 : Assume that $\Delta (\neq \emptyset) \subseteq D_{e_0}$. Let $\varepsilon > 0$ be arbitrary and choose $\mathcal{H} \in \Delta$ and $\varphi_1, \varphi_2 \in U$ such that $|\varphi_2 - \varphi_1| \leq \varepsilon$ and $\varphi_2 \geq \varphi_1$.

Now,

$$\begin{aligned} |(\mathcal{S}\mathcal{H})(\varphi_2) - (\mathcal{S}\mathcal{H})(\varphi_1)| &= |\Psi(\varphi_2, \mathcal{J}(\varphi_2, \mathcal{H}(\varphi_2)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_2)) - \Psi(\varphi_1, \mathcal{J}(\varphi_1, \mathcal{H}(\varphi_1)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1))| \\ &\leq |\Psi(\varphi_2, \mathcal{J}(\varphi_2, \mathcal{H}(\varphi_2)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_2)) - \Psi(\varphi_2, \mathcal{J}(\varphi_2, \mathcal{H}(\varphi_2)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1))| \\ &\quad + |\Psi(\varphi_2, \mathcal{J}(\varphi_2, \mathcal{H}(\varphi_2)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1)) - \Psi(\varphi_2, \mathcal{J}(\varphi_1, \mathcal{H}(\varphi_1)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1))| \\ &\quad + |\Psi(\varphi_2, \mathcal{J}(\varphi_1, \mathcal{H}(\varphi_1)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1)) - \Psi(\varphi_1, \mathcal{J}(\varphi_1, \mathcal{H}(\varphi_1)), ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1))| \\ &\leq \beta_2 |({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1)| + \beta_1 |\mathcal{J}(\varphi_2, \mathcal{H}(\varphi_2)) - \mathcal{J}(\varphi_1, \mathcal{H}(\varphi_1))| + \mu_\Psi(U, \varepsilon) \\ &\leq \beta_2 |({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1)| + \beta_1 \beta_3 |\mathcal{H}(\varphi_2) - \mathcal{H}(\varphi_1)| + \mu_\Psi(U, \varepsilon), \end{aligned}$$

where

$$\mu_\Psi(U, \varepsilon) = \sup \left\{ |\Psi(\varphi_2, \mathcal{J}, U_1) - \Psi(\varphi_1, \mathcal{J}, U_1)| : |\varphi_2 - \varphi_1| \leq \varepsilon; \varphi_1, \varphi_2 \in U; \mathcal{J} \in [-\hat{\mathcal{J}}, \hat{\mathcal{J}}]; U_1 \in [-\hat{U}, \hat{U}] \right\}.$$

Also,

$$\begin{aligned} |({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega, h, \sigma}\mathcal{H})(\varphi_1)| &= \left| \frac{1}{h^\omega \Gamma(\omega)} \int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2) - \sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \right. \\ &\quad \left. - \frac{1}{h^\omega \Gamma(\omega)} \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1) - \sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \right| \\ &\leq \frac{1}{h^\omega \Gamma(\omega)} \left| \int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2) - \sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \right. \end{aligned}$$

$$\begin{aligned}
 & - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \Big| \\
 & \leq \frac{1}{h^\omega \Gamma(\omega)} \Big| \int_0^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \\
 & - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \Big| \\
 & + \frac{1}{h^\omega \Gamma(\omega)} \Big| \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \\
 & - \int_0^{\varphi_1} e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \mathcal{H}(\vartheta) \sigma'(\vartheta) d\vartheta \Big| \\
 & \leq \frac{1}{h^\omega \Gamma(\omega)} \int_{\varphi_1}^{\varphi_2} e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} |\mathcal{H}(\vartheta)| \sigma'(\vartheta) d\vartheta \\
 & + \frac{1}{h^\omega \Gamma(\omega)} \int_0^{\varphi_1} \Big| \left(e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} - e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \right) \mathcal{H}(\vartheta) \sigma'(\vartheta) \Big| d\vartheta \\
 & \leq \frac{-e^{\frac{(h-1)I}{h}}}{h^{\omega-1}(h-1)\Gamma(\omega)} \|\mathcal{H}\| (\varphi_2 - \varphi_1)^{\omega-1} \\
 & + \frac{\|\mathcal{H}\|}{h^\omega \Gamma(\omega)} \int_0^{\varphi_1} \Big| \left(e^{\frac{(h-1)(\sigma(\varphi_2)-\sigma(\vartheta))}{h}} (\sigma(\varphi_2) - \sigma(\vartheta))^{\omega-1} - e^{\frac{(h-1)(\sigma(\varphi_1)-\sigma(\vartheta))}{h}} (\sigma(\varphi_1) - \sigma(\vartheta))^{\omega-1} \right) \sigma'(\vartheta) \Big| d\vartheta.
 \end{aligned}$$

As $\varepsilon \rightarrow 0$, then $\varphi_2 \rightarrow \varphi_1$ and so, $|({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_1)| \rightarrow 0$. Hence,

$$|(\mathcal{S}\mathcal{H})(\varphi_2) - (\mathcal{S}\mathcal{H})(\varphi_1)| \leq \beta_2 |({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_1)| + \beta_1 \beta_3 \mu(\mathcal{H}, \varepsilon) + \mu_\Psi(U, \varepsilon),$$

gives

$$\mu(\mathcal{S}\mathcal{H}, \varepsilon) \leq \beta_2 |({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_2) - ({}_0U^{\omega,h,\sigma}\mathcal{H})(\varphi_1)| + \beta_1 \beta_3 \mu(\mathcal{H}, \varepsilon) + \mu_\Psi(U, \varepsilon).$$

By the uniform continuity of Ψ on $U \times [-\hat{\mathcal{J}}, \hat{\mathcal{J}}] \times [-\hat{\mathcal{U}}, \hat{\mathcal{U}}]$ we have $\mu_\Psi(U, \varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$.

Taking $\sup_{\mathcal{H} \in \Delta}$ and $\varepsilon \rightarrow 0$ we get,

$$\mu_0(\mathcal{S}\Delta) \leq \beta_1 \beta_3 \mu_0(\Delta).$$

Thus by Corollary 2.5, \mathcal{S} has a fixed point in $\Delta \subseteq D_{e_0}$ i.e. equation (4.1) has a solution in \mathfrak{J} . \square

Example 4.2. Consider the equation below

$$\mathcal{H}(\varphi) = \frac{\mathcal{H}(\varphi)}{9 + \varphi^4} + \frac{({}_0U^{1, \frac{1}{3}, \varphi}\mathcal{H})(\varphi)}{20} \quad (4.2)$$

for $\varphi \in [0, 3] = U$.

We have

$$\begin{aligned}
 \sigma(\varphi) &= \varphi; \\
 ({}_0U^{1, \frac{1}{3}, \varphi}\mathcal{H})(\varphi) &= \frac{3}{\Gamma(1)} \int_0^\varphi e^{-2(\varphi-\vartheta)} \mathcal{H}(\vartheta) d\vartheta.
 \end{aligned}$$

Also, $\Psi(\varphi, \mathcal{J}, U_1) = \mathcal{J} + \frac{U_1}{20}$ and $\mathcal{J}(\varphi, \mathcal{H}) = \frac{\mathcal{H}}{9 + \varphi^4}$. It is trivial that both Ψ , \mathcal{J} are continuous satisfying

$$|\mathcal{J}(\varphi, L_1) - \mathcal{J}(\varphi, L_2)| \leq \frac{|L_1 - L_2|}{9},$$

and

$$|\Psi(\varphi, \mathcal{J}, U_1) - \Psi(\varphi, \bar{\mathcal{J}}, \bar{U}_1)| \leq |\mathcal{J} - \bar{\mathcal{J}}| + \frac{1}{20} |U_1 - \bar{U}_1|.$$

Therefore, $\beta_1 = 1$, $\beta_2 = \frac{1}{20}$, $\beta_3 = \frac{1}{9}$ and $\beta_1\beta_3 = \frac{1}{9} < 1$. If $\|\mathcal{H}\| \leq e_0$ then

$$\hat{\mathcal{J}} = \frac{e_0}{9}$$

and

$$\hat{U} = \frac{3e_0}{2} \left(1 - \frac{1}{e^6}\right).$$

Further,

$$|\Psi(\varphi, \mathcal{J}, U_1)| \leq \frac{e_0}{9} + \frac{3e_0}{40} \left(1 - \frac{1}{e^6}\right) \leq e_0.$$

If we choose $e_0 = 3$ then

$$\hat{\mathcal{J}} = \frac{1}{3}, \quad \hat{U} = \frac{9}{2} \left(1 - \frac{1}{e^6}\right),$$

which gives

$$\bar{\Psi} \leq 3.$$

For $e_0 = 3$, however, assumption (E) is also satisfied. We can see that all of Theorem 4.1's assumptions are achieved, from (A) to (E). Equation (4.2), according to Theorem 4.1, has a solution in $\mathfrak{Z} = C(U)$.

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