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The q-analog of Kostant's partition function for $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$

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Abstract

In this paper, we consider the q-analog of Kostant's Partition Function of Lie algebras $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$ and present a closed formula for the values of these functions.

Keywords: symplectic Lie algebra, positive root, q-analog of Kostant's partition function
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1 Introduction

Let L be a finite dimensional semi-simple Lie algebra of rank ℓ with a Cartan subalgebra H. Suppose that Φ is a root system corresponding to (L, H) with a simple roots $\Pi = \{R_1, ..., R_\ell\}$ and let Φ^+ denote the positive roots respect to Π [1]. If λ is an integral dominant weight of L and $V(\lambda)$ is the corresponding irreducible L-module, then for any integral dominant weight μ, the multiplicity of μ in λ denoted by $m(\lambda, \mu)$. To compute $m(\lambda, \mu)$, we use the Kostant's weight multiplicity formula [4]:

$$m(\lambda, \mu) = \sum_{\sigma \in \mathcal{W}} \varepsilon(\sigma) \mathfrak{P}(\sigma(\lambda + \rho) - (\mu + \rho)), \tag{1.1}$$

where $\mathcal{W}$ is the Weyl group of L, $\varepsilon(\sigma)$ is the sing of σ, $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ and $\mathfrak{P}$ is the Kostant's Partition Function. The q-analog of Kostant's weight multiplicity formula is a generalization of Kostant's weight multiplicity formula that defined by Luszting in [5] as follows:

$$m_q(\lambda, \mu) = \sum_{\sigma \in \mathcal{W}} \varepsilon(\sigma) \mathfrak{P}_q(\sigma(\lambda + \rho) - (\mu + \rho)). \tag{1.2}$$

In this formula, $\mathfrak{P}_q$ denoting the q-analog of Kostant's partition function defined on $\xi \in H^*$ by $\mathfrak{P}_q(\xi) = \sum c_i q^i$, where c_i is the number of ways to write the weight ξ as a sum of exactly i positive roots.

Note that, if we take $q = 1$ in (1.2), we recover the (1.1). In general, there is no known closed formula for the Kostant's partition function and q-analog of this function. However, various works have been done in this field (see for instance [2, 3, 6, 7]). The aim of this paper is to find a closed formula for $\mathfrak{P}_q$ in Lie algebras $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$.

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2 Preliminaries

In this section, we introduce some notations of Lie algebras $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$. Let

$$H = \{h = diag(a_1, a_2, a_3, a_4) \mid a_i \in \mathbb{C}, \quad a_1 + a_2 + a_3 + a_4 = 0\}$$

be a Cartan subalgebra for $\mathfrak{sl}_4(\mathbb{C})$ and for all $i = 1, ..., 4$, define a functional $\mu_i : H \longrightarrow \mathbb{C}$ by $\mu_i(h) = a_i$. Then the set

$$\Phi = \{\mu_i - \mu_j \mid 1 \leq i \neq j \leq 4\}$$

is the root system for $\mathfrak{sl}_4(\mathbb{C})$. We choose the set

$$\Pi = \{R_1 = \mu_1 - \mu_2, \quad R_2 = \mu_2 - \mu_3, \quad R_3 = \mu_3 - \mu_4\}$$

as a basis for Φ. So the positive roots is the set

$$\Phi^+ = \{\mu_i - \mu_j \mid 1 \leq i < j \leq 4\}.$$

For abbreviation, we will denote the elements of Φ^+ by $\beta_1, ..., \beta_6$. Therefor we can write

$$\beta_1 = R_1, \quad \beta_2 = R_2, \quad \beta_3 = R_3, \quad \beta_4 = R_1 + R_2, \quad \beta_5 = R_2 + R_3, \quad \beta_6 = R_1 + R_2 + R_3.$$

Similarly, for the $\mathfrak{sp}_6(\mathbb{C})$, a Cartan subalgebra is

$$H = \{h = diag(a_1, a_2, a_3, -a_1, -a_2, -a_3) \mid a_i \in \mathbb{C}\}$$

and the functional μ_i defined by same manner. So we can write a root system, simply roots and positive roots as follows:

$$\Phi = \{\pm\mu_i \pm \mu_j \mid 1 \leq i, j \leq 3\} - \{0\},$$
$$\Pi = \{R_1 = \mu_1 - \mu_2, \quad R_2 = \mu_2 - \mu_3, \quad R_3 = 2\mu_3\},$$
$$\Phi^+ = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9\},$$

where

$$\beta_1 = R_1, \quad \beta_2 = R_2, \quad \beta_3 = R_3, \quad \beta_4 = R_1 + R_2, \quad \beta_5 = R_2 + R_3,$$
$$\beta_6 = R_1 + R_2 + R_3, \quad \beta_7 = R_1 + 2R_2 + R_3, \quad \beta_8 = 2R_1 + 2R_2 + R_3, \quad \beta_9 = 2R_2 + R_3.$$

3 Some elementary results

To obtain an exact formula for values of q-analog of Kostant's partition function, we will need to some special results of systems of linear equalities and inequalities. In this section we introduce several systems and their solutions.

Lemma 3.1. If $F(m)$ is the number of non-negative integer solutions of the equality $x + y + 2z = m$, then

$$F(m) = \begin{cases} \frac{1}{4}(m+2)^2; & \text{if m is even} \\ \\ \frac{1}{4}(m+1)(m+3); & \text{if m is odd} \end{cases}$$

Proof . If $z = i$ is a part of a solution of this equality, then we must have $0 \leq i \leq [\frac{1}{2}z]$ and $x + y = m - 2i$, ([ ]=integral part function). Let $[\frac{1}{2}z] = k$. It is easy to check that

$$F(m) = \sum_{i=0}^{k} (m - 2i + 1).$$

Applying these sums, the lemma follows. $\square$

Corollary 3.2. If $F(m,n)$ is the number of non-negative integer solutions of the system

$$\begin{cases} x+y+2z=m, \\ x+y+z\leq n, \end{cases}$$

then

$$F(m,n)=\begin{cases} 0 \quad ; \quad m>2n \\ \\ F(m) \quad ; \quad m\leq n \\ \\ F(2n-m) \quad ; \quad n\leq m\leq 2n \end{cases}$$

Lemma 3.3. If $H(m)$ is the number of non-negative integer solutions of the equality $x+2y=m$, then

$$H(m)=\begin{cases} \frac{1}{2}(m+2) \quad ; \quad \text{if m is even} \\ \\ \frac{1}{2}(m+1) \quad ; \quad \text{if m is odd} \end{cases}$$

Proof . Similar to the proof of Lemma 3.1, the proof is straightforward. □

Corollary 3.4. If $H(m,n)$ is the number of non-negative integer solutions of the system

$$\begin{cases} x+2y\leq m, \\ y\leq n, \end{cases}$$

then

$$H(m,n)=\begin{cases} H(m) \quad ; \quad m<2n \\ \\ n+1 \quad ; \quad m\geq 2n \end{cases}$$

Lemma 3.5.

$$\sum_{i=1}^{m} H(i)=\begin{cases} \frac{1}{4}m(m+4) \quad ; \quad \text{if m is even} \\ \\ \frac{1}{4}(m+1)(m+3)-1 \quad ; \quad \text{if m is odd} \end{cases}$$

Proof . If $m=2k$ then we have

$$\begin{aligned} \sum_{i=1}^{2k} H(i) &= 1+2+2+...+k+k+(k+1) \\ &= \frac{(k+1)(k+2)}{2}+\frac{k(k+1)}{2}-1 \\ &= (k+1)^2-1 \\ &= \frac{1}{4}m(m+4). \end{aligned}$$

The same reasoning applies to the case $m=2k+1$. □

Corollary 3.6.

$$\sum_{i=k+1}^{m} H(i)=\sum_{i=1}^{m} H(i)-\sum_{i=1}^{k} H(i).$$

Lemma 3.7. Let $G(m,n)$ be the number of non-negative integer solutions of the system

$$\begin{cases} x+y+2z=m, \\ x+z\leq n, \end{cases} \tag{3.1}$$

if m is even then

$$G(m,n)=\begin{cases}\frac{1}{4}(m+2)^2; & m\leq n\\ \frac{1}{4}m(m+4)-\frac{1}{2}(m-n+1)(m-n-2); & n<m<2n\\ \frac{1}{2}(n+1)(n+2); & m\geq 2n\end{cases}$$

and if m is odd then

$$G(m,n)=\begin{cases}\frac{1}{4}(m+1)(m+3); & m\leq n\\ \frac{1}{4}(m+1)(m+3)-\frac{1}{2}(m-n+1)(m-n-2)-1; & n<m<2n\\ \frac{1}{2}(n+1)(n+2); & m\geq 2n\end{cases}$$

Proof . We give the proof in the two cases:

Case 1. $m\leq n$. In this case, by Lemma 3.1, we have $G(m,n)=F(m)$ as claimed.

Case 2. $n<m$. If $x=i$ is a part of a solution of the system (3.1), then we must have $0\leq i\leq n$ and

$$\begin{cases} y+2z=m-i,\\ z\leq n-i.\end{cases} \tag{3.2}$$

Hence, the number of non-negative integer solutions of system (3.1) is equal to the number of non-negative integer solutions of the system (3.2) for $i=0,1,...,n$. This follows

$$G(m,n)=\sum_{i=0}^{n}H(m-i,n-i).$$

Now if $m<2n$, then we have

$$\begin{aligned}G(m,n)&=\sum_{i=0}^{n}H(m-i,n-i)\\ &=\sum_{i=0}^{2n-m-1}H(m-i,n-i)+\sum_{i=2n-m}^{n}H(m-i,n-i)\\ &=\sum_{i=0}^{2n-m-1}H(m-i)+\sum_{i=2n-m}^{n}(n-i+1)\\ &=\sum_{i=2m-2n+1}^{m}H(i)+\frac{1}{2}(m-n+1)(m-n+2)\\ &=\sum_{i=1}^{m}H(i)-\sum_{i=1}^{2m-2n}H(i)+\frac{1}{2}(m-n+1)(m-n+2)\\ &=\sum_{i=1}^{m}H(i)-\frac{1}{2}(m-n+1)(m-n-2),\end{aligned}$$

and if $m\geq 2n$, then we have

$$\begin{aligned}G(m,n)&=\sum_{i=0}^{n}H(m-i,n-i)\\ &=\sum_{i=0}^{n}(n-i+1)\\ &=\frac{1}{2}(n+1)(n+2).\end{aligned}$$

□

4 The q-analog of Kostant's partition function for $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$

In this section, our maim results are stated and proved. To make different notations for q-analog of Kostant's partition function in Lie algebras $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$, we use the notations $\mathfrak{P}_q^{\mathfrak{sl}}$ and $\mathfrak{P}_q^{\mathfrak{sp}}$ respectively.

Theorem 4.1. Let $\gamma = aR_1 + bR_2 + cR_3$ *be a weight of the Lie algebra* $\mathfrak{sl}_4(\mathbb{C})$ *where* a, b and c are non-negative integers and $\mathfrak{P}_q^{\mathfrak{sl}}(\gamma) = \sum c_i q^i$. Let $m_i = a + b + c - i$.

(i) If $b \leq a$ and $b \leq c$, then
$$c_i = F(m_i, b).$$

(ii) If $a \leq c \leq b$, then
$$c_i = \begin{cases} F(m_i); \quad m_i \leq a \\ G(m_i, a); \quad a < m_i \leq c \\ ([\frac{m_i}{2}] + 1)(a + c - m_i + 1); \quad \frac{m_i}{2} \leq a, \quad c < m_i \leq b \\ (a+1)(a + c - m_i + 1); \quad a < \frac{m_i}{2} \leq c, \quad c < m_i \leq b \\ 0; \quad c < \frac{m_i}{2} \leq b, \quad c < m_i \leq b \\ ([\frac{m_i}{2}] - m_i + b + 1)(a + c - m_i + 1); \quad \frac{m_i}{2} \leq a, \quad m_i > b \\ (a + b - m_i + 1)(a + c - m_i + 1); \quad a < \frac{m_i}{2} \leq c, \quad m_i > b \\ (a + b - m_i + 1); \quad c < \frac{m_i}{2} \leq b, \quad m_i > b \end{cases}$$

(iii) If $c \leq a \leq b$, then
$$c_i = \begin{cases} F(m_i); \quad m_i \leq c \\ G(m_i, c); \quad c < m_i \leq a \\ ([\frac{m_i}{2}] + 1)(a + c - m_i + 1); \quad \frac{m_i}{2} \leq c, \quad a < m_i \leq b \\ (c+1)(a + c - m_i + 1); \quad c < \frac{m_i}{2} \leq a, \quad a < m_i \leq b \\ 0; \quad a < \frac{m_i}{2} \leq b, \quad a < m_i \leq b \\ ([\frac{m_i}{2}] - m_i + b + 1)(a + c - m_i + 1); \quad \frac{m_i}{2} \leq c, \quad m_i > b \\ (c + b - m_i + 1)(a + c - m_i + 1); \quad c < \frac{m_i}{2} \leq a, \quad m_i > b \\ (c + b - m_i + 1); \quad a < \frac{m_i}{2} \leq b, \quad m_i > b \end{cases}$$

(iv) If $a \leq b \leq c$, then
$$c_i = \begin{cases} F(m_i); \quad m_i \leq a \\ G(m_i, a); \quad a < m_i \leq b \\ \frac{1}{8}(2b - m_i + 2)(4a + 2b - 3m_i + 4); m_i \text{ is even and } \frac{m_i}{2} \leq a, m_i > b \\ \frac{1}{8}(2b - m_i + 1)(4a + 2b - 3m_i + 5); m_i \text{ is odd and } \frac{m_i}{2} \leq a, m_i > b \\ \frac{1}{2}(a + b - m_i + 1)(a + b - m_i + 2); \frac{m_i}{2} > a, m_i > b \end{cases}$$

(v) If $c \leq b \leq a$, then

$$c_i = \begin{cases} F(m_i); \quad m_i \leq c \\ G(m_i, c); \quad c < m_i \leq b \\ \frac{1}{8}(2b - m_i + 2)(4c + 2b - 3m_i + 4); m_i \text{ is even and } \frac{m_i}{2} \leq c, m_i > b \\ \frac{1}{8}(2b - m_i + 1)(4c + 2b - 3m_i + 5); m_i \text{ is odd and } \frac{m_i}{2} \leq c, m_i > b \\ \frac{1}{2}(c + b - m_i + 1)(c + b - m_i + 2); \frac{m_i}{2} > c, m_i > b \end{cases}$$

Proof . Let $\gamma = aR_1 + bR_2 + cR_3$ be a weight of the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$. If we can write γ as a linear combination of positive roots with non-negative integer coefficients, then we will have

$$\begin{aligned} aR_1 + bR_2 + cR_3 &= r_1\beta_1 + ... + r_6\beta_6 \\ &= (r_1 + r_4 + r_6)R_1 + (r_2 + r_4 + r_5 + r_6)R_2 + (r_3 + r_5 + r_6)R_3. \end{aligned}$$

Hence, we must have

$$\begin{cases} r_1 + r_4 + r_6 = a, \\ r_2 + r_4 + r_5 + r_6 = b, \\ r_3 + r_5 + r_6 = c. \end{cases} \tag{4.1}$$

By the definition of q-analog of Kostant partition function, if $\mathfrak{P}_q^{\mathfrak{sl}}(\gamma) = \sum c_i q^i$, then we can write

$$c_i = |\{(r_1, r_2, ..., r_6) : r_j \in \mathbb{Z}, r_j \geq 0, \gamma = \sum_{j=1}^{6} r_j\beta_j, \sum_{j=1}^{6} r_j = i\}|.$$

Equivalently, c_i is the number of ordered 6-tuples of non-negative integer $(r_1, ..., r_6)$ such that $(r_1, ..., r_6)$ is a solution of (4.1) and satisfies in the equality $r_1 + ... + r_6 = i$. Therefore, c_i is the number of non-negative integer solutions of the following system.

$$\begin{cases} r_1 + r_2 + ... + r_6 = i, \\ r_1 + r_4 + r_6 = a, \\ r_2 + r_4 + r_5 + r_6 = b, \\ r_3 + r_5 + r_6 = c. \end{cases} \tag{4.2}$$

According to the system (4.1), if the three non-negative integer r_4, r_5 and r_6 are part of a solution of (4.1), then r_1, r_2 and r_3 are obtained uniquely from the following formula:

$$r_1 = a - r_4 - r_6, \qquad r_2 = b - r_4 - r_5 - r_6, \qquad r_3 = c - r_5 - r_6.$$

On the other hand, if r_4, r_5 and r_6 are part of a solution of (4.1), then we must have

$$\begin{cases} r_4 + r_5 + r_6 \leq b, \\ r_4 + r_6 \leq min\{a, b\}, \\ r_5 + r_6 \leq min\{b, c\}. \end{cases} \tag{4.3}$$

This shows that the number of non-negative integer solutions of (4.1) is equal to the number of non-negative integer solutions of (4.3). For simplicity of notation, let $r_4 = x$, $r_5 = y$ and $r_6 = z$. Now if $(r_1, ..., r_6)$ is a solution of (4.1) which obtained by the above method, then according to the system (4.2), we must have $r_1 + ... + r_6 = i$ and this give $x + y + 2z = a + b + c - i$. Summarizing, we conclude that c_i is equal to the number of non-negative integer solutions of the following system.

$$\begin{cases} x + y + 2z = a + b + c - i, \\ x + y + z \leq b, \\ x + z \leq \min\{a, b\}, \\ y + z \leq \min\{b, c\}. \end{cases} \tag{4.4}$$

Taking $m_i = a + b + c - i$, we consider all possible cases.

(i) If $b \leq a$ and $b \leq c$, then the system (4.4) becomes the following system

$$\begin{cases} x+y+2z = m_i, \\ x+y+z \leq b. \end{cases}$$

This shows that $c_i = F(m_i, b)$, by Corollary 3.2.

(ii) If $a \leq c \leq b$, then the system (4.4) becomes the following system

$$\begin{cases} x+y+2z = m_i, \\ x+y+z \leq b, \\ x+z \leq a, \\ y+z \leq c. \end{cases} \tag{4.5}$$

According to the values of m_i, we have divided the proof of this case into several part:

(M1) If $m_i \leq a$, then it is easy to check that the number of non-negative integer solutions of (4.5) is equal to the number of non-negative integer solutions of the equality $x+y+2z = m_i$, which establishes the formula.

(M2) If $a < m_i \leq c$, then the system (4.5) becomes the following system

$$\begin{cases} x+y+2z = m_i, \\ x+z \leq a. \end{cases}$$

This finishes the proof, by Lemma 3.7.

(M3) If $c < m_i \leq b$, then the number of non-negative integer solutions of (4.5) is equal to the number of non-negative integer solutions of the following system.

$$\begin{cases} x+y+2z = m_i, \\ x+z \leq a, \\ y+z \leq c. \end{cases} \tag{4.6}$$

Now, we compute the number of non-negative integer solutions of (4.6) in 3 cases:

(M3.1) $\frac{m_i}{2} \leq a$. For any solution (x, y, z) of system (4.6), if we take $z = k, (0 \leq k \leq \frac{m_i}{2})$, then (x, y) obtain from the following system

$$\begin{cases} x+y = m_i - 2k, \\ x \leq a-k, \\ y \leq c-k. \end{cases} \tag{4.7}$$

Clearly, this system has a solution if $a+c \geq m_i$. If this condition holds, then the number of non-negative integer solutions of (4.7) is equal to $a+c-m_i+1$. Therefore, in this case we have

$$c_i = \sum_{k=0}^{[\frac{m_i}{2}]} (a+c-m_i+1) = ([\frac{m_i}{2}]+1)(a+c-m_i+1).$$

(M3.2) $a < \frac{m_i}{2} \leq c$. Similarly, we apply (M3.1) for $0 \leq k \leq a$.

(M3.3) $c < \frac{m_i}{2} \leq b$. As was described in (M3.1), $c_i = 0$ because $a+c < m_i$.

(M4) If $m_i > b$ then c_i is equal to the number of non-negative integer solutions of the system (4.5). If $z = k, (0 \leq k \leq \frac{m_i}{2})$ is a part of a solution of (4.5), then the solution remaining obtain from the following system.

$$\begin{cases} x+y = m_i - 2k, \\ x+y \leq b-k, \\ x \leq a-k, \\ y \leq c-k. \end{cases} \tag{4.8}$$

Clearly, the system (4.8) has a solution if and only if $m_i - b \leq k$ and $m_i \leq a + c$. If these conditions hold, the system (4.8) becomes the following system

$$\begin{cases} x + y = m_i - 2k, \\ x \leq a - k, \\ y \leq c - k. \end{cases}$$

It is easy to check that the number of non-negative integer solutions of this system is equal to $a + c - m_i + 1$. Now, we give the proof in three cases:

(M4.1) $\frac{m_i}{2} \leq a$. In this case we have $m_i - b \leq k \leq \frac{m_i}{2}$, so

$$c_i = \sum_{i=m_i-b}^{[\frac{m_i}{2}]} (a + c - m_i + 1) = ([\frac{m_i}{2}] - m_i + b + 1)(a + c - m_i + 1).$$

(M4.2) $a < \frac{m_i}{2} \leq c$. In this case we have $m_i - b \leq k \leq a$. Hence

$$c_i = \sum_{i=m_i-b}^{a} (a + c - m_i + 1) = (a - m_i + b + 1)(a + c - m_i + 1).$$

(M4.3) $c < \frac{m_i}{2} \leq b$. In this case we have again $m_i - b \leq k \leq a$ and the conditions for the existence of solution imply that $m_i = a + c$. Therefore

$$c_i = \sum_{i=m_i-b}^{a} 1 = a - m_i + b + 1$$

(iii) The proof of this case is similar to (ii).

(iv) If $a \leq b \leq c$, then the system (4.4) becomes the following system

$$\begin{cases} x + y + 2z = m_i, \\ x + y + z \leq b, \\ x + z \leq a. \end{cases} \tag{4.9}$$

We compute the number of non-negative integer solutions of (4.9) in three cases:

(N1) If $m_i \leq a$, then c_i is equal to the number of non-negative integer solutions of equation $x + y + 2z = m_i$. Thus $c_i = F(m_i)$, by Lemma 3.1.

(N2) If $a < m_i \leq b$, then the system (4.9) becomes the following system

$$\begin{cases} x + y + 2z = m_i, \\ x + z \leq a. \end{cases}$$

Therefore $c_i = G(m_i, a)$, by lemma 3.7.

(N3) If $m_i > b$, then by taking $z = k, (0 \leq k \leq \frac{m_i}{2})$, as a part of solution of system (4.9), we conclude that the solution remaining obtain from the following system.

$$\begin{cases} x + y = m_i - 2k, \\ x + y \leq b - k, \\ x \leq a - k. \end{cases} \tag{4.10}$$

Clearly, the condition for having a solution is that $m_i - b \leq k$. We have divided the proof of this case into several parts:

(N3.1) $\frac{m_i}{2} \leq a$. In this case we have $m_i - b \leq k \leq [\frac{m_i}{2}]$ and the system (4.10) becomes the following system

$$\begin{cases} x + y = m_i - 2k, \\ x \leq a - k. \end{cases}$$

This shows that

$$c_i = \sum_{i=m_i-b}^{[\frac{m_i}{2}]} (a - k + 1).$$

Applying these sums in two cases $m_i = 2\ell$ and $m_i = 2\ell + 1$, the desired conclusion follows.

(N3.2) $\frac{m_i}{2} > a$. In this case we have $m_i - b \leq k \leq a$. Hence

$$\begin{aligned} c_i &= \sum_{k=m_i-b}^{a} (a - k + 1) \\ &= \frac{1}{2}(a + b - m_i + 1)(a + b - m_i + 2). \end{aligned}$$

(v) The proof of this case is similar to (iv).

□

Theorem 4.2. Let $\gamma = aR_1 + bR_2 + cR_3$ be a weight of the Lie algebra $\mathfrak{sp}_6(\mathbb{C})$. Let

$$\begin{aligned} \hat{m} &= min\{a, [\frac{1}{2}b], c\}, \quad ([\ \ ] = \ integral\ part\ function) \\ \hat{n} &= min\{[\frac{1}{2}(a - m)], [\frac{1}{2}b] - m, c - m\}, \\ \hat{k} &= min\{[\frac{1}{2}b] - m - n, c - m - n\}, \\ \gamma(m, n, k) &= (a - m - 2n)R_1 + (b - 2m - 2n - 2k)R_2 + (c - m - n - k)R_3, \end{aligned}$$

and suppose that $\mathfrak{P}_q^{\mathfrak{sl}}(\gamma(m, n, k)) = \sum c_i q^i$. If $\mathfrak{P}_q^{\mathfrak{sp}}(\gamma) = \sum d_i q^i$, then

$$d_i = \sum_{m=0}^{\hat{m}} \sum_{n=0}^{\hat{n}} \sum_{k=0}^{\hat{k}} \hat{c}(m, n, k)_i,$$

where

$$\hat{c}(m, n, k)_i = c_{\hat{i}}, \ \hat{i} = i - m - n - k.$$

Proof . If $\beta_1, ..., \beta_9$ are positive roots of $\mathfrak{sp}_6(\mathbb{C})$ and $\gamma = \sum_{i=1}^{9} r_i \beta_i$, then we must have

$$\begin{cases} r_1 + r_4 + r_6 + r_7 + 2r_8 = a, \\ r_2 + r_4 + r_5 + r_6 + 2r_7 + 2r_8 + 2r_9 = b, \\ r_3 + r_5 + r_6 + r_7 + r_8 + r_9 = c. \end{cases}$$

By definition, d_i is equal to the number of 9-tuples $(r_1, ..., r_9)$ of non-negative integers such that $(r_1, ..., r_9)$ is a solution of the following system.

$$\begin{cases} r_1 + r_2 + ... + r_9 = i, \\ r_1 + r_4 + r_6 + r_7 + 2r_8 = a, \\ r_2 + r_4 + r_5 + r_6 + 2r_7 + 2r_8 + 2r_9 = b, \\ r_3 + r_5 + r_6 + r_7 + r_8 + r_9 = c. \end{cases} \tag{4.11}$$

By comparing the positive roots of $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sp}_6(\mathbb{C})$, we see that the positive roots $\beta_1, ..., \beta_6$ of $\mathfrak{sp}_6(\mathbb{C})$ are similar to positive roots of $\mathfrak{sl}_4(\mathbb{C})$. Suppose that (m, n, k) be a triple of non-negative integer such that the 9-tuple $(r_1, ..., r_6, m, n, k)$ is a non-negative integer solution of (4.11). then we have

$$\begin{cases} r_1 + r_2 + ... + r_6 = i - m - n - k, \\ r_1 + r_4 + r_6 = a - m - 2n, \\ r_2 + r_4 + r_5 + r_6 = b - 2m - 2n - 2k, \\ r_3 + r_5 + r_6 = c - m - n - k. \end{cases} \tag{4.12}$$

Clearly, for each triple (m, n, k), the number of non-negative integer solution of (4.11) in the form $(r_1, ..., r_6, m, n, k)$ is equal to the number of non-negative integer solution of (4.12). On the other hand, we have

$$\gamma = \sum_{i=1}^{6} r_i\beta_i + m\beta_7 + n\beta_8 + k\beta_9.$$

Since

$$m\beta_7 + n\beta_8 + k\beta_9 = (m + 2n)R_1 + (2m + 2n + 2k)R_2 + (m + n + k)R_3,$$

we have

$$\sum_{i=1}^{6} r_i\beta_i = (a - m - 2n)R_1 + (b - 2m - 2n - 2k)R_2 + (c - m - n - k)R_3.$$

According to the system (4.12), we take $\hat{i} = i - m - n - k$ and

$$\gamma(m, n, k) = (a - m - 2n)R_1 + (b - 2m - 2n - 2k)R_2 + (c - m - n - k)R_3.$$

If $\mathfrak{P}_q^{\mathfrak{sl}}(\gamma(m, n, k)) = \sum c_i q^i$, then the number of non-negative integer solution of (4.12) is equal to $c_{\hat{i}}$. To complete the proof, we need to determine what the triple (m, n, k) is the part of a solution of (4.11). It is easy to check that, we must have

$$0 \leq m \leq \min\{a, [\frac{1}{2}b], c\},$$
$$0 \leq n \leq \min\{[\frac{1}{2}(a - m)], [\frac{1}{2}b] - m, c - m\},$$
$$0 \leq k \leq \min\{[\frac{1}{2}b] - m - n, c - m - n\},$$

and the proof is complete. □

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