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A new faster iteration process to fixed points of generalized α-nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce a new iterative scheme to approximate the fixed point of generalized α-nonexpansive mappings. we first prove that the proposed iteration process is faster than all of Picard, Mann, Ishikawa, Noor, Agarwal, Abbas and Thakur processes for contractive mappings. We also obtain some weak and strong convergence theorems for generalized α-nonexpansive mappings. Using the example presented in [R. Pant and R. Shukla, Approximating fixed point of generalized α-nonexpansive mappings in Banach spaces, J. Numer. Funct. Anal. Optim. 38(2017) 248-266.], we compare the convergence behavior of the new iterative process with other iterative processes.

Keywords: Uniformly convex Banach space, Convergence theorem, Generalized α-nonexpansive mapping, Opial property, Iterative process
2020 MSC: 47H05, 47H09, 47H20

1 Introduction

Throughout the paper, we denote by N the set of positive integers and by R the set of real numbers. Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. It is well known that a mapping $T : C \to C$ is said to be nonexpansive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is called quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$, where $F(T)$ is the set of fixed points of T i.e., $F(T) = \{x \in C : Tx = x\}$. It is well known that if C is a closed, bounded and convex subset of a uniformly convex Banach space E, then $F(T)$ is nonempty for a nonexpansive mapping. In 2008, Suzuki [13] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition(C) and obtained some existence and convergence Theorems for such mappings. Let C be a nonempty subset of a Banach space E, a mapping $T : C \to C$ is said to satisfy condition(C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. It is obvious that every condition(C) mapping with a fixed point is quasi-nonexpansive mapping.

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Recently, Aoyama and Kohsaka in [3], introduced the class of α-nonexpansive mappings in Banach spaces and obtained a fixed point Theorems for such mappings. Let C be a nonempty subset of a Banach space E, a mapping $T: C \to C$ is said to be α-nonexpansive if for a given real number $\alpha < 1$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2, \quad \forall x, y \in C.$$

Ariza-Puiz et al. in [4] showed that the concept of α-nonexpansive is trivial for $\alpha < 0$. It is obvious that every nonexpansive mapping is 0-nonexpansive and also every α-nonexpansive mapping with a fixed point is quasi-nonexpansive. In general condition(C) and α-nonexpansive mapping are not continuous mappings (See [13] and [10]).

Recently Pant and Shukla in [10], introduced the class of generalized α-nonexpansive mappings which contains the condition (C) mappings. Let C be a nonempty subset of a Banach space E, a mapping $T: C \to C$ is said to be generalized α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|, \quad \forall x, y \in C.$$

There exists some iteration processes which is often used to approximate fixed points of nonexpansive mappings. For example, Picard iteraion, Mann iteration and Ishikawa iteration.

During the recent years, the Picard, Mann and Ishikawa iterative processes have been studied by many mathematicians. We know that the Picard, Mann [7] and Ishikawa [6] iteration processes are defined respectively as:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = Tx_n, \qquad n \in N, \end{cases} \tag{1.1}$$

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \qquad n \in N, \end{cases} \tag{1.2}$$

and

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \qquad n \in N, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$.

In 2000, Noor [8] introduced the following three-step iteration process.

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n)x_n + \beta_n Tz_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \qquad n \in N, \end{cases} \tag{1.4}$$

where introduced the following there $\{\alpha_n\}$ and $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$. In 2007, Agarwal et al. [2] introduced the following iteration process.

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \qquad n \in N, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Sahu in [11] recently proved that this process converges at a rate faster than both Picard and Mann iterations for contractive mappings. Recently, Abbas and Nazir [1] introduced the following three-step iteration process.

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_n Tz_n \\ y_n = (1 - \beta_n)Tx_n + \beta_n Tz_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \qquad n \in N, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$. They proved that this process converges faster than iteration process (1.5) for contractive mappings. Recently, Thakur et al. [15] introduced the following three-step iteration process.

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Tz_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n)z_n + \beta_n Tz_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \qquad n \in N, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$. They proved that this process converges faster than iteration process (1.6) for contractive mappings. Now, we pose the following question:

Question1: Is it possible to develop an iteration process which rate of convergence is even faster than the iteration process (1.7)?

As an answer, we introduce the following three-step iteration process.

$$\begin{cases} x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n \\ y_n = T((1-\beta_n)x_n + \beta_n z_n) \\ z_n = T((1-\gamma_n)x_n + \gamma_n Tx_n), \qquad n \in N, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$.

In this paper first, we prove that our iteration process converges faster than iteration process (1.7) for contractive mappings and we present numerical example in support of the proof. Also, we prove some weak and strong convergence theorem for generalized α-nonexpansive mappings in a uniformly convex Banach spaces and we using a example of [10] for generalized α-nonexpansive mappings compare the convergence behavior our iteration process with other processes.

2 Preliminaries

Assume that E be a Banach space. We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. A Banach space E is called uniformly convex if for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. It is well known that a Banach space E satisfy the Opial property [9], if for each $\{x_n\}$ in E such that $x_n \rightharpoonup x$ and $x \neq y$, then

$$\lim_{n\to\infty} inf\|x_n - x\| < \lim_{n\to\infty} inf\|x_n - y\|.$$

(see for more details [14]). Let C be a nonempty closed convex subset of a Banach space E and let $\{x_n\}$ be a bounded sequence in E. For $x \in E$, we set

$$r(x, \{x_n\}) = \lim_{n\to\infty} sup\|x - x_n\|.$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\}\}.$$

It is known that in a uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one point. Also, $A(C, \{x_n\})$ is nonempty and convex when C is weakly compact and convex (for more details see [14] and [**?**]). The following results for generalized α-nonexpansive mappings can be found in [10].

Proposition 1. Every mapping satisfying condition(C) is a generalized α-nonexpansive mapping, but the converse is not true.

Proposition 2. Let K be a nonempty subset of a Banach space E and $T : K \to K$ a generalized α-nonexpansive mapping with a fixed point $y \in K$. Then T is quasi-nonexpansive.

Proposition 3. Let K be a nonempty subset of a Banach space E and $T : K \to K$ a generalized α-nonexpansive mapping. Then for all $x, y \in K$.

$$\|x - T(y)\| \leq \frac{(3+\alpha)}{(1-\alpha)}\|x - T(x)\| + \|x - y\|.$$

Proposition 4. (Demiclosedness principle). Let K be a nonempty closed subset of a banach space E with the Opial property and $T : K \to K$ generalized α-nonexpansive mapping. If $\{x_n\}$ converges weakly to a point z and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$, then $T(z) = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.

The following Theorem is in [13].

Theorem 1. Let C be a weakly compact convex subset of a uniformaly convex Banach space E. Let T be a mapping on C. Assume that T satisfies condition (C). Then T has a fixed point.

The following lemma is important in the proof crucial theorem of section (4) in this paper.

Lemma 1. ([12]). Let E be a uniformly convex Banach space and $0 < a \leq l_n \leq b < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be the two sequences such that

$$\lim_{n\to\infty} sup\|x_n\| \leq r,$$

$$\lim_{n\to\infty} sup\|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|l_n x_n + (1-l_n)y_n\| = r$$

hold for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = r.$$

The following definitions about the rate of convergence are in [11] and [5].

Definition 1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively. If

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2. Suppose that for two fixed point iteration processes $\{x_n\}$ and $\{u_n\}$, both converging to the same fixed point p, the error estimates

$$\begin{cases} \|x_n - p\| \leq a_n, & \forall n \in N \\ \|u_n - p\| \leq b_n, & \forall n \in N \end{cases}$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ converges faster than $\{u_n\}$ to p.

3 Rate of convergence

In this section, we show our process (1.8) converges faster than (1.7).

Theorem 2. Let C be a nonempty closed convex subset of a uniformly convex Banach space. Let T be a contractive mapping with a contraction factor $k \in (0,1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process

$$\begin{cases} u_1 = x \in C \\ u_{n+1} = (1-\alpha_n)Tw_n + \alpha_n Tv_n \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases}$$

and $\{x_n\}$ by (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$ and and $\{\gamma_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for any $n \in N$ and some ε in $(0,1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$.

Proof . As proved in Theorem 3.1 of [15], we have

$$\|u_{n+1} - p\| \leq k^n[1-(1-k)\gamma]^n \|u_1 - p\|,$$

for any $n \in N$. Let

$$a_n = k^n[1-(1-k)\gamma]^n \|u_1 - p\|.$$

Now with notion to process (1.8), we have

$$\begin{aligned}\|z_n - p\| &= \|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\| \\ &\leq k\|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\leq k[(1-\gamma_n)\|x_n - p\| + k\gamma_n\|x_n - p\|] \\ &= k[1-(1-k)\gamma_n]\|x_n - p\|\end{aligned}$$

so that

$$\begin{aligned}\|y_n - p\| &= \|T((1-\beta_n)x_n + \beta_n z_n) - p\| \\ &\leq k\|(1-\beta_n)x_n + \beta_n z_n - p\| \\ &\leq k(1-\beta_n)\|x_n - p\| + k\beta_n\|z_n - p\| \\ &\leq k(1-\beta_n)\|x_n - p\| + k^2\beta_n[1-(1-k)\gamma_n]\|x_n - p\| \\ &< [k(1-\beta_n) + k\beta_n(1-(1-k)\gamma_n]\|x_n - p\| \\ &= k[1-(1-k)\gamma_n\beta_n]\|x_n - p\|.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)Ty_n + \alpha_n Tz_n) - p\| \\ &\leq (1-\alpha_n)k\|y_n - p\| + \alpha_n k\|z_n - p\| \\ &\leq [(1-\alpha_n)k^2(1-(1-k)\gamma_n\beta_n) + \alpha_n k^2(1-(1-k)\gamma_n)]\|x_n - p\| \\ &< [k(1-(1-k)\gamma_n\beta_n)]\|x_n - p\|.\end{aligned}$$

Let

$$b_n = k^n[1-((1-k))\gamma\beta]^n\|x_1 - p\|.$$

Then

$$\frac{b_n}{a_n} = \frac{k^n[1-(1-k)\gamma\beta]^n\|x_1 - p\|}{k^n[1-(1-k)\gamma]^n\|u_1 - p\|} = [\frac{1-(1-k)\gamma\beta}{1-(1-k)\gamma}]^n\frac{\|x_1 - p\|}{\|u_n - p\|} \to 0 \qquad as \qquad n\to\infty.$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

Now, we present an example which shows that our iteration process (1.8) converges at a rate faster than Picard iteration process (1.1), Mann iteration process(1.2), Ishikawa iteration (1.3), Noor iteration process (1.4), Agarwaletal et al. iteration process (1.5) , Abbas and Naziris iteration process (1.6) and Thakur et al iteration process (1.7).

Example 1. Let $E = R$ and $C = [1, 50]$. Let $T : C \to C$ be a mapping defined by $T(x) = \sqrt{x^2 - 7x + 28}$ for any $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, with the initial value $x_1 = 30$. It is obvious that $x = 4$ is fixed point of T. The table below show behavior all the iterations processes mentioned to fixed point of T in 38 iteration.

We see that the new iteration converges faster than the known iterations.

4 Convergence theorems in uniformly convex Banach spaces

In this section, we prove weak and strong convergence theorems for our iteration process (1.8) in uniformly convex Banach spaces. The following Lemma will be useful to prove our main theorem in this section.

Lemma 2. Let C be a nonempty closed convex subset of a Banach space E and $T : C \Rightarrow C$ be a generalized α-nonexpansive mapping. Suppose the sequence $\{x_n\}$ be by (1.8), then $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$.

Proof . If $F(T) \neq \emptyset$, Let $p \in F(T)$. By Proposition (2), since T is quasi-nonexpansive, we have

$$\begin{aligned}\|z_n - p\| &= \|T((1-\gamma_n)x_n + \gamma_n Tx_n) - p\| \\ &\leq \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.1}$$

No. of iteration	Picard	Mann	Ishikaw	Noor	Agarwal	Abbas	Thakur	New
1	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
2	26.7955	28.3978	27.6060	27.2153	26.0038	24.0316	24.4262	21.6876
3	23.6312	26.8049	25.2357	24.4651	22.0757	18.2478	19.0080	13.8361
4	20.5186	25.2227	22.8939	21.7586	18.2432	12.7944	13.8530	7.1987
5	17.4752	23.6524	20.5874	19.1083	14.5525	8.0381	9.2090	4.1723
6	14.5278	22.0960	18.3251	16.5328	11.0879	4.9187	5.6998	4.0017
7	11.7202	20.5553	16.1196	14.0599	8.0152	4.0665	4.2164	4.0000
8	9.1280	19.0332	13.9888	11.7322	5.6559	4.0029	4.0135	4.0000
9	6.8866	17.5327	11.9590	9.6151	4.3951	4.0001	4.0007	4.0000
10	5.2172	16.0579	10.0689	7.8003	4.0519	4.0000	4.0000	4.0000
11	4.3242	14.6139	8.3738	6.3844	4.0053	4.0000	4.0000	4.0000
12	4.0533	13.2076	6.9444	5.4026	4.0005	4.0000	4.0000	4.0000
13	4.0070	11.8475	5.8444	4.7903	4.0001	4.0000	4.0000	4.0000
14	4.0009	10.5452	5.0862	4.4343	4.0000	4.0000	4.0000	4.0000
15	4.0001	9.3157	4.6133	4.2354	4.0000	4.0000	4.0000	4.0000
16	4.0000	8.1782	4.3378	4.1266	4.0000	4.0000	4.0000	4.0000
17	4.0000	7.1565	4.1836	4.0679	4.0000	4.0000	4.0000	4.0000
18	4.0000	6.2764	4.0991	4.0363	4.0000	4.0000	4.0000	4.0000
19	4.0000	5.5599	4.0532	4.0194	4.0000	4.0000	4.0000	4.0000
20	4.0000	5.0156	4.0286	4.0103	4.0000	4.0000	4.0000	4.0000
21	4.0000	4.6319	4.0153	4.0055	4.0000	4.0000	4.0000	4.0000
22	4.0000	4.3794	4.0082	4.0029	4.0000	4.0000	4.0000	4.0000
23	4.0000	4.2221	4.0044	4.0016	4.0000	4.0000	4.0000	4.0000
24	4.0000	4.1280	4.0023	4.0008	4.0000	4.0000	4.0000	4.0000
25	4.0000	4.0730	4.0013	4.0004	4.0000	4.0000	4.0000	4.0000
26	4.0000	4.0414	4.0007	4.0002	4.0000	4.0000	4.0000	4.0000
27	4.0000	4.0234	4.0004	4.0001	4.0000	4.0000	4.0000	4.0000
28	4.0000	4.0132	4.0002	4.0001	4.0000	4.0000	4.0000	4.0000
29	4.0000	4.0074	4.0001	4.0000	4.0000	4.0000	4.0000	4.0000
30	4.0000	4.0042	4.0001	4.0000	4.0000	4.0000	4.0000	4.0000
31	4.0000	4.0024	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
32	4.0000	4.0013	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
33	4.0000	4.0007	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
34	4.0000	4.0004	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
35	4.0000	4.0002	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
36	4.0000	4.0001	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
37	4.0000	4.0001	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
38	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000

So that

$$\begin{aligned}\|y_n - p\| &= \|T((1-\beta_n)x_n + \beta_n z_n) - p\| \\ &\leq \|(1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.2}$$

Hence, by (4.1) and (4.2), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|T((1-\alpha_n)Ty_n - \alpha_n T z_n) - p\| \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\leq \|x_n - p\|.\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded and nonincreasing for all $p \in F(T)$. Hence $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Now, we prove the following theorem which is useful for further theorems.

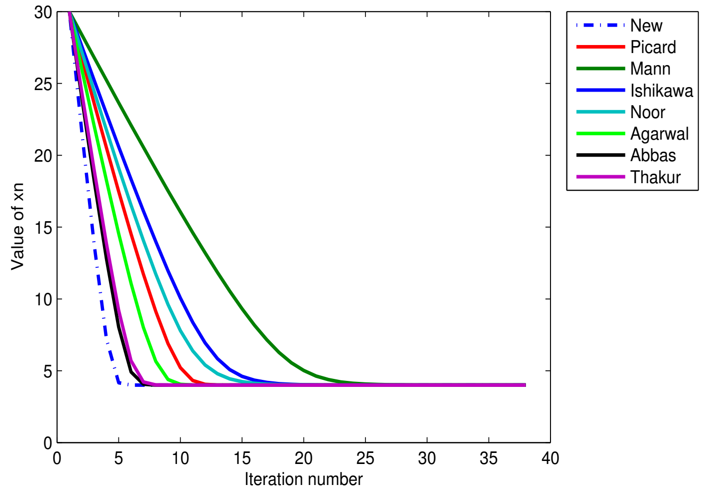

Figure 1: Comparison of iteration processes converges.

Theorem 3. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be a generalized α-nonexpansive mapping. Suppose the sequence $\{x_n\}$ be generated by (1.8). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Proof . Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma (2), $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|x_n - p\| = k. \tag{4.3}$$

By the proof Lemma (2), we have

$$\lim_{n\to\infty} \sup \|z_n - p\| \leq \lim_{n\to\infty} \sup \|x_n - p\| = k. \tag{4.4}$$

So, since T is quasi-nonexpansive, we have

$$\lim_{n\to\infty} \sup \|Tx_n - p\| \leq \lim_{n\to\infty} \sup \|x_n - p\| = k. \tag{4.5}$$

We know again by the proof Lemma (2) that $\|y_n - p\| \leq \|x_n - p\|$. So, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|. \end{aligned}$$

It follows

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \leq \|z_n - p\| - \|x_n - p\|.$$

So, we have

$$\|x_{n+1} - p\| \leq \|z_n - p\|.$$

From (4.3), we have

$$k \leq \lim_{n\to\infty} \inf \|z_n - p\|. \tag{4.6}$$

Hence, from (4.4) and (4.6), we have

$$k = \lim_{n\to\infty} \|z_n - p\|.$$

Therefore, since T is quasi-nonexpansive and from (4.3), we have

$$\begin{aligned} k &= \lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|T(1-\gamma_n)x_n + \gamma_n Tx_n) - p\| \\ &\leq \lim_{n\to\infty}\|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\ &\leq \lim_{n\to\infty}(1-\gamma_n)\|x_n - p\| + \lim_{n\to\infty}\sup \gamma_n\|Tx_n - p\| \\ &\leq k. \end{aligned} \tag{4.7}$$

Hence

$$k = \lim_{n\to\infty}\|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\|. \tag{4.8}$$

Now, from (4.4), (4.5), (4.8) and Lemma (1), we conclude that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Let $p \in A(C, \{x_n\})$. By proposition(3), we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \lim_{n\to\infty}\sup\|x_n - Tp\| \\ &\leq (\frac{3+\alpha}{1-\alpha})\lim_{n\to\infty}\sup\|Tx_n - x_n\| + \lim_{n\to\infty}\sup\|x_n - p\| \\ &= \lim_{n\to\infty}\sup\|x_n - p\| \\ &= r(p, \{x_n\}) \end{aligned}$$

Hence, we conclude that $Tp \in A(C, \{x_n\})$. Since E is uniformly convex, $A(C, \{x_n\})$ consist of a uniquence member. Thus, we have $Tp = p$. □

Using Theorem(3), we have the following weak convergence theorem.

Theorem 4. Let C,E,T and $\{x_n\}$ be as in Theorem(3). Suppose E with the opial property and $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges weakly to a fixed point of T.

Proof . By Theorem(3), the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since E is uniformly convex, E is reflexive. So, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $z_1 \in C$. By proposition(4), $p \in F(T)$. Sufficient, we show that $\{x_n\}$ converges weakly to z_1. In fact, if $\{x_n\}$ does not converges weakly to z_1. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z_2 \in C$ such that $\{x_{n_k}\}$ converges weakly to z_2 and $z_1 \neq z_2$. Again, by Proposition (4), $z_2 \in F(T)$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. By the Opial property, we have

$$\begin{aligned} \lim_{n\to\infty}\|x_n - z_1\| &= \lim_{j\to\infty}\|x_{n_j} - z_1\| \\ &< \lim_{j\to\infty}\|x_{n_j} - z_2\| \\ &= \lim_{n\to\infty}\|x_n - z_2\| \\ &= \lim_{k\to\infty}\|x_{n_k} - z_2\| \\ &< \lim_{k\to\infty}\|x_{n_k} - z_1\| \\ &= \lim_{n\to\infty}\|x_n - z_1\|. \end{aligned}$$

This is a contradiction. So, we have $z_1 = z_2$. Thus $\{x_n\}$ converges weakly to $z_1 \in F(T)$. Now, we prove a strong convergence theorem for mappings with condition(C). □

Theorem 5. Let C be a nonempty compact convex subset of a uniformly convex Banach space E and T be a mapping on C such that satisfying condition (C). If $\{x_n\}$ be generated bye (1.8) . Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof . By Theorem(1), $F(T) \neq \emptyset$. Therefore by Theorem(3), $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to z \in C$. Since a generalized 0-nonexpansive mapping satisfying condition(C), then from proposition(3), we have

$$\|x_{n_j} - Tz\| \leq 3\|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - z\|,$$

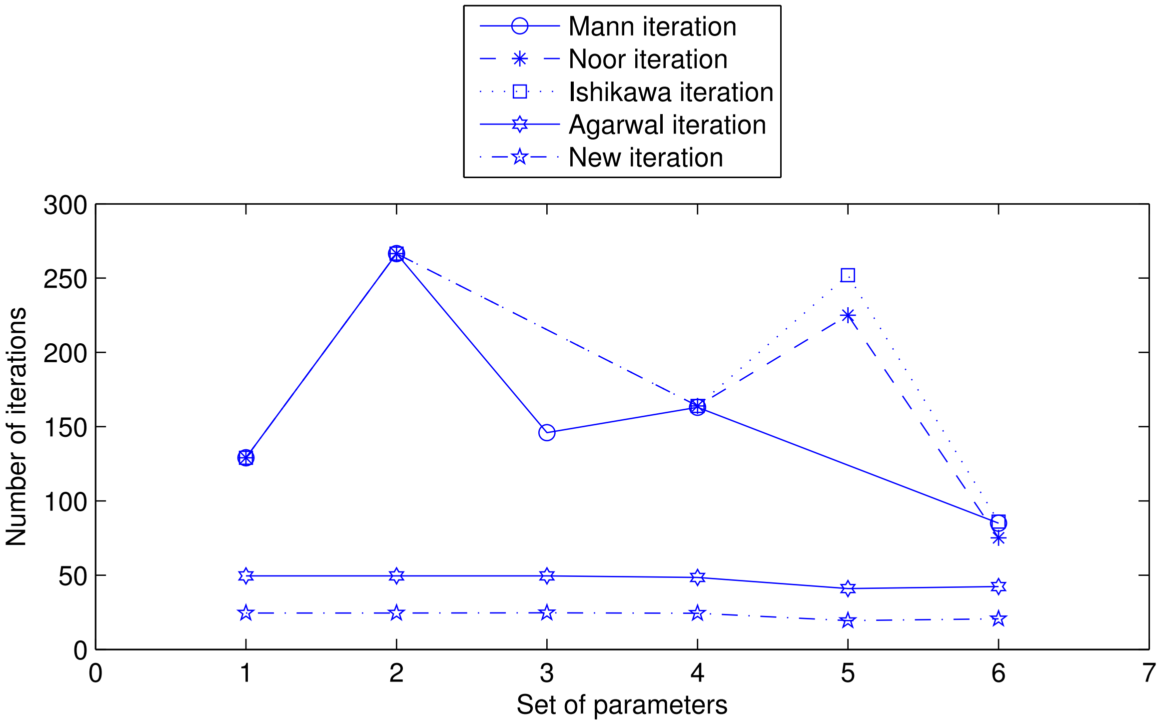

Figure 2: Average number of iterations under distinct parameters.

as $j \to \infty$, we conclude that $\{x_{n_j}\}$ converges to Tz, so $Tz = z$. Therefore by Lemma(2), $x_n \to z$. $\square$ In this paper, we have the following open problem. Now, we using Example 6.1 in [10] for generalized α-nonexpansive mappings, we compare convergence behavior of some well known iteration processes with our iteration (1.8).

Example 2. ([10]). Let the set $K = [-1, 1]$ be equipped with the usual norm $\|.\|$ and let $T : K \to K$ be defined as:

$$T(x) = \begin{cases} \frac{x}{2}, & if \quad x \in [-1, 0) = A \\ -x, & if \quad x \in [0, 1] \setminus \{\frac{1}{2}\} = B \\ 0, & if \quad x = \frac{1}{2} \end{cases} \tag{4.9}$$

Then

(1)T does not satisfy condition (C)

(2)T is a generalized α-nonexpansive mapping.

Initial points	*Mann iteration*	*Ishikawa iteration*	*Noor iteration*	*Agarwal iteration*	*New iteration*
-0.4	271	269	269	49	**24**
-0.2	265	263	263	48	**24**
-0.1	259	257	257	47	**23**
-0.05	253	251	251	46	**23**
0.05	56	59	59	47	**23**
0.1	57	60	60	48	**24**
0.2	59	61	61	49	**24**
0.4	60	63	63	50	**25**

Now, we have some open problems in the end of paper.

Open problem. Is it possible that in general case, Theorem(5) is true for generalized α-nonexpansive mappings?

Open problem. Is it possible to develop an iteration process which rate of convergence is even faster than the iteration process (1.8)?

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