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A remark on the Hyers-Ulam-Rassias stability of n-Jordan $*$-homomorphisms on C^*-algebras

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Abstract

In the Hyers-Ulam-Rassias stability depending on the type of the function whose stability we want to verify a suitable functional equation is used. The authors in [6] want to investigate the Hyers-Ulam-Rassias stability of n-Jordan C^*-homomorphisms on C^*-algebras, but they used a quadratic functional equation while we know that the homomorphisms are linear on C^*-algebras. In this paper, we correct the main results of [6] by removing the quadratic functional equation and replacing the linear one and removing some extra conditions. We also show that by using some other multi-variable linear functional equations, the estimation becomes better and more accurate.

Keywords: Hyers-Ulam-Rassias stability, n-Jordan C^*-homomorphism, n-Jordan homomorphism, C^*-algebra
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1 Introduction

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation, and every solution of this quadratic functional equation is said to be a quadratic function, see for example [5, 12, 14, 17, 18, 19] and the references therein. Note that any solution of (1.1) in the space of real numbers is of the form $g(x)=ax^2$ for all $x\in\mathbb{R}$, where $a\in\mathbb{R}$. A generalized stability problem for the quadratic functional equation (1.1) was proved by Skof [21] for mappings from a normed space to a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain is replaced by an Abelian group. Park [20] proved the generalized stability of the quadratic functional equation in Banach modules over a C^*-algebra. Czerwik et al. [2, 3] established the stability of the quadratic functional equations in normed and Lipschitz spaces.

In the category of C^*-algebras each C^*-homomorphism has three properties. Let A and B be two C^*-algebras. The mapping $f:A\to B$ is C^*-homomorphism if f is linear, multiplicative, and $*$-preserving. In the Hyers-Ulam-Rassias stability, the mapping $f:A\to B$ is is said an approximate C^*-homomorphism if there exits a $\delta>0$ such that

$$\|f(\mu x+y)-\mu f(x)-f(y)\|\le\delta(\|x\|^p+\|y\|^p),\quad \|f(xy)-f(x)f(y)\|\le\delta(\|x\|^p+\|y\|^p),\quad \|f(x^*)-f(x)^*\|\le\delta\|x\|^p$$

for all $x,y\in A$, all $\mu\in\mathbb{C}$, and some suitable $p\in\mathbb{R}$.

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Ghaffary et al. [6] investigated the Hyers-Ulam-Rassias stability of n-Jordan C^*-homomorphisms on C^*-algebras. We first review the definition of n-Jordan C^*-homomorphisms on C^*-algebras from [6]. Let $n \in N$ and let A, B be two algebras. A linear map $h : A \to B$ is called n-Jordan homomorphism if

$$h(x^n) = h(x)^n$$

for all $x \in A$, see some more properties in [7, 8, 9]. Let A, B be two C^*-algebras. An n-Jordan homomorphism $h : A \to B$ is called n-Jordan C^*-homomorphism if

$$h(x^*) = h(x)^*$$

for all $x \in A$. In Section 2, we correct the main results of [6] by removing the quadratic functional equation and replacing the linear one and removing some extra conditions. In Section 3, we demonstrate by using some other multi-variable linear functional equations our estimations can be improved and in Section 4, we give some conclusions.

2 Main corrected results

In this section, we correct Theorems 2.1, 2.2 and Corollary 2.3 from [6]. We now review the main results of [6].

Theorem 2.1. [6, Theorem 2.1] Let A and B be two C^*-algebras, let δ, ϵ, p, and q be real numbers such that $p, q < 1$ or $p, q > 1$, and that $q > 0$. Assume that $f : A \to B$ satisfies the functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_B \leq \epsilon(\|x\|_A^p + \|y\|_A^p), \tag{2.1}$$
$$\|f(x^n) - f(x)^n\|_B \leq \delta\|x\|_A^{nq}, \tag{2.2}$$
$$\|f(x^*) - f(x)^*\|_B \leq \delta\|x^*\|_A^q \tag{2.3}$$

for all $x, y \in A$. Then, there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{| 2 - 2^p |}\|x\|_A^p \tag{2.4}$$

for all $x \in A$.

Theorem 2.2. [6, Theorem 2.2] Let A and B be two C^*-algebras, let δ, ϵ, p, and q be real numbers such that $p < 1$ or $q < 0$. If $f : A \to B$ is a mapping with $f(0) = 0$ and the inequalities (2.1), (2.2), and (2.3) are valid. Then, there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{| 2 - 2^p |}\|x\|_A^p$$

for all $x \in A$.

Corollary 2.3. [6, Corollary 2.3] Let A and B be two C^*-algebras, let $\delta, \epsilon \geq 0$, and let p, q be real numbers such that $(p-1)(q-1) > 0$, $q < 0$ or that $(p-1)(q-1) > 0$, $q \geq 0$ and $f(0) = 0$. Assume that $f : A \to B$ satisfies the functional inequalities (2.1), (2.2), and (2.3). Then, there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{| 2 - 2^p |}\|x\|_A^p$$

for all $x \in A$.

By using the direct method, the authors defined in Theorem 2.1 the unique n-Jordan C^*-homomorphism $h : A \to B$ by

$$h(x) = \lim_{m\to\infty} \frac{f(2^{sm}x)}{2^{sm}}$$

for all $x \in A$, where $s = -\mathrm{sgn}(p-1) = -\mathrm{sgn}(q-1)$. Then they claimed that h is an additive map which satisfies (2.4). We show that the condition (2.1) in Theorem 2.1 is a wrong condition, since in this situation one can deduce the mapping h is a quadratic function as follows:

$$\begin{aligned}
&\|\frac{f(2^{sm}(x+y)}{2^{sm}} + \frac{f(2^{sm}(x-y))}{2^{sm}} - 2\frac{f(2^{sm}x)}{2^{sm}} - 2\frac{f(2^{sm}y)}{2^{sm}}\|_B \\
&= \frac{1}{2^{sm}}\|f(2^{sm}(x+y) + f(2^{sm}(x-y)) - 2f(2^{sm}x) - 2f(2^{sm}y)\|_B \\
&\le \epsilon\frac{1}{2^{sm}}(\|2^{sm}x\|_A^p + \|2^{sm}y\|_A^p) \\
&= 2^{sm(p-1)}\epsilon(\|x\|_A^p + \|y\|_A^p). \qquad (2.5)
\end{aligned}$$

We have either $p-1<0$, $s>0$ or $p-1>0$, $s<0$ and so $sm(p-1)<0$. Hence as $m \to \infty$ we conclude that

$$h(x+y) + h(x-y) - 2h(x) - 2h(y) = 0.$$

This indicates that h is a quadratic function while the authors claimed that h is linear. We are now in a position to correct the main result of [6]. Another approach was given by Khodaei [11] to correct and improve the main results of [6]. Khodaei [11] replaced the functional inequality (2.1) with

$$\|f(x+y) + f(x-y) - 2f(x)\|_B \le \theta + \delta(\|x\|_A^p + \|y\|_A^p)$$

and he proved that for some $\theta, \delta \ge 0$, $p<1$ and for all $x, y \in A\setminus\{0\}$ there exists a unique additive mapping $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \theta + \|f(0)\|_B + \frac{2\delta}{2-2^p}\|x\|_A^p$$

for all $x \in A$, see [11, Lemma 2.2]. He also showed in [11, Lemma 2.5] that with $\theta = 0, f(0) = 0, p > 1$ and for all $x, y \in A$ there exists a unique additive mapping $h : A \to B$ such that for all $x \in A$,

$$\|f(x) - h(x)\|_B \le \frac{2\delta}{2^p - 2}\|x\|_A^p.$$

We replace the functional inequality (2.1) with (2.6) in Theorem 2.1 and since $\|x^*\|_A = \|x\|_A$ we replace the functional inequality (2.3) with (2.8) and remove the extra condition $q>0$. We arrange the revised theorem as follows.

Theorem 2.4. Let A and B be two C^*-algebras, let δ, ϵ, p, and q be real numbers such that $p, q < 1$ or $p, q > 1$. Assume that $f : A \to B$ satisfies the functional inequalities

$$\|f(\mu x + y) - \mu f(x) - f(y)\|_B \le \epsilon(\|x\|_A^p + \|y\|_A^p), \qquad (2.6)$$

$$\|f(x^n) - f(x)^n\|_B \le \delta\|x\|_A^{nq}, \qquad (2.7)$$

$$\|f(x^*) - f(x)^*\|_B \le \delta\|x\|_A^q \qquad (2.8)$$

for all $x, y \in A$ and $\mu \in \mathbb{T} := \{z \in \mathbb{C} : \mid z \mid = 1\}$. Then, there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \frac{2\epsilon}{\mid 2 - 2^p \mid}\|x\|_A^p \qquad (2.9)$$

for all $x \in A$.

Proof . Setting $x = y$ and $\mu = 1$ in (2.6) one has

$$\|f(2x) - 2f(x)\|_B \le 2\epsilon\|x\|_A^p. \qquad (2.10)$$

Dividing both sides of (2.10) by 2 we get

$$\|\frac{1}{2}f(2x) - f(x)\|_B \le \epsilon\|x\|_A^p. \qquad (2.11)$$

Replacing x with $2x$ in (2.11) and dividing both sides of the resulting inequality by 2 we find

$$\|\frac{1}{2^2}f(2^2x)-\frac{1}{2}f(2x)\|_B \leq \epsilon\frac{1}{2}\|2x\|_A^p. \tag{2.12}$$

Summing the inequalities (2.11) and (2.12) we reach

$$\|\frac{1}{2^2}f(2^2x)-f(x)\|_B \leq \epsilon\|x\|_A^p+\epsilon\frac{1}{2}\|2x\|_A^p. \tag{2.13}$$

By induction on m we have

$$\|\frac{1}{2^m}f(2^mx)-f(x)\|_B \leq \epsilon\|x\|_A^p\left(\sum_{k=0}^{m}2^{k(p-1)}\right). \tag{2.14}$$

On the other hand, replacing x with $\frac{x}{2}$ in (2.10) we get

$$\|f(x)-2f(\frac{x}{2})\|_B \leq 2\epsilon\|\frac{x}{2}\|_A^p. \tag{2.15}$$

Replacing x with $\frac{x}{2}$ in (2.15) and multiplying both sides of the resulting inequality by 2 we deduce

$$\|2f(\frac{x}{2})-2^2f(\frac{x}{2^2})\|_B \leq 2^2\epsilon\|\frac{x}{2^2}\|_A^p. \tag{2.16}$$

Summing the inequalities (2.15) and (2.16) we reach

$$\|f(x)-2^2f(\frac{x}{2^2})\|_B \leq 2\epsilon\|\frac{x}{2}\|_A^p+2^2\epsilon\|\frac{x}{2^2}\|_A^p. \tag{2.17}$$

By induction on m one can deduce

$$\|f(x)-2^mf(\frac{x}{2^m})\|_B \leq \epsilon\|x\|_A^p\left(\sum_{k=1}^{m}2^{k(1-p)}\right). \tag{2.18}$$

So, the sequence $\{\frac{f(2^{sm}x)}{2^{sm}}\}$ is Cauchy, where $s=-\mathrm{sgn}(p-1)=-\mathrm{sgn}(q-1)$ and hence it is convergent. Define $h: A \to B$ by

$$h(x)=\lim_{m\to\infty}\frac{f(2^{sm}x)}{2^{sm}}.$$

Let $p-1<0$. Then $s=1$. Letting $m\to\infty$ it follows from (2.14) that

$$\|h(x)-f(x)\|_B \leq \frac{2\epsilon}{2-2^p}\|x\|_A^p. \tag{2.19}$$

Let $p-1>0$. Then $s=-1$. Using (2.18) and letting $m\to\infty$ we deduce

$$\|f(x)-h(x)\|_B \leq \frac{2\epsilon}{2^p-2}\|x\|_A^p. \tag{2.20}$$

We now conclude from (2.19) and (2.20) that the desired inequality (2.9) holds. We prove h is linear. The inequality (2.6) ensures that

$$\begin{aligned}
&\|\frac{f(2^{sm}(\mu x+y))}{2^{sm}}-\mu\frac{f(2^{sm}x)}{2^{sm}}-\frac{f(2^{sm}y)}{2^{sm}}\|_B \\
&=\frac{1}{2^{sm}}\|f(2^{sm}(\mu x+y))-\mu f(2^{sm}x)-f(2^{sm}y)\|_B \\
&\leq \frac{\epsilon}{2^{sm}}(\|2^{sm}x\|_A^p+\|2^{sm}y\|_A^p) \\
&=2^{sm(p-1)}\epsilon(\|x\|_A^p+\|y\|_A^p)
\end{aligned} \tag{2.21}$$

for all $x, y \in A$ and $\mu \in \mathbb{T}$. We know that $sm(p-1) < 0$ either $p-1<0$ or $p-1>0$. Letting $m \to \infty$ we obtain

$$h(\mu x + y) - \mu h(x) - h(y) = 0$$

for all $x, y \in A$ and $\mu \in \mathbb{T}$. Using [15, Lemma 1.2] one can deduce h is linear. By the similar way we can prove $h(x^n) = h(x)^n$ and $h(x^*) = h(x)^*$ for all $x \in A$. $\square$

We remark that p and q in Theorem 2.1 should simultaneously be less than one or greater than one. So, Theorem 2.2 has low conditions. Indeed, the conditions $p < 1$ or $q < 0$ are not enough for this theorem and if we correct its conditions it will be Theorem 2.4. The conditions $q < 0$ or $q \geq 0$ in Corollary 2.3 are extra conditions and so they can be removed. If we correct the conditions of Corollary 2.3 it will also be Theorem 2.4.

To obtain some sharp estimations for n-Jordan C^*-homomorphisms one can use the following multi-variable linear functional equations (2.22) and (2.23):

$$\begin{aligned}\sum_{k=2}^{m}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1, i\neq i_1,\ldots,i_{n-k+1}}^{m}\mu x_i - \sum_{r=1}^{n-k+1}\mu x_{i_r}\Big)\\ + f\Big(\sum_{i=1}^{m}\mu x_i\Big) = 2^{m-1}f(\mu x_1) = 0 \qquad (2.22)\end{aligned}$$

and

$$\sum_{i=1}^{m} f\Big(m\mu x_i + \sum_{j=1, j\neq i}^{m}\mu x_j\Big) + f\Big(\sum_{i=1}^{m}\mu x_i\Big) - 2m\mu\sum_{i=1}^{m} f(x_i) = 0 \qquad (2.23)$$

for all $x_i \in A$, $(1 \leq i \leq m)$, and $\mu \in \mathbb{T}$, see for example [15, 16]. In the next section we apply (2.22) to show that our estimations in Section 2 can be even better and more accurate.

3 Improvement of the results

Throughout this section for given mapping $f : A \to B$ we define the difference operator $D_\mu f : A^m \to B$ by setting

$$\begin{aligned}D_\mu f(x_1, \ldots, x_m) := \sum_{k=2}^{m}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{m-k}+1}^{m}\Big)f\Big(\sum_{i=1, i\neq i_1,\ldots,i_{m-k+1}}^{m}\mu x_i - \sum_{r=1}^{m-k+1}\mu x_{i_r}\Big)\\ + f\Big(\sum_{i=1}^{m}\mu x_i\Big) - 2^{m-1}f(\mu x_1)\end{aligned}$$

for all $\mu \in \mathbb{T}$ and $m \geq 2$, see [16, 10]. It was proved in [10, Theorem 2.1] if a function $f : A \to B$ satisfies the functional equation

$$D_\mu f(x_1, \ldots, x_m) = 0,$$

then f is linear. In this section by using the fixed point method and applying the linear functional equation (2.22) we give better and more accurate estimations for our results in the prior section. In more concrete terms, the results of [6] will improve. A fundamental result in the fixed point theory is the following theorem. We recall this theorem which is given by Diaz and Margolis [13].

Theorem 3.1. Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that

1. $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$,
2. the sequence $\{J^n x\}$ converges to a fixed point y^* of J,
3. y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$,
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

We now ready to provide the main result of this section.

Theorem 3.2. Let A and B be two C^*-algebras, let δ, ϵ, p, and q be real numbers such that $p, q > 1$. Assume that $f : A \to B$ satisfies the functional inequalities

$$\|D_\mu f(x_1, ..., x_m)\|_B \leq \epsilon(\|x_1\|_A^p + ... + \|x_m\|_A^p), \tag{3.1}$$

$$\|f(x^n) - f(x)^n\|_B \leq \delta\|x\|_A^{nq}, \tag{3.2}$$

$$\|f(x^*) - f(x)^*\|_B \leq \delta\|x\|_A^q \tag{3.3}$$

for all $x, x_1, ..., x_m \in A$, $\mu \in \mathbb{T}$ and $m \geq 2$. Then there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{1}{2^{m-2}}\frac{2\epsilon}{2^p - 2}\|x\|_A^p \tag{3.4}$$

for all $x \in A$.

Proof . Consider $\varphi(x_1, ..., x_m) = \epsilon(\|x_1\|_A^p + ... + \|x_m\|_A^p)$. Put $\mu = 1, x_1 = x_2 = x$, and $x_3 = x_4 = ... = x_n = 0$ in (3.1) to reach

$$\|\frac{\alpha}{2}f(2x) - \alpha f(x)\|_B \leq \varphi(x, , x, 0, ..., 0)$$

for all $x \in A$ where $\alpha = 2^{m-1}$, $m \geq 2$ and so

$$\begin{aligned}
\|2f(\frac{x}{2}) - f(x)\|_B &\leq \frac{2}{\alpha}\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0)\\
&= \frac{2}{\alpha}\epsilon(\|\frac{x}{2}\|_A^p + \|\frac{x}{2}\|_A^p)\\
&= \frac{1}{\alpha}\frac{1}{2^{p-1}}\epsilon(\|x\|_A^p + \|x\|_A^p)\\
&= \frac{1}{\alpha}\frac{1}{2^{p-1}}\varphi(x, , x, 0, ..., 0).
\end{aligned} \tag{3.5}$$

Denote the set of all functions f from A to B by F. This set of functions is a generalized complete metric space by the following metric

$$\rho(f, g) := \inf\{c \in [0, \infty] : \|f(x) - g(x)\|_B \leq c\varphi(x, x, 0, ..., 0), x \in A\}.$$

Consider the mapping $(\Gamma f)(x) := 2f(\frac{x}{2})$ for all $f \in F$ and $x \in A$. Use [4, Lemma 1.3] and (3.5) to see that Γ is a strictly contractive mapping with the Lipschitz constant $L = \frac{1}{2^{p-1}}$. It follows from (3.5) that $\rho(\Gamma f, f) \leq \frac{1}{\alpha}\frac{1}{2^{p-1}}$. Therefore according to Theorem 3.1, the sequence $\{\Gamma^k f\}$ converges to a fixed point h such that $h(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k})$. Note that h is the unique fixed point of Γ and

$$\begin{aligned}
\rho(h, f) &\leq \frac{1}{1 - L}\rho(\Gamma f, f)\\
&\leq \frac{L}{\alpha(1 - L)}\\
&= \frac{1}{2^{m-1}}\frac{2}{2^p - 2}.
\end{aligned}$$

This means that

$$\begin{aligned}
\|f(x) - h(x)\|_B &\leq \frac{1}{2^{m-1}}\frac{2}{2^p - 2}\varphi(x, , x, 0, ..., 0)\\
&= \frac{1}{2^{m-2}}\frac{2\epsilon}{2^p - 2}\|x\|_A^p.
\end{aligned}$$

for all $x \in A$ and $m \geq 2$. Consequently, the inequality (3.4) holds for all $x \in A$. For the linearity of h we have

$$\begin{aligned}
\|D_\mu h(x_1, ..., x_m)\|_B &= \lim_{k\to\infty} 2^k\|D_\mu f(\frac{x_1}{2^k}, ..., \frac{x_m}{2^k})\|\\
&\leq \lim_{k\to\infty} 2^k\epsilon(\|\frac{x_1}{2^k}\|_A^p + ... + \|\frac{x_m}{2^k}\|_A^p)\\
&= \lim_{k\to\infty} \frac{1}{2^{k(p-1)}}\epsilon(\|x_1\|_A^p + ... + \|x_m\|_A^p)\\
&= 0.
\end{aligned} \tag{3.6}$$

This ensures that the function h satisfies the functional equation $D_\mu h(x_1, ..., x_m) = 0$ and so h is linear. The rest of the proof is similar to that of (2.21) and we omit it. □

We now compare Theorem 2.4 and Theorem 3.2. In Theorem 2.4 for $p > 1$ we found a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{2^p - 2}\|x\|_A^p \tag{3.7}$$

for all $x \in A$ and in Theorem 3.2 we found a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{1}{2^{m-2}}\frac{2\epsilon}{2^p - 2}\|x\|_A^p \tag{3.8}$$

for all $x \in A$ and $m \geq 2$. Indeed, for $m = 2$ the two result coincide, but for $m > 2$ in the later result the distance between the function f and the n-Jordan C^*-homomorphism h is reduced up to $\frac{1}{2^{m-2}}$.

Theorem 3.3. Let A and B be two C^*-algebras, let δ, ϵ, p, and q be real numbers such that $0 < p < 1$ and $q < 1$. Assume that $f : A \to B$ satisfies the functional inequalities

$$\|D_\mu f(x_1, ..., x_m)\|_B \leq \epsilon(\|x_1\|_A^p + ... + \|x_m\|_A^p) \tag{3.9}$$
$$\|f(x^n) - f(x)^n\|_B \leq \delta\|x\|_A^{nq}, \tag{3.10}$$
$$\|f(x^*) - f(x)^*\|_B \leq \delta\|x\|_A^q \tag{3.11}$$

for all $x, x_1, ..., x_m \in A$, $\mu \in \mathbb{T}$ and $m \geq 2$. Then there exists a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{1}{2^{m-2}}\frac{2\epsilon}{2 - 2^p}\|x\|_A^p \tag{3.12}$$

for all $x \in A$.

Proof . Consider $\varphi(x_1, ..., x_m) = \epsilon(\|x_1\|_A^p + ... + \|x_m\|_A^p)$. It follows from (3.9) that

$$\begin{aligned}\|\frac{1}{2}f(2x) - f(x)\|_B &\leq \frac{1}{\alpha}\varphi(x, x, 0, ..., 0)\\ &= \frac{2^{p-1}}{\alpha}\frac{\epsilon}{2^{p-1}}(\|x\|_A^p + \|x\|_A^p)\\ &= \frac{2^{p-1}}{\alpha}2\epsilon(\|\frac{x}{2}\|_A^p + \|\frac{x}{2}\|_A^p)\\ &= \frac{2^{p-1}}{\alpha}2\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0) \qquad (3.13)\end{aligned}$$

for all $x \in A$ where $\alpha = 2^{m-1}$, $m \geq 2$. Consider the generalized complete metric (F, ρ) with the generalized metric ρ defined by

$$\rho(f, g) := \inf\{c \in [0, \infty] : \|f(x) - g(x)\|_B \leq c\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0), x \in A\}.$$

Define the mapping $(Jf)(x) := \frac{1}{2}f(2x)$ for all $f \in F$ and $x \in A$. Apply [4, Lemma 1.2] to find that J is a strictly contractive mapping with the Lipschitz constant $L = 2^{p-1}$. It follows from (3.13) that $\rho(Jf, f) \leq \frac{2L}{\alpha}$. Applying Theorem 3.1, we get the sequence $\{J^k f\}$ converges to a unique fixed point h of J such that $h(x) = \lim_{k\to\infty} \frac{1}{2^k}f(2^k x)$ and

$$\rho(h, f) \leq \frac{1}{1 - L}\rho(Jf, f) \leq \frac{2L}{\alpha(1 - L)}.$$

This entails that

$$\begin{aligned}\|f(x) - h(x)\|_B &\leq \frac{2L}{\alpha(1 - L)}\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0)\\ &= \frac{1}{2^{m-2}}\frac{2\epsilon}{2 - 2^p}\|x\|_A^p\end{aligned}$$

for all $x \in A$ and $m \geq 2$. So, the inequality (3.12) holds for all $x \in A$. The proof of the linearity of h is similar to that of (3.6) and also the rest of the proof is similar to that of (2.21). □

Comparison of Theorem 2.4 and Theorem 3.3 shows that in Theorem 2.4 for $0 < p < 1$ we found a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{2\epsilon}{2 - 2^p}\|x\|_A^p \tag{3.14}$$

for all $x \in A$ where in Theorem 3.3 we found a unique n-Jordan C^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{1}{2^{m-2}}\frac{2\epsilon}{2 - 2^p}\|x\|_A^p \tag{3.15}$$

for all $x \in A$ and $m \geq 2$. In fact for $m = 2$ the bounds for $\|f(x) - h(x)\|_B$ are again the same where for $m > 2$ in the later result (3.15) the distance between the function f and the n-Jordan C^*-homomorphism h is reduced up to $\frac{1}{2^{m-2}}$. It should also be mentioned that in Theorem 2.4, we can take p as negative, but in Theorem 3.3, we have to consider $p > 0$ to avoid $\|0\|^p$, unless we can accept the certain convenient $\|0\|^p = 0$. These results discover that using the linear functional equation (2.22) one can reach some sharp estimations for n-Jordan C^*-homomorphisms and for enough large m these estimations can be improved.

4 Conclusion

It is common to use a functional equation in the Hyers-Ulam-Rassias stability depending on the type of the function whose the stability we want to verify. For example, if the function in our question is linear, a linear functional equation is used, or if the function in our question is quadratic, a quadratic functional equation is used, etc. In the Hyers-Ulam-Rassias stability of n-Jordan C^*-homomorphisms, since the function in our question is linear, a linear functional equation should be used. We remark that the functional equation (1.1) is used in [6], which can be shown by common techniques in the Hyers-Ulam-Rassias stability (as we showed in (2.5)), the function in our question is quadratic, while this is not true, since the function that we are seeking is an n-Jordan C^*-homomorphism and so is linear.

We corrected Theorems 2.1, 2.2 and Corollary 2.3 from [6] by removing the condition (2.1) in Theorem 2.1 and replacing the condition (2.6) and removing some extra conditions. Moreover, we showed that Theorem 2.2 and Corollary 2.3 from [6] are indeed the same as our corrected Theorem 2.4. Finally, by using the fixed point method and applying the multi-variable linear functional equation (2.22) we gave better and more accurate estimations for our results and so the results of [6] were also improved.

Declarations

The author declares that there is no funding available for this article. The author also declares that availability of data 'Not applicable' for this research and state no conflict of interest.

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