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Fuzzy HUR stability of partitioned functional equations

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Abstract

In this paper, we establish the Hyers-Ulam-Rassias stability of the following functional equation

$$(4p)^n f\left(\frac{x_1+\cdots+x_{(4p)^n}}{(4p)^n}\right)+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)=2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right)$$

in fuzzy Banach spaces.

Keywords: Fuzzy relations, Partitioned functional equation, Hyers-Ulam-Rassias stability, Fuzzy normed space
2020 MSC: 39B52, 46S40, 26E50

1 Introduction

The stability problem of functional equations originated from a question of Ulam [49] in 1940, concerning the stability of group homomorphisms. *Let* $(G_1, .)$ *be a group and let* $(G_2, *)$ *be a metric group with the metric* $d(.,.)$. *Given* $\epsilon > 0$, *does there exist a* $\delta 0$, *such that if a mapping* $h : G_1 \longrightarrow G_2$ *satisfies the inequality* $d(h(x.y), h(x)*h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \longrightarrow G_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [15] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x)-T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Th. M. Rassias [36] proved the following theorem.

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Theorem 1.1. [36]: Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

In 1991, Z. Gajda [13] answered the question about the Rassias Theorem for the case $p > 1$, which was raised by Rassias. This new concept is known as the generalized Hyers–Ulam stability of functional equations.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [47] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [6] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8]–[11], [14, 16, 30, 31], [37]–[44], [46, 51]).

Recently, Trif [48, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n \; {}_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

In [33], Park conjectured the following, and gave a partial answer for the conjecture.

Conjecture. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$p^n f\left(\frac{x_1 + \cdots + x_{p^n}}{p^n}\right) + (pk - p) \sum_{i=1}^{p^{n-1}} f\left(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}\right) = k \sum_{i=1}^{p^n} f\left(\frac{x_i + \cdots + x_{i+k-1}}{k}\right)$$

for all $x_1 = x_{p^n+1}, \cdots, x_{k-1} = x_{p^n+k-1}, x_k, \cdots, x_{p^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$ and each positive integer p.

Fuzzy normed spaces have been the focus of attention for several decades and many related problems have been studied in the field of fuzzy functional analysis. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [12], [19]– [32]). In particular, in 1984, Katsaras [17] introduced the idea of fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. In 1991, Biswas [3] defined and studied fuzzy inner product spaces in linear spaces. After that, some mathematicians have defined fuzzy norms on a linear vector space from special points of view. In 1994, Cheng and Mordeson [4] defined another type of fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is in the sense of Kramosil and Michalek [18]. This idea has been improved by Bag and Samanta [1] in 2003. Also, in 2005, Bag and Samanta [2] established a decomposition theorem of a fuzzy norm and investigated some properties of fuzzy normed linear space. In 2008, Mirmostafaee et al. [26] proved a generalized Hyers- Ulam-Rassias stability theorem in the fuzzy sense. Fuzzy stability of a functional equation associated with inner product spaces has been studied by Park [32] in 2009. In 2010, Saadati and Park [41] investigated the stability of special functional equations on non-Archimedean L-fuzzy normed spaces. In 2013, Mursaleen and Ansari [29] determined some stability results concerning a cubic functional equation in the setting of intuitionistic fuzzy normed spaces. The Hyers-Ulam stability of some functional equations in fuzzy Banach spaces have been proved by Seo et al. [45] in 2015, and also by Park et al. [34] in 2016. In 2018, Ren et al. [40] gave some Hyers-Ulam stability results of Hermite fuzzy differential equations. In 2020, Liu and O'Regan [20]considered Ulam stability concepts for first impulsive fuzzy differential equations. In 2021, Ramdoss et al. [35] have investigated the Hyers-Ulam stability of a new generalized n-variable mixed type of additive and quadratic functional equations in fuzzy modular spaces by using fixed point method. Furthermore, the Hyers-Ulam-Rassias stability of additive mappings in fuzzy normed spaces has been studied by Wu and LU [50] in 2021. In 2022, H. Dutta et al. [7] considered various classical stabilities of a new hexic functional equation in different fuzzy spaces. Also, fuzzy Hyers-Ulam-Rassias stability for generalized additive functional equations has been investigated by Zamani et al. [51] in 2022.

In this paper, we consider the following functional equation

$$(4p)^n f\left(\frac{x_1+\cdots+x_{(4p)^n}}{(4p)^n}\right)+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)=2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right) \tag{1.1}$$

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.1) in fuzzy Banach spaces.

2 Preliminaries

Definition 2.1. *(Bag and Samanta) Let X be a real vector space. A function $N : X\times\mathbb{R}\to[0,1]$ is called a fuzzy norm on X if for all $x, y\in X$ and all $s, t\in\mathbb{R}$,*
$(N1)$ $N(x,t)=0$ *for* $t\le 0$;
$(N2)$ $x=0$ *if and only if* $N(x,t)=1$ *for all* $t>0$;
$(N3)$ $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ *if* $c\neq 0$;
$(N4)$ $N(x+y,c+t)\ge min\{N(x,s),N(y,t)\}$;
$(N5)$ $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty}N(x,t)=1$;
$(N6)$ *for* $x\neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed vector space.

Example 2.2. *Let $(X,\|.\|)$ be a normed linear space and $\alpha,\beta>0$. Then*

$$N(x,t)=\begin{cases}\frac{\alpha t}{\alpha t+\beta\|x\|} & t>0, x\in X\\ 0 & t\le 0, x\in X\end{cases}$$

is a fuzzy norm on X.

Definition 2.3. *(Bag and Samanta) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x\in X$ such that $\lim_{t\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N-\lim_{t\to\infty}x_n=x$.*

Definition 2.4. *(Bag and Samanta) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon>0$ and each $t>0$ there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\epsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X\to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x\in X$ if for each sequence $\{x_n\}$ converging to $x_0\in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X\to Y$ is continuous at each $x\in X$, then $f : X\to Y$ is said to be continuous on X (see [2]).

Definition 2.5. *Let X be a set. A function $d : X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:*

(1) $d(x,y)=0$ *if and only if* $x=y$ *for all* $x,y\in X$;

(2) $d(x,y)=d(y,x)$ *for all* $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ *for all* $x,y,z\in X$.

Theorem 2.6. Let (X,d) be a complete generalized metric space and $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then, for all $x\in X$, either

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty$ for all $n_0\ge n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y=\{y\in X : d(J^{n_0}x,y)<\infty\}$;

(4) $d(y,y^*)\le \frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

3 Fuzzy stability of functional equation (1.1): a fixed point method

Theorem 3.1. Let V and W be vector spaces. A mapping $f:V\to W$ with $f(0)=0$ satisfies the functional equation

$$(4p)^n f\left(\frac{x_1+\cdots+x_{(4p)^n}}{(4p)^n}\right)+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)=2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right) \tag{3.1}$$

for all $x_1=x_{(4p)^n+1}, x_2,\cdots,x_{(4p)^n}\in V$ if and only if the mapping $f:V\to W$ satisfies the Cauchy equation $f(x+y)=f(x)+f(y)$ for all $x,y\in V$.

In this section, using the fixed point alternative approach we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this section, assume that X is a vector space and that (Y,N) is a fuzzy Banach space. Let $\frac{(4p)^n r}{d}\neq 1$.

Theorem 3.2. Let $\varphi: X^{(4p)^n}\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi\left(\frac{(4p)^n r x_1}{d},\frac{(4p)^n r x_2}{d},\cdots,\frac{(4p)^n r x_{(4p)^n}}{d}\right)\le \frac{(4p)^n rL}{d}\varphi(x_1,x_2,\cdots,x_{(4p)^n})$$

for all $x_1=x_{(4p)^n+1}, x_2,\cdots,x_{(4p)^n}\in X$ and positive integers r and d. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned} N\Bigg(\frac{d}{r}f\left(\frac{rx_1+\cdots+rx_{(4p)^n}}{d}\right) =&4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)-2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right),t\Bigg)\\ \ge&\frac{t}{t+\varphi(x_1,x_2,\cdots,x_{(4p)^n})} \end{aligned} \tag{3.2}$$

for all $x_1=x_{(4p)^n+1}, x_2,\cdots,x_{(4p)^n}\in X$ and all $t>0$. Then, the limit

$$A(x):=N\text{-}\lim_{j\to\infty}\left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)$$

exists for each $x\in X$ and defines a unique additive mapping $A:X\to Y$ such that

$$N(f(x)-A(x),t)\ge \frac{((4p)^n-(4p)^nL)t}{((4p)^n-(4p)^nL)t+\varphi(x,x,\cdots,x)}. \tag{3.3}$$

Proof . Putting $x_1 = x_2 = \cdots = x_{(4p)^n} = x$ in (3.2), we have

$$N\left(\frac{d}{r}f\left(\frac{(4p)^n rx}{d}\right) + (4p)^n f(x) - 2(4p)^n f(x), t\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)} \tag{3.4}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \to Y \ ; \ g(0) = 0\}$$

and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}, \forall x \in X,\, t>0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see [22, Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{d}{(4p)^n r} f\left(\frac{(4p)^n rx}{d}\right)$$

for all $x \in X$. Let $g,h \in S$ be such that $d(g,h) = \epsilon$. Then

$$N(g(x)-h(x),\epsilon t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
N(Jg(x)-Jh(x),L\epsilon t) &= N\left(\frac{d}{(4p)^n r}g\left(\frac{(4p)^n rx}{d}\right) - \frac{d}{(4p)^n r}h\left(\frac{(4p)^n rx}{d}\right), L\epsilon t\right) \\
&= N\left(g\left(\frac{(4p)^n rx}{d}\right) - h\left(\frac{(4p)^n rx}{d}\right), \frac{L(4p)^n r\epsilon t}{d}\right) \\
&\geq \frac{\frac{L(4p)^n rt}{d}}{\frac{L(4p)^n rt}{d} + \varphi\left(\frac{(4p)^n rx}{d}, \frac{(4p)^n rx}{d}, \cdots, \frac{(4p)^n rx}{d}\right)} \\
&\geq \frac{\frac{L(4p)^n rt}{d}}{\frac{L(4p)^n rt}{d} + \frac{(4p)^n rL\varphi(x_1,x_2,\cdots,x_m)}{d}} \\
&= \frac{t}{t+\varphi(x,x,\cdots,x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) = \epsilon$ implies that $d(Jg,Jh) \leq L\epsilon$. This means that $d(Jg,Jh) \leq Ld(g,h)$ for all $g,h \in S$. It follows from (3.4) that

$$N\left(f(x) - \frac{d}{(4p)^n r}f\left(\frac{(4p)^n rx}{d}\right), \frac{t}{(4p)^n}\right) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}.$$

So

$$d(f,Jf) \leq \frac{1}{(4p)^n}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{(4p)^n rx}{d}\right) = \frac{(4p)^n rA(x)}{d} \tag{3.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.5) such that there exists $\mu \in (0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^j f, A) \to 0$ as $j \to \infty$. This implies the equality

$$N\text{-}\lim_{j\to\infty} \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{1}{(4p)^n - (4p)^n L}.$$

This implies that the inequality (3.3) holds. Furthermore,

$$\begin{aligned}
&N\left(\frac{d}{r}A\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) + 4p\sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i + x_{i+1}}{2}\right), t\right) \\
&= N - \lim_{j\to\infty}\left(\left(\frac{d}{(4p)^n r}\right)^j \left[\frac{d}{r}f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right)\right.\right. \\
&\left.\left.+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_i + x_{i+1}}{2}\right)\right], t\right) \\
&\geq \lim_{n\to\infty} \frac{\left(\frac{(4p)^n r}{d}\right)^j t}{\left(\frac{(4p)^n r}{d}\right)^j t + \varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1, \left(\frac{(4p)^n r}{d}\right)^j x_2, \cdots, \left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right)} \\
&\geq \lim_{n\to\infty} \frac{\left(\frac{(4p)^n r}{d}\right)^j t}{\left(\frac{(4p)^n r}{d}\right)^j t + \left(\frac{(4p)^n rL}{d}\right)^j \varphi(x_1, x_2, \cdots, x_{(4p)^n})}
\end{aligned}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Since

$$\lim_{n\to\infty} \frac{\left(\frac{(4p)^n r}{d}\right)^j t}{\left(\frac{(4p)^n r}{d}\right)^j t + \left(\frac{(4p)^n rL}{d}\right)^j \varphi(x_1, x_2, \cdots, x_{(4p)^n})} = 1$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$, we deduce that

$$N\left(\frac{d}{r}A\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) + 4p\sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i + x_{i+1}}{2}\right), t\right) = 1$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Putting $x_1 = x_2 = \cdots = x_{(4p)^n} = x$ in the above equality, we find

$$N\left(\frac{d}{r}A\left(\frac{(4p)^n rx}{d}\right) - (4p)^n A(x), t\right) = 1 \tag{3.6}$$

for all $x \in X$ and all $t > 0$. So

$$A\left(\frac{(4p)^n rx}{d}\right) = \frac{(4p)^n r}{d}A(x) \tag{3.7}$$

for all $x \in X$. Since

$$\begin{aligned} \frac{d}{r}A\left(\frac{rx_1+\cdots,rx_{(4p)^n}}{d}\right) &= \frac{d}{r}A\left(\frac{(4p)^n r(x_1+\cdots,x_{(4p)^n})}{(4p)^n d}\right) \\ &= \frac{d}{r}\frac{(4p)^n r}{d}A\left(\frac{x_1+\cdots,x_{(4p)^n}}{(4p)^n}\right) \\ &= (4p)^n A\left(\frac{x_1+\cdots,x_{(4p)^n}}{(4p)^n}\right) \end{aligned} \tag{3.8}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$,

$$\begin{aligned} (4p)^n A\left(\frac{x_1+\cdots+x_{(4p)^n}}{(4p)^n}\right) &+ 4p\sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right) \\ &= 2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i+x_{i+1}}{2}\right) \end{aligned}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$. By Theorem 4.1, the mapping $A : X \to Y$ is additive, as desired. □

Corollary 3.3. Let $\theta \geq 0$ and s be a real number with $s > 2$. Let X be a normed vector space with norm $\|.\|$. If $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and

$$N\left(\frac{d}{r}f\left(\frac{rx_1+\cdots+rx_{(4p)^n}}{d}\right)+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)-2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right),t\right) \geq \frac{t}{t+\theta\left(\sum_{i=1}^{(4p)^n}\|x_i\|^s\right)} \tag{3.9}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$, then the limit

$$A(x) := N\text{-}\lim_{j\to\infty}\left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \geq \begin{cases} \frac{(d^s-(4p)^{ns}r^s)t}{(d^s-(4p)^{ns}r^s)t+d^s\theta\|x\|^s} & \text{if } \frac{(4p)^n r}{d} < 1, \\ \frac{((4p)^{ns}r^s-d^s)t}{((4p)^{ns}r^s-d^s)t+(4p)^{ns}\theta\|x\|^s} & \text{if } \frac{(4p)^n r}{d} > 1 \end{cases}$$

for all $x \in X$.

Proof . The proof follows from Theorem 3.2 by taking $\varphi(x_1, x_2, \cdots, x_{(4p)^n}) := \theta\left(\sum_{i=1}^{(4p)^n}\|x_i\|^s\right)$ for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$. Then we can choose

$$L = \begin{cases} \left(\frac{(4p)^n r}{d}\right)^s & \text{if } \frac{(4p)^n r}{d} < 1, \\ \left(\frac{d}{(4p)^n r}\right)^s & \text{if } \frac{(4p)^n r}{d} > 1 \end{cases}$$

and we get the desired result. □

Theorem 3.4. Let $\varphi : X^{(4p)^n} \to [0,\infty)$ be a function such that there exists an $L < 1$ with

$$\varphi\left(\frac{dx}{(4p)^n r},, \frac{dx}{(4p)^n r}, \cdots, \frac{dx}{(4p)^n r}\right) \leq \frac{dL}{(4p)^n r}\varphi(x_1, x_2, \cdots, x_{(4p)^n})$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and positive integers r and d. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Then, the limit

$$A(x) := N\text{-}\lim_{j\to\infty}\left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{((4p)^n - (4p)^n L)t}{((4p)^n - (4p)^n L)t + L\varphi(x, x, \cdots, x)}. \tag{3.10}$$

Proof . Let (S, d) be the generalized metric space defined as in the proof of Theorem 3.2. Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{(4p)^n r}{d} g\left(\frac{dx}{(4p)^n r}\right)$$

for all $x \in X$. Proceeding as in the proof of Theorem 3.2, we see that if $g, h \in S$ be such that $d(g, h) = \epsilon$ then $d(Jg, Jh) \leq L\epsilon$. It follows from (3.4) that

$$\begin{aligned} N\left(f(x) - \frac{(4p)^n r}{d} f\left(\frac{dx}{(4p)^n r}\right), \frac{rt}{d}\right) &\geq \frac{t}{t + \varphi\left(\frac{dx}{(4p)^n}, \frac{dx}{(4p)^n}, \cdots, \frac{dx}{(4p)^n}\right)} \\ &\geq \frac{t}{t + \frac{dL\varphi(x,x,\cdots,x)}{(4p)^n r}} \\ &= \frac{\frac{(4p)^n rt}{dL}}{\frac{(4p)^n rt}{dL} + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. So

$$N\left(f(x) - \frac{(4p)^n r}{d} f\left(\frac{dx}{(4p)^n}\right), \frac{Lt}{(4p)^n}\right) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}.$$

Therefore

$$d(f, Jf) \leq \frac{L}{(4p)^n}.$$

By Theorem 2.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{dx}{(4p)^n r}\right) = \frac{dA(x)}{(4p)^n r} \tag{3.11}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (3.11) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^j f, A) \to 0$ as $j \to \infty$. This implies the equality

$$N\text{-}\lim_{j\to\infty} \left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{L}{(4p)^n - (4p)^n L}.$$

This implies that the inequality (3.10) holds. The rest of the proof is similar to that of the proof of Theorem 3.2. $\square$

Corollary 3.5. Let $\theta \geq 0$ and s be a real number with $0 < s < \frac{2}{(4p)^n}$. Let X be a normed vector space with norm $\|.\|$. If $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and

$$N\left(\frac{d}{r}f\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) + 4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i + x_{i+1}}{2}\right), t\right) \geq \frac{t}{t + \theta\left(\prod_{i=1}^m \|x_i\|^s\right)} \tag{3.12}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$, then the limit

$$A(x) := N\text{-}\lim_{j\to\infty}\left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \begin{cases} \frac{(4p)^n(d^s - (4p)^{ns}r^s)t}{(4p)^n(d^s - (4p)^{ns}r^s)t + (4p)^{ns}r^s\theta\|x\|^{(4p)^n s}}, & \text{if } \frac{(4p)^n r}{d} < 1, \\ \frac{(4p)^n((4p)^{ns}r^s - d^s)t}{(4p)^n((4p)^{ns}r^s - d^s)t + d^s\theta\|x\|^{(4p)^n s}}, & \text{if } \frac{(4p)^n r}{d} > 1 \end{cases}$$

for all $x \in X$.

Proof . The proof follows from Theorem 3.4 by taking $\varphi(x_1, x_2, \cdots, x_{(4p)^n}) := \theta\left(\prod_{i=1}^{(4p)^n} \|x_i\|^s\right)$ for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$. Then we can choose

$$L = \begin{cases} \left(\frac{(4p)^n r}{d}\right)^s, & \text{if } \frac{(4p)^n r}{d} < 1, \\ \left(\frac{(4p)^n r}{d}\right)^{-s}, & \text{if} \frac{(4p)^n r}{d} > 1 \end{cases}$$

and we get the desired result. □

Remark 3.6. There is a natural difference between normed spaces and fuzzy normed spaces. It's clear from the definition of fuzzy normed space that $N(x,t) \leq 1$ for all $x \in X$ and all $t \geq 0$. If, we wanted to prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in normed spaces, we must considered the control functions $\theta\left(\sum_{i=1}^{(4p)^n} \|x_i\|^s\right)$ and $\theta\left(\prod_{i=1}^{(4p)^n} \|x_i\|^s\right)$, instead of, respectively, $\frac{t}{t+\theta\left(\sum_{i=1}^{(4p)^n} \|x_i\|^s\right)}$ and $\frac{t}{t+\theta\left(\prod_{i=1}^{(4p)^n} \|x_i\|^s\right)}$.

4 Fuzzy stability of functional equation (1.1): a direct method

In this section, using direct method, we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed spaces.

Theorem 4.1. Assume that a mapping $f : X \to Y$ satisfies $f(0) = 0$ and

$$\begin{aligned} N\left(\frac{d}{r}f\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right)\right. &+ 4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) \\ &- 2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i + x_{i+1}}{2}\right), t\Bigg) \geq N'(\varphi(x_1, \cdots, x_{(4p)^n}), t) \end{aligned} \tag{4.1}$$

and $\varphi : X^{(4p)^n} \to Z$ is a mapping for which there is a constant $\alpha \in \mathbb{R}$ satisfying $0 < |\alpha| < \frac{(4p)^n r}{d}$ such that

$$N'\left(\varphi\left(\frac{(4p)^n rx_1}{d}, \frac{(4p)^n rx_2}{d}, \cdots, \frac{(4p)^n rx_{(4p)^n}}{d}\right), t\right) \geq N'\left(\varphi(x_1, \cdots, x_{(4p)^n}), \frac{t}{|\alpha|}\right) \tag{4.2}$$

for all $x_1 = x_{(4p)^n+1}, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Then we can find a unique additive mapping $A : X \to Y$ satisfies (1.1) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{r\varphi(x, x, \cdots, x)}{(4p)^n r - d|\alpha|}, t\right) \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof . It follows from (4.2) that

$$N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1, \left(\frac{(4p)^n r}{d}\right)^j x_2, \cdots, \left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right), t\right) \geq N'\left(\varphi(x_1, x_2, \cdots, x_{(4p)^n}), \frac{t}{|\alpha|^j}\right) \quad (4.4)$$

for all $x_1, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. So

$$N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1, \left(\frac{(4p)^n r}{d}\right)^j x_2, \cdots, \left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right), |\alpha|^j t\right) \geq N'\left(\varphi(x_1, x_2, \cdots, x_{(4p)^n}), t\right) \quad (4.5)$$

for all $x_1, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Substituting $x_1 = x_2 = \cdots = x_{(4p)^n} = x$ in (4.1), we obtain

$$N\left(f(x) - \frac{d}{(4p)^n r} f\left(\frac{(4p)^n r x}{d}\right), \frac{t}{(4p)^n}\right) \geq N'\left(\varphi(x, x, \cdots, x), t\right) \quad (4.6)$$

and all $t > 0$. Replacing x by $\left(\frac{(4p)^n r}{d}\right)^j x$ in (4.6), we have

$$\begin{aligned}
&N\left(\left(\frac{d}{(4p)^n r}\right)^{j+1} f\left(\left(\frac{(4p)^n r}{d}\right)^{j+1} x\right) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), \frac{d^j t}{(4p)^{n(j+1)} r^j}\right) \\
&\geq N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1, \left(\frac{(4p)^n r}{d}\right)^j x_2, \cdots, \left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right), t\right) \\
&\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|\alpha|^j}\right)
\end{aligned}$$

for all $x \in X$, $t > 0$ and any integer $j \geq 0$. So

$$\begin{aligned}
&N\left(f(x) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), \sum_{l=0}^{j-1} \frac{d^l |\alpha|^l t}{(4p)^{n(l+1)} r^l}\right) \\
&= N\left(\sum_{l=0}^{j-1}\left[\left(\frac{d}{(4p)^n r}\right)^{l+1} f\left(\left(\frac{(4p)^n r}{d}\right)^{l+1} x\right) - \left(\frac{d}{(4p)^n r}\right)^l f\left(\left(\frac{(4p)^n r}{d}\right)^l x\right)\right], \sum_{l=0}^{j-1} \frac{d^l |\alpha|^l t}{(4p)^{n(l+1)} r^l}\right) \\
&\geq \min_{0 \leq l \leq j-1}\left\{N\left(\left(\frac{d}{(4p)^n r}\right)^{l+1} f\left(\left(\frac{(4p)^n r}{d}\right)^{l+1} x\right) - \left(\frac{d}{(4p)^n r}\right)^l f\left(\left(\frac{(4p)^n r}{d}\right)^l x\right), \frac{d^l |\alpha|^l t}{(4p)^{n(l+1)} r^l}\right)\right\} \\
&\geq N'(\varphi(x, x, \cdots, x), t)
\end{aligned}$$

which yields

$$\begin{aligned}
&N\left(\left(\frac{d}{(4p)^n r}\right)^{j+k} f\left(\left(\frac{(4p)^n r}{d}\right)^{j+k} x\right) - \left(\frac{d}{(4p)^n r}\right)^k f\left(\left(\frac{(4p)^n r}{d}\right)^k x\right), \sum_{l=0}^{j-1} \frac{d^{l+k} |\alpha|^l t}{(4p)^{n(l+k+1)} r^{l+k}}\right) \\
&\geq N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^k x, \left(\frac{(4p)^n r}{d}\right)^k x, \cdots, \left(\frac{(4p)^n r}{d}\right)^k x\right), t\right) \\
&\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|\alpha|^k}\right)
\end{aligned}$$

for all $x \in X$, $t > 0$ and any integers $j > 0$, $k \geq 0$. Hence one obtains

$$\begin{aligned}
&N\left(\left(\frac{d}{(4p)^n r}\right)^{j+k} f\left(\left(\frac{(4p)^n r}{d}\right)^{j+k} x\right) - \left(\frac{d}{(4p)^n r}\right)^k f\left(\left(\frac{(4p)^n r}{d}\right)^k x\right), t\right) \\
&\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{l=0}^{j-1} \frac{d^{l+k} |\alpha|^{l+k}}{(4p)^{n(l+k+1)} r^{l+k}}}\right)
\end{aligned} \quad (4.7)$$

for all $x \in X$, $t > 0$ and any integers $j > 0$, $k \geq 0$. Since

$$\sum_{l=0}^{\infty} \frac{d^{l+k}|\alpha|^{l+k}}{(4p)^{n(l+k+1)}r^{l+k}} < \infty$$

is convergent series, we see by taking the limit $j \to \infty$ in the last inequality that a sequence $\left\{ \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right) \right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{n\to\infty} \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)$ is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(A(x) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), t\right) = 1 \tag{4.8}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (4.7) that

$$N\left(f(x) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), t\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{l=0}^{j-1} \frac{d^l|\alpha|^l}{(4p)^{n(l+1)}r^l}}\right)$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} N(f(x) - A(x), t) \quad &\geq \quad \min\Bigg\{N\left(f(x) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), (1-\epsilon)t\right), \\ &\qquad N\left(A(x) - \left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right), \epsilon t\right)\Bigg\} \\ &\geq \quad N'\left(\varphi(x, x, \cdots, x), \frac{t}{\sum_{l=0}^{j-1} \frac{d^l|\alpha|^l}{(4p)^{n(l+1)}r^l}}\right) \geq N'\left(\varphi(x, x, \cdots, x), \frac{((4p)^n r - d|\alpha|)\epsilon t}{r}\right) \end{aligned}$$

for sufficiently large j and for all $x \in X$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain $N(f(x) - A(x), t) \geq N'\left(\varphi(x, x, \cdots, x), \frac{((4p)^n r - d|\alpha|)t}{r}\right)$, for all $x \in X$ and $t > 0$. It follows from (4.1) that

$$\begin{aligned} &N\Bigg(\left(\frac{d}{(4p)^n r}\right)^j \Bigg[\frac{d}{r} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) \\ &+4p \sum_{i=1}^{(4p)^{n-1}} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_i + x_{i+1}}{2}\right)\Bigg], t\Bigg) \\ &\geq N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1, \left(\frac{(4p)^n r}{d}\right)^j x_2, \cdots, \left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right), \left(\frac{(4p)^n r}{d}\right)^j t\right) \\ &\geq N'\left(\varphi(x_1, x_2, \cdots, x_{(4p)^n}), \left(\frac{(4p)^n r}{d|\alpha|}\right)^j t\right) \to 1 \text{ as } j \to \infty \end{aligned}$$

for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Therefore, we obtain in view of (4.8)

$$\begin{aligned} &N\Bigg(\frac{d}{r} A\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) + 4p \sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) - 2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i + x_{i+1}}{2}\right), t\Bigg) \\ &\geq \min\Bigg\{N\Bigg(\frac{d}{r} A\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) + 4p \sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) \\ &-2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i + x_{i+1}}{2}\right) - \left(\frac{d}{(4p)^n r}\right)^j \Bigg[\frac{d}{r} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) \end{aligned}$$

$$+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)-2\sum_{i=1}^{(4p)^n} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_i+x_{i+1}}{2}\right)\Bigg],\frac{t}{2}\Bigg),$$

$$N\Bigg(\left(\frac{d}{(4p)^n r}\right)^j\Bigg[\frac{d}{r}f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{rx_1+\cdots+rx_{(4p)^n}}{d}\right)$$

$$+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)-2\sum_{i=1}^{(4p)^n} f\left(\left(\frac{(4p)^n r}{d}\right)^j \frac{x_i+x_{i+1}}{2}\right)\Bigg],\frac{t}{2}\Bigg)$$

$$\geq N'\left(\varphi(x_1,x_2,\cdots,x_m),\left(\frac{(4p)^n r}{d|\alpha|}\right)^j t\right)\to 1 \quad \text{as } j\to\infty$$

which implies

$$\frac{d}{r}A\left(\frac{rx_1+\cdots+rx_{(4p)^n}}{d}\right)+4p\sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)=2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i+x_{i+1}}{2}\right)$$

for all $x_1,x_2,\cdots,x_{(4p)^n}\in X$. Proceeding as in the proof of Theorem 3.2, we have

$$(4p)^n A\left(\frac{x_1+\cdots+x_{(4p)^n}}{(4p)^n}\right)+4p\sum_{i=1}^{(4p)^{n-1}} A\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)=2\sum_{i=1}^{(4p)^n} A\left(\frac{x_i+x_{i+1}}{2}\right)$$

for all $x_1=x_{(4p)^n+1},x_2,\cdots,x_{(4p)^n}\in X$. Thus $A:X\to Y$ is an additive mapping satisfying (4.3).
To prove the uniqueness, assume there is another mapping $L:X\to Y$ which satisfies the inequality (4.3). Since for all $x\in X$ and all integers $j>0$

$$\left(\frac{(4p)^n r}{d}\right)^j A(x)=A\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)\quad\text{and}\quad\left(\frac{(4p)^n r}{d}\right)^j L(x)=L\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)$$

we obtain

$$N(A(x)-L(x),t)=N\left(\left(\frac{d}{(4p)^n r}\right)^j A\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)-\left(\frac{d}{(4p)^n r}\right)^j L\left(\left(\frac{(4p)^n r}{d}\right)^j x\right),t\right)$$

$$\geq \min\Bigg\{N\left(\left(\frac{d}{(4p)^n r}\right)^j A\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)-\left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right),\frac{t}{2}\right),$$

$$N\left(\left(\frac{d}{(4p)^n r}\right)^j f\left(\left(\frac{(4p)^n r}{d}\right)^j x\right)-\left(\frac{d}{(4p)^n r}\right)^j L\left(\left(\frac{(4p)^n r}{d}\right)^j x\right),\frac{t}{2}\right)\Bigg\}$$

$$\geq N'\left(\varphi\left(\left(\frac{(4p)^n r}{d}\right)^j x_1,\left(\frac{(4p)^n r}{d}\right)^j x_2,\cdots,\left(\frac{(4p)^n r}{d}\right)^j x_{(4p)^n}\right),\left(\frac{(4p)^n r}{d}\right)^j\frac{((4p)^n-d|\alpha|)t}{2r}\right)$$

$$\geq N\left(\varphi(x,x,\cdots,x),\left(\frac{(4p)^n r}{d}\right)^j\frac{((4p)^n-d|\alpha|)t}{|\alpha|^j 2r}\right)\to 1\quad\text{as } j\to\infty \text{ by (N5)}$$

for all $x\in X$ and $t>0$. Therefore $A(x)=L(x)$ for all $x\in X$, which completes the proof. □

Corollary 4.2. Let X be a normed space and $(\mathbb{R},N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta\geq 0$ and $0<s<2$ such that a mapping $f:X\to Y$ satisfies $f(0)=0$ and

$$N\Bigg(\frac{d}{r}f\left(\frac{rx_1+\cdots+rx_{(4p)^n}}{d}\right)+4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1}+\cdots+x_{4pi}}{4p}\right)-2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i+x_{i+1}}{2}\right),t\Bigg)\geq N'\left(\theta\left(\sum_{j=1}^{(4p)^n}\|x_j\|^s\right),t\right) \tag{4.9}$$

for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfies (1.1) and the inequality

$$N(f(x) - A(x), t) \geq \begin{cases} N'\left(\frac{(4p)^n r\theta\|x\|^s}{(4p)^n r - d\left(\frac{(4p)^n r}{d}\right)^{1+\frac{s}{2}}}, t\right) & \text{if } \frac{(4p)^n r}{d} < 1, \\ N'\left(\frac{(4p)^n r\theta\|x\|^s}{(4p)^n r - d}, t\right) & \text{if } \frac{(4p)^n r}{d} > 1 \end{cases}$$

for all $x \in X$ and $t > 0$.

Proof . The proof follows from Theorem 4.1 by taking $\varphi(x_1, x_2, \cdots, x_{(4p)^n}) := \theta\left(\sum_{i=1}^{(4p)^n} \|x_i\|^s\right)$ for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$. Then we can choose

$$|\alpha| = \begin{cases} \left(\frac{(4p)^n r}{d}\right)^{1+\frac{s}{2}} & \text{if } \frac{(4p)^n r}{d} < 1, \\ 1 & \text{if } \frac{(4p)^n r}{d} > 1 \end{cases}$$

and we get the desired result. □

Theorem 4.3. Assume that a mapping $f : X \to Y$ satisfying the inequality (4.1), $f(0) = 0$ and $\varphi : X^{(4p)^n} \to Z$ is a mapping for which there is a constant $\alpha \in \mathbb{R}$ satisfying $0 < |\alpha| < \frac{d}{(4p)^n r}$ such that

$$N'\left(\varphi(x_1, \cdots, x_{(4p)^n}), \frac{t}{|\alpha|}\right) \leq N'\left(\varphi\left(\frac{dx_1}{(4p)^n r}, \frac{dx_2}{(4p)^n r}, \cdots, \frac{dx_{(4p)^n}}{(4p)^n r}\right), t\right) \tag{4.10}$$

for all $x_1 = x_{(4p)^n+1}, \cdots, x_{(4p)^n} \in X$ and all $t > 0$. Then we can find a unique additive mapping $A : X \to Y$ that satisfies(1.1) and the following inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{r|\alpha|\varphi(x, x, \cdots, x)}{d - |\alpha|(4p)^n r}, t\right). \tag{4.11}$$

for all $x \in X$ and all $t > 0$.

Proof . It follows from (4.6) that

$$N\left(f(x) - \frac{(4p)^n r}{d} f\left(\frac{dx}{(4p)^n r}\right), \frac{rt}{d}\right) \geq N'\left(\varphi\left(\frac{dx}{(4p)^n}, \frac{dx}{(4p)^n}, \cdots, \frac{dx}{(4p)^n}\right), t\right)$$

for all $x \in X$ and all $t > 0$. Replacing x by $\left(\frac{d}{(4p)^n}\right)^j x$ in the above inequality, we obtain

$$\begin{aligned} &N\left(\left(\frac{(4p)^n r}{d}\right)^{j+1} f\left(\left(\frac{d}{(4p)^n r}\right)^{j+1} x\right) - \left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right), \left(\frac{(4p)^n r}{d}\right)^j \frac{rt}{d}\right) \\ &\geq N'\left(\varphi\left(\left(\frac{d}{(4p)^n r}\right)^{j+1} x, \left(\frac{d}{(4p)^n r}\right)^{j+1} x, \cdots, \left(\frac{d}{(4p)^n r}\right)^{j+1} x\right), t\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{|\alpha|^{j+1}}\right). \end{aligned}$$

So

$$\begin{aligned} &N\left(\left(\frac{(4p)^n r}{d}\right)^{j+1} f\left(\left(\frac{d}{(4p)^n r}\right)^{j+1} x\right) - \left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right), \left(\frac{(4p)^n r}{d}\right)^j \frac{r|\alpha|^{j+1} t}{d}\right) \\ &\geq N'(\varphi(x, x, \cdots, x), t) \end{aligned}$$

for all $x \in X$, all $t > 0$ and all integers $j > 0$. Proceeding as in the proof of Theorem 3.2, we obtain that

$$N\left(f(x) - \left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right), \frac{r|\alpha|}{d} \sum_{l=0}^{j-1} \left(\frac{|\alpha|(4p)^n r}{d}\right)^l t\right) \geq N'(\varphi(x, x, \cdots, x), t)$$

for all $x \in X$, all $t > 0$ and any integer $j > 0$. So

$$\begin{aligned} N\left(f(x) - \left(\frac{(4p)^n r}{d}\right)^j f\left(\left(\frac{d}{(4p)^n r}\right)^j x\right), t\right) &\geq N'\left(\varphi(x, x, \cdots, x), \frac{t}{\frac{r|\alpha|}{d}\sum_{l=0}^{j-1}\left(\frac{|\alpha|(4p)^n r}{d}\right)^l}\right) \\ &\geq N'\left(\varphi(x, x, \cdots, x), \frac{(d - |\alpha|(4p)^n r)t}{r|\alpha|}\right) \end{aligned} \tag{4.12}$$

for sufficiently large j and for all $x \in X$, $t > 0$. The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 4.4. Let X be a normed space and $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $s_i \in \mathbb{R}^+$ with $0 < s = \sum_{j=1}^{(4p)^n} s_i < 2$ such that a mapping $f : X \to Y$ satisfies $f(0) = 0$ and

$$\begin{aligned} N\bigg(\frac{d}{r} f\left(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}\right) &+ 4p\sum_{i=1}^{(4p)^{n-1}} f\left(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}\right) \\ &- 2\sum_{i=1}^{(4p)^n} f\left(\frac{x_i + x_{i+1}}{2}\right), t\bigg) \geq N'\left(\theta\left(\prod_{j=1}^{(4p)^n}\|x_j\|^s\right), t\right) \end{aligned} \tag{4.13}$$

for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \to Y$ that satisfies (1.1) and the inequality

$$N(f(x) - A(x), t) \geq \begin{cases} N'\left(\frac{r\left(\frac{d}{(4p)^n r}\right)^{1+\frac{s}{2}}\theta\|x\|^s}{d - (4p)^n r\left(\frac{d}{(4p)^n r}\right)^{1+\frac{s}{2}}}, t\right) & \text{if } \frac{d}{(4p)^n r} < 1, \\ N'\left(\frac{r\theta\|x\|^s}{d-(4p)^n r}, t\right) & \text{if } \frac{d}{(4p)^n r} > 1 \end{cases}$$

for all $x \in X$ and $t > 0$.

Proof . The proof follows from Theorem 4.3 by taking $\varphi(x_1, x_2, \cdots, x_{(4p)^n}) := \theta\left(\prod_{i=1}^{(4p)^n}\|x_i\|^{s_i}\right)$ for all $x_1, x_2, \cdots, x_{(4p)^n} \in X$. Then we can choose

$$|\alpha| = \begin{cases} \left(\frac{d}{(4p)^n r}\right)^{1+\frac{s}{2}} & \text{if } \frac{d}{(4p)^n r} < 1, \\ 1 & \text{if } \frac{d}{(4p)^n r} > 1 \end{cases}$$

and we get the desired result. □

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