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Existence and controllability for non-instantaneous impulsive stochastic integro-differential equations with noncompact semigroups

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Abstract

This paper deals with the existence and exact controllability of a class of non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions in a Hilbert space under the assumption that the semigroup generated by the linear part is noncompact. A set of sufficient conditions are generated using the stochastic analysis technique, Kuratowskii's measure of non-compactness, a resolvent operator and a generalized Darbo's fixed point theorem to obtain existence and controllability results of mild solutions for the considered system. Examples are also given to illustrate the effectiveness of controllability results obtained.

Keywords: Stochastic integro-differential equations, non-instantaneous impulses, resolvent operator, measure of non-compactness, fixed point theory
2020 MSC: 60H10, 34A37, 47G20, 47H10

1 Introduction

In this article, we discuss the existence of mild solutions for the following non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions in the abstract form:

$$\begin{cases} dx(t) = Ax(t)dt + \int_0^t \Upsilon(t-s)x(t)dsdt + f(t,x(t))d\mathbb{W}t, t \in \bigcup\limits_{k=0}^{m}(s_k, t_{k+1}], \\ x(t) = g_k(t, x(t_k^-)), t \in \bigcup_{k=1}^{m}(t_k, s_k], \\ x(0) + h(x) = x_0 \in \mathbb{H}, \end{cases} \tag{1.1}$$

where the state $x(\cdot)$ takes values in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. Υ is a closed linear operator on $\mathbb{H}$ with domain $D(A) \subset D(\Upsilon)$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{K}}$ and

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norm $\| \, . \, \|_{\mathbb{K}}$. Assume that $\{\mathbb{W}(t) : t \geq 0\}$ is a given $\mathbb{K}$-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the Wiener process $\mathbb{W}$. f is a given nonlinear function satisfying some assumption to be specified latter. Let $0 = s_0 = t_0 < t_1 \leq s_1 < t_2 < \ldots \leq s_m < t_{m+1} = b$, where $b > 0$ is a constant and $g_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is called non-instantaneous impulsive function, for all $k = 1, 2 \ldots, m$. x_0 is an $\mathcal{F}_0$-measurable random variable with $E||x_0||^2 < \infty$.

Stochastic differential and integro-differential equations have attracted a lot of attentions of works because of potential applications in many problems in control theory, physics, biology, mechanics and etc. A lot of qualitative properties such as existence, uniqueness and stability for various stochastic integro-differential systems have been obtained, see for instance [23, 9, 13, 14, 15, 21, 26, 38, 37, 4, 31] and the references therein.

Controllability is one of the fundamental concepts of mathematical control theory. First introduced by Kalman [20] in 1963, it has since received great influence both in differential equations and in the theory of stochastic processes. For different types of controllability such as exact, approximate or null controllability, the problem is to find a control function which steers the solution from the initial state to a desired final state. The controllability of nonlinear stochastic integro-differential equations has recently received a lot of attentions (see [35, 3, 18]). Yan and Jia [34] presented the controllability of the controlled fractional impulsive stochastic partial integro-differential systems with non-instantaneous impulses. Youssef and El Hassan [35] studied the controllability of a class of impulsive neutral stochastic integro-differential systems driven by fractional Brownian motion and Poisson process in a separable Hilbert space with infinite delay. Liu et al. [24] studied the existence and approximate controllability for a type of non-instantaneous impulsive stochastic evolution equation excited by fractional Brownian motion with Hurst index $H \in (0, \frac{1}{2})$. Alnafisah and Ahmed [2] investigated the sufficient conditions for null controllability of non-instantaneous impulsive Hilfer fractional stochastic integro-differential system with the Rosenblatt process and Poisson jump. Sunkavilli [33] examined the controllability for a class of multi-valued Sobolev type neutral stochastic differential equations driven by fractional Brownian motion B_t^H with non-instantaneous impulses for $H \in (\frac{1}{2}, 1)$.

On the other side, the state of many evolutionary processes experiences suffered with small abrupt changes at certain moments and it is expressed as impulses. According to the duration of the change, there are two specific cases for this impulse. One is called instantaneous impulse in which the duration of changes is relatively short compared to the overall duration of the whole process (see [22, 28]). The other is called non-instantaneous impulse (see [1, 19]), i.e., impulse starts at any fixed point and remains active in a finite period. Several authors have investigated controllability of impulsive stochastic integro-differential equations. For instance, we refer the reader to [35, 30, 3, 29].

In 1990, Byszewski and Lakshmikantham [8] introduced nonlocal problems for abstract evolution equations. As is noted in [7], nonlocal problems have better effects in applications than classical Cauchy problems, i.e., the nonlocal condition $x(0)+g(t_1, t_2, \cdots, t_k), x(t_1), x(t_2), \cdots, x(t_k)) = x_0$, $0 < t_1 < \cdots < t_k \leq T$ is usually more precise for physical measurements than the classical Cauchy condition $x(0) = x_0$. In [11], the nonlocal condition is used to describe the diffusion phenomena of a small amount of gas in a transport tube. Meraj and Pandey [27] given the existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions by using noncompact semigroup and Darbo-sadovskii fixed point theorem.

In recent years, several papers related to stochastic differential equations have been given without the compact semigroup assumption. Zhang et al.[36], studied the mild solution of stochastic partial differential equation with nonlocal conditions using an equicontinuous semigroup and a generalized fixed point theorem introduced by Liu et al. [25]. In [27] the existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions is given by using noncompact semigroup and Darbo-sadovskii fixed point theorem. Recently, Diop et al. [13] established the existence and controllability results for nonlocal stochastic integro-differential equations with noncompact semigroup.

However, the exact controllability of non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions with noncompact semigroup has not been discussed in the standard literature. Motivated by the above consideration, in this paper, we consider the existence and exact controllability of a class of non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions with the assumption of noncompact semigroup. Our approach here is based on a generalized Darbo's fixed point theorem based on the technique of measure of non-compactness and combined with the resolvent operators theory.

The rest of this paper is organized as follows. In section 2, we present some preliminaries. In section 3, we prove the existence of mild solutions of (1.1) using a generalized Darbo's fixed point theorem. In section 4, an application is given to ensure the exact controllability of the problem (4.1).

2 Preliminaries

Let $(\mathbb{H}, (\cdot,\cdot), \| \,.\, \|)$, and $(\mathbb{K}, (\cdot,\cdot)_{\mathbb{K}}, \| \,.\, \|_{\mathbb{K}})$ be two real separable Hilbert spaces. $\mathrm{L}(\mathbb{K},\mathbb{H})$ be the space of all bounded linear operators mapping $\mathbb{K}$ into $\mathbb{H}$ and $\mathrm{L}(\mathbb{H})$ whenever $\mathbb{K}=\mathbb{H}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e it is right continuous increasing family and $\mathcal{F}_0$ contains all $\mathcal{P}$ null sets), let $\mathbb{W}$ be a Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with the covariance operator Q such that $\mathrm{Tr}\ Q<\infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers $\{\lambda_i\}_{n=1}^{\infty}$ such that $Qe_n=\lambda_n e_n$, $n\in\mathbb{N}$ and a sequence β_n of independent Brownian motions such that

$$\mathbb{W}(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n e_n, \quad t\in[0,b],$$

and $\mathcal{F}_t=\mathcal{F}_t^{\mathbb{W}}$, where $\mathcal{F}_t^{\mathbb{W}}$ is the σ-algebra generated by $\{\mathbb{W}(t): 0\leq s\leq t\}$. For ψ, $\varphi\in \mathrm{L}(\mathbb{K},\mathbb{H})$, we define $\mathrm{L}_Q=\mathrm{L}_2(Q^{1/2}\mathbb{K},\mathbb{H})$ the space of all Q-Hilbert-Schmidt operators from $Q^{1/2}\mathbb{K}$ to $\mathbb{H}$ with the inner product $(\varphi,\psi)_Q=Tr(\varphi Q\psi^*)$, where ψ^* is the adjoint of the operator ψ. Clearly, for any bounded operator $\psi\in\mathrm{L}(\mathbb{K},\mathbb{H})$, we have

$$\|\psi\|_Q^2=Tr(\psi Q\psi^*)=\sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\psi e_n\|.$$

The collection of all strongly measurable square integrable, $\mathbb{H}$ valued random variables denoted by $\mathrm{L}^2(\Omega,\mathbb{H})$ is a Banach space equipped with the norm

$$\|x\|_{\mathrm{L}^2(\Omega,\mathbb{H})}=(\mathrm{E}\|x\|^2)^{1/2},$$

where $\mathrm{E}(x)=\displaystyle\int_{\Omega}x(\cdot)d\mathcal{P}(\cdot)$. The subspace $\mathrm{L}_0^2(\Omega,\mathbb{H})$ is given by

$$\mathrm{L}_0^2(\Omega,\mathbb{H})=\Big\{f\in\mathrm{L}^2(\Omega,\mathbb{H})\ : f\ is\ \mathcal{F}_0\text{-measurable}\Big\}.$$

Now, we define the space of piecewise continuous functions $\mathcal{PC}([0,b],\mathbb{H})$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{H}$ valued stochastic process $\{x(t): t\in[0,b]\}$ such that x is continuous at $t\neq t_k$, $x(t_k^-)=x(t_k)$ and $x(t_k^+)$ exist for all $k=1,2,3\dots,m$ endowed with the norm

$$\|x\|_{\mathcal{PC}}=\Big(\sup_{0\leq t\leq b}\mathrm{E}\|x(t)\|^2\Big)^{\frac{1}{2}},$$

it is easy to see that $(\mathcal{PC}([0,b],\mathbb{H}),\|\cdot\|_{\mathcal{PC}})$ is a Banach space. The following result is very important to prove our main results.

Lemma 2.1. ([10]) For $T>0$, let

$$M(\mathbb{K},\mathbb{H})=\Big\{\Phi(\cdot,\cdot):\Phi \text{ is an } \mathrm{L}(\mathbb{K},\mathbb{H})\text{-valued stochastic process on } [0,T]\times\Omega \text{ such that}$$

$$\Phi(t) \text{ is measurable relative to } \mathcal{F}_t \text{ for all } t\in[0,T],\ \int_0^T E\|\Phi(t)\|^2dt<\infty\Big\}.$$

If Φ is an element of $M(\mathbb{K},\mathbb{H})$, then we have the following property

$$E\Big\|\int_0^T\Phi(s)d\mathbb{W}s\Big\|^2\leq TrQ\int_0^T E\big\|\Phi(s)\big\|^2ds.$$

Now, we introduce some basics about the Kuratowskii measure of non-compactness. Which will be needed throughout this paper.

Definition 2.2. ([5]) The Kuratowskii measure of non-compactness $\alpha(\cdot)$ defined on a bounded set U of Hilbert space $\mathbb{H}$ by

$$\alpha(U)=\inf\Big\{\delta>0\ :\ U=\bigcup_{n=1}^{m}U_n \quad \text{with } diam(U_n)\le\delta, \quad \text{for } n=1,2,\dots,m\Big\}.$$

Theorem 2.3. ([5]) Let $\mathbb{H}$ be a Hilbert space and $U,V\subset\mathbb{H}$ be bounded, then the following properties are satisfied:

(a) $\alpha(U)=0\Leftrightarrow\overline{U}$ is compact;
(b) $\alpha(U)=\alpha(\overline{U})=\alpha(\mathrm{co}(U))$, where $\mathrm{co}(U)$ means the convex hull of U;
(c) $\alpha(\lambda U)=|\lambda|\alpha(U)$, for any $\lambda\in\mathbb{R}$;
(d) $\alpha(U)\le\alpha(V)$, when $U\subset V$;
(e) $\alpha(U\cup V)=\max\{\alpha(U),\alpha(V)\}$;
(f) $\alpha(U+V)\le\alpha(U)+\alpha(V)$, where $U+V=\{x|x=y+z,y\in U,z\in V\}$;
(g) $\alpha(U+x)=\alpha(U)$, for all $x\in\mathbb{H}$;
(h) if the map $Q:D(Q)\subset\mathbb{H}\longrightarrow\mathbb{K}$ is lipschitz continuous with constant k, then $\alpha(Q(U))\le k\alpha(U)$ for any bounded subset $U\subset D(Q)$, and $\mathbb{K}$ is another Hilbert space.

The notation $\alpha(\cdot)$ $\alpha(\cdot)_{\mathcal{C}}$ $\alpha(\cdot)_{\mathcal{PC}}$ are the Kuratowskii measure of non-compactness on the bounded set of $\mathbb{H}$, $\mathcal{C}([0,b],\mathbb{H})$, and $\mathcal{PC}([0,b],\mathbb{H})$, respectively.

For more details see([5]).

Lemma 2.4. ([5]) If $U\subset\mathcal{PC}([0,b],\mathbb{H})$ is bounded, then $\alpha(U(t))\le\alpha_{\mathcal{PC}}(U)$ for all $t\in[0,b]$, where $U(t)=\{x(t):x\in U\}\subseteq\mathbb{H}$. Furthermore, if U is piecewise equicontinuous on $[0,b]$, then $U(t)$ is continuous for $t\in[0,b]$, and $\alpha_{\mathcal{PC}}(U)=\sup_{t\in[0,b]}\alpha(U(t))$.

Definition 2.5. ([12]) A continuous map $Q:U\subseteq\mathbb{H}\to\mathbb{H}$ is said to be α-contraction if there exists a positive constant $k\in[0,1)$ such that for any bounded set $\Omega\subset U$

$$\alpha(Q(\Omega))\le k\alpha(\Omega).$$

Theorem 2.6. (Generalized Darbo's fixed point theorem [25, 32]) Let E be a closed and convex subset of a real Banach space $\mathbb{H}$. Suppose that $Q:E\to E$ is a continuous operator and $Q(E)$ is bounded, for any bounded subset $D\subset E$,

$$Q^1(D)=Q(D),\qquad Q^n(D)=Q(\overline{co}(Q^{n-1}(D))),n=2,3,\dots,m.$$

If there exists a constant $0\le\delta<1$, and a positive integer n_0 such that for any bounded subset $D\subset E$.

$$\alpha(Q^{n_0}(D))\le\delta\alpha(Q(D)).$$

Then Q has at least one fixed point in D.

Theorem 2.7. (Darbo's fixed point theorem [5]) Let $\mathbb{H}$ be a Banach space. If $U\subset\mathbb{H}$ is a bounded closed and convex subset, the continuous map $Q:U\to U$ is a α-contraction.

Then Q has at least one fixed point in U.

In this part, we introduce some basic notions about resolvent operators for integro-differential equations.

In what follows, H is a Banach space, A and $\Upsilon(t)$ are closed linear operators on H. And $\mathbb{K}$ be the Banach space $D(A)$ equipped with the graph norm defined by

$$\|y\|_{\mathbb{K}}=\|Ay\|+\|y\|,\ y\in\mathbb{K}.$$

Let us consider the following Cauchy problem

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t \Upsilon(t-s)x(s)ds, \quad t \geq 0, \\ x(0) &= x_0 \in \mathbb{H}. \end{aligned} \tag{2.1}$$

Definition 2.8. ([16]) A resolvent operator for problem (2.1) is a bounded linear operator $R(t) \in \mathrm{L}(\mathbb{H})$ for $t \geq 0$, having the following properties:

(i) $R(0) = I$ (The Identity operator of $\mathbb{H}$) and $\|R(t)\| \leq Me^{\beta t}$ for some constants $M > 0$ and $\beta \in \mathbb{R}$.
(ii) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) For $x \in \mathbb{K}$, $R(\cdot)x \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{H}) \cap \mathcal{C}(\mathbb{R}^+, \mathbb{K})$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \Upsilon(t-s)R(s)x\ ds \\ &= R(t)Ax + \int_0^t R(t-s)\Upsilon(s)x\ ds, \text{for } t \geq 0. \end{aligned}$$

Next, we make the following hypotheses:

(H_1) The operator A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$.

(H_2) For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from $D(A)$ to $\mathbb{H}$ and $\Upsilon(t) \in \mathrm{L}(\mathbb{K}, \mathbb{H})$. For any $x \in \mathbb{K}$, the map $t \to \Upsilon(t)x$ is bounded, differentiable and the derivative $t \to \Upsilon^{'}(t)x$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.9. ([16]) Assume that (H_1)-(H_2) hold. Then there exists a unique resolvent operator to the Cauchy problem(2.1).

More details can be found in [16, 17].

Definition 2.10. ([36]) A semigroup $T(t)(t \geq 0)$ in $\mathbb{H}$ is said to be equicontinuous if the operator $T(t)$ is uniformly continuous by operator norm for every $t > 0$.

Theorem 2.11. ([14]) Let A be the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ and let $\Upsilon(t)(t \geq 0)$ satisfy (H_2). Then the resolvent operator $R(t)(t > 0)$ is operator norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $T(t)(t \geq 0)$ is operator norm continuous for $t \geq 0$.

Now, we can get the definition of mild solution for our main problem.

Definition 2.12. A $\mathcal{F}_t$-adapted stochastic process $x(t) : [0, b] \to \mathbb{H}$ is called a mild solution of(1.1) if $x(0) + h(x) = x_0 \in \mathbb{H}$, $x \in \mathcal{PC}([0, b], \mathbb{H})$ and

$$x(t) = \begin{cases} R(t)\big(x_0 - h(x)\big) + \displaystyle\int_0^t R(t-s)f(s, x(s))\ d\mathbb{W}s, & t \in [0, t_1], \\ g_k(t, x(t_k^-)), & t \in \bigcup_{k=1}^m (t_k, s_k], \\ R(t-s_k)g_k(s_k, x(t_k^-)) + \displaystyle\int_{s_k}^t R(t-s)f(s, x(s))\ d\mathbb{W}s, & t \in \bigcup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{2.2}$$

3 Existence results

In this section, we prove the existence of mild solutions for the system (1.1). The following assumptions will be needed throughout the paper:

(H_3) The resolvent operator $R(t), t \geq 0$ is continuous in operator norm topology, and there exists a constant $M > 0$ such that
$$\|R(t)\| \leq M.$$

(H_4) The nonlinear function $f : [0, b] \times \mathbb{H} \longrightarrow \mathrm{L}(\mathbb{K}, \mathbb{H})$ satisfying the following conditions:

1. $f(\cdot, u)$ is strongly measurable for $x \in \mathbb{H}$.
2. $f(t, \cdot)$ is continuous for any $t \in [0, b]$.
3. For some positive number $q > 0$, there exists a constant $\rho > 0$, Lebesgue integrable function $\varphi : [0, b] \to [0, +\infty)$ and a non-decreasing continuous function $\psi_f : [0, +\infty) \to (0, +\infty)$ such that

$$E(\|f(t, x)\|^2) \leq \varphi(t)\psi_f(E\|x\|^2), \qquad \lim_{n\to+\infty} \inf \frac{\psi_f(q)}{q} = \rho < +\infty.$$

(H_5) The impulsive function $g_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is continuous and compact, and there exist constants $N_{g_k} > 0, k = 1, 2, 3, \ldots, m$, such that for all $x \in \mathbb{H}$
$$E\|g_k(t, x)\|^2 \leq N_{g_k} E\|x\|^2.$$

(H_6) The nonlocal function $h : \mathcal{PC}([0, b], \mathbb{H}) \longrightarrow \mathbb{H}$ is continuous and compact, and there exists a constant $N_h > 0$, such that for all $x \in \mathbb{H}$
$$E\|h(x)\|^2 \leq N_h.$$

(H_7) There exists a positive constant L such that for any bounded set $U \subset \mathbb{H}$
$$\alpha(f(t, U)) \leq L\alpha(U).$$

For simplicity of notations, we denote
$$N_g = \max_{k=1,2,\ldots,m} N_{g_k}, \quad \Lambda = \max_{k=0,1,2,\ldots,m} \|\varphi\|_{L[s_k,t_{k+1}]}.$$

Theorem 3.1. Assume that the conditions (H_1)-(H_7) are satisfied, then the problem (1.1) has at least one mild solution provided that
$$2M^2 \left(TrQ\ \rho\Lambda + N_g\right) < 1. \tag{3.1}$$

Proof . Consider the operator $\Xi : \mathcal{PC}([0, b], \mathbb{H}) \to \mathcal{PC}([0, b], \mathbb{H})$ defined by
$$\Xi x(t) = \begin{cases} R(t)\big(x_0 - h(x)\big) + \displaystyle\int_0^t R(t - s)f(s, x(s))\, d\mathbb{W}s, & t \in [0, t_1], \\ g_k(t, x(t_k^-)), & t \in \bigcup_{k=1}^m (t_k, s_k], \\ R(t - s_k)g_k(s_k, x(t_k^-)) + \displaystyle\int_{s_k}^t R(t - s)f(s, x(s))\, d\mathbb{W}s, & t \in \bigcup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{3.2}$$

Obviously, the fixed point of $\Xi x(t)$ is the solution of the problem (1.1). For each finite constant $r > 0$, let
$$\Omega_r = \Big\{x \in \mathcal{PC}([0, b], \mathbb{H}) : \|x\|^2_{\mathcal{PC}} \leq r\Big\}.$$

It is clear that Ω_r is a bounded closed and convex set in $\mathcal{PC}([0, b], \mathbb{H})$. The proof falls naturally into four steps.

Step 1: We prove that there exists a constant $r > 0$ such that $\Xi(\Omega_r) \subset \Omega_r$. Assuming the opposite, for each $r > 0$, there would exist $x_r \in \Omega_r$ and $t_r \in [0, b]$ such that $E\|\Xi(x_r)(t_r)\|^2 > r$. For that, we consider three cases.

Case 1: For $t_r \in [0, t_1]$, by Lemma 2.1, (3.2) and assumptions (H_3)-(H_4), (H_6), we obtain

$$\begin{aligned}
E\|(\Xi x_r)(t_r)\|^2 &\leq 2M^2 E\|x_0 - h(x_r)\|^2 + 2M^2 E\Big\|\int_0^{t_r} f(s, x_r(s))\ d\mathbb{W}s\Big\|^2 \\
&\leq 2M^2\big(E\|x_0\|^2 + E\|h(x_r)\|^2\big) + 2M^2\ TrQ \int_0^{t_r} E\|f(s, x_r(s))\|^2\ ds \\
&\leq 2M^2\big(E\|x_0\|^2 + N_h\big) + 2M^2\ TrQ \int_0^{t_r} \varphi(s)\psi_f(E\|x_r\|^2)\ ds \\
&\leq 2M^2\ \big(E\|x_0\|^2 + N_h\big) + 2M^2\ TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[0,t_1]},
\end{aligned}$$

so we have

$$E\|(\Xi x_r)(t_r)\|^2 \leq 2M^2\Big(E\|x_0\|^2 + N_h + TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[0,t_1]}\Big). \tag{3.3}$$

Case 2: For $t_r \in (t_k, s_k], k = 1, 2, \ldots, m$, by(3.2) and assumption (H5), we get

$$\begin{aligned}
E\|(\Xi x_r)(t_r)\|^2 &= E\|g_k(t_r, x_r(t_k^-))\|^2 \\
&\leq N_{g_k} E\|x_r(t_r)\|^2 \\
&\leq N_g\ r.
\end{aligned}$$

Then

$$E\|(\Xi x_r)(t_r)\|^2 \leq N_g\ r. \tag{3.4}$$

Case 3: For $t_r \in (s_k, t_{k+1}], k = 1, 2, \ldots, m$, by Lemma 2.1, (3.2), and assumptions (H_3)-(H_5), we obtain

$$\begin{aligned}
E\|(\Xi x_r)(t_r)\|^2 &\leq 2M^2\ E\|g_k(s_k, x(t_k^-))\|^2 + 2M^2\ TrQ \int_{s_k}^{t_r} E\|f(s, x_r(s))\|^2\ ds \\
&\leq 2M^2 N_g r + 2M^2\ TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,t_{k+1}]}.
\end{aligned}$$

Hence

$$E\|(\Xi x_r)(t_r)\|^2 \leq 2M^2\Big(N_g\ r + TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,t_{k+1}]}\Big), \tag{3.5}$$

from (3.3), (3.4) and (3.5), we have for a.e $t \in [0, b]$

$$r < E\|(\Xi x_r)(t_r)\|^2 \leq 2M^2\ \Big(E\|x_0\|^2 + N_h + N_g r + TrQ\ \psi_f(r)\Lambda\Big).$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, we get

$$1 \leq 2M^2\Big(TrQ\ \rho\Lambda + N_g\Big),$$

which contradict with condition (3.1), hence $\Xi(\Omega_r) \subset \Omega_r$.

Step 2: We prove that the operator Ξ is continuous in Ω_r. Let us consider a sequence $\{x_n\}_{n=1}^{+\infty} \subset \mathcal{PC}([0, b], \mathbb{H})$ such that $\lim_{n\to+\infty} x_n = x \in \mathcal{PC}([0, b], \mathbb{H})$. Since f is a Carathédory function and using the fact that the nonlocal function h, and g_k are continuous, we have

$$\lim_{n\to+\infty} f(s, x_n(s)) = f(s, x(s)), \tag{3.6}$$

$$\lim_{n\to+\infty} g_k(s, x_n(t_k^-)) = g_k(s, x(t_k^-)), \tag{3.7}$$

$$\lim_{n\to+\infty} h(x_n) = h(x). \tag{3.8}$$

By assumption (H_4), for a.e $t \in [0, b]$, we obtain

$$E\Big\|f(s, x_n(s)) - f(s, x(s))\Big\|^2 \leq 2E\big\|f(s, x_n(s))\big\|^2 + 2E\big\|f(s, x(s))\big\|^2 \leq 4\varphi(s)\psi_f(r). \tag{3.9}$$

Case 1: For $t \in [0, t_1]$, using the fact that the function $s \to 4\varphi(s)\psi_f(r)$ is Lebesgue integrable for $s \in [0,t]$ and $t \in (0, t_1]$, so by Lemma 2.1, (3.6), (3.8) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} E\big\|(\Xi x_n)(t) - (\Xi x)(t)\big\|^2 &\leq 2M^2\Big(E\big\|h(x_n) - h(x)\big\|^2 + TrQ\int_{s_k}^t E\big\|f(s, x_n(s)) - f(s, x(s))\big\|^2 ds\Big) \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \end{aligned}$$

Case 2: For $t \in (t_k, s_k], k = 1, 2, \dots, m$, by (3.7), we get

$$E\big\|(\Xi x_n)(t) - (\Xi x)(t)\big\|^2 \leq E\big\|g_k(s, x_n(t_k^-)) - g_k(s, x(t_k^-))\big\|^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Case 3: For $t \in (s_k, t_{k+1}], k = 1, 2, \dots, m$, by Lemma 2.1, (3.6), (3.7), (3.9) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} E\big\|(\Xi x_n)(t) - (\Xi x)(t)\big\|^2 \leq& 2M^2 E\big\|g_k(s, x_n(t_k^-)) - g_k(s, x((t_k^-)))\big\|^2 + 2M^2 TrQ\int_{s_k}^t E\big\|f(s, x_n(s)) - f(s, x(s))\big\|^2\, ds \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \end{aligned}$$

Thus

$$\|\Xi x_n - \Xi x\|^2_{\mathcal{PC}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Therefore Ξ is continuous in Ω_r.

Step 3: Now, we prove that the operator $\Xi : \Omega_r \to \Omega_r$ is equicontinuous. Since the impulsive function g_k is compact, then $\Xi(\Omega_r)$ is equicontinuous on $(t_k, s_k], k = 1, 2, \dots, m$.

Case 1: For any $x \in \Omega_r$ and $0 < \tau_1 < \tau_2 \leq t_1$, by Lemma 2.1, (H_3)-(H_4) and H(6), we obtain

$$\begin{aligned} E\Big\|(\Xi x)(\tau_2) - (\Xi x)(\tau_1)\Big\|^2 \leq& 3E\Big\|\Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)[x_0 - h(x)]\Big\|^2 + 3E\Big\|\int_{\tau_1}^{\tau_2} R(\tau_2 - s)\ f(s, x(s))\ d\mathbb{W}s\Big\|^2 \\ &+ 3E\Big\|\int_0^{\tau_1}\Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)\ f(s, x(s))\ d\mathbb{W}s\Big\|^2 \\ \leq& 3\Big\|R(\tau_2 - s) - R(\tau_1 - s)\Big\|^2\Big(E\|x_0\|^2 + E\|h(x)\|^2\Big) + 3M^2 TrQ\int_{\tau_1}^{\tau_2} E\|f(s, x(s))\|^2 ds \\ &+ 3TrQ\int_0^{\tau_1}\Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)E\|f(s, x(s))\|^2 ds \\ \leq& I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= 3\Big\|R(\tau_2 - s) - R(\tau_1 - s)\Big\|^2\Big(E\|x_0\|^2 + N_h\Big), \\ I_2 &= 3M^2\ TrQ\int_{\tau_1}^{\tau_2}\varphi(s)\psi_f(r)ds, \\ I_3 &= 3TrQ\int_{s_k}^{\tau_1}\Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)\varphi(s)\psi_f(r)ds. \end{aligned}$$

In order to prove that $E\Big\|(\Xi x)(\tau_2) - (\Xi x)(\tau_1)\Big\|^2 \to 0$ as $\tau_2 - \tau_1 \to 0$, we only need to check independently of $x \in \Omega_r$ when $\tau_2 - \tau_1 \longrightarrow 0$. For I_1, since the resolvent operator solution is continuous in operator norm topology for $t \geq 0$ and the nonlocal function h is compact, we can easily see that $I_1 \to 0$ as $\tau_2 - \tau_1 \to 0$. For I_2, using the fact that the function $s \to 4\varphi(s)\psi_f(r)$ is Lebesgue integrable, we get

$$I_2 = 3M^2\ TrQ\ \psi_f(r)\int_{\tau_1}^{\tau_2}\varphi(s)\ ds \longrightarrow 0 \qquad \text{as} \quad \tau_2 - \tau_1 \longrightarrow 0.$$

For I_3, since the resolvent operator $R(t)(t>0)$ is operator norm continuous, and using the fact that the function $s \to 4\varphi(s)\psi_f(r)$ is Lebesgue integrable, we get

$$I_3 \quad = \quad 2TrQ \int_{s_k}^{\tau_1} \left\| \Big(R(\tau_2 - s) - R(\tau_1 - s) \Big) \right\|^2 \varphi(s)\psi_f(r)\mathrm{d}s \longrightarrow 0 \quad \text{as } \tau_2 - \tau_1 \longrightarrow 0.$$

Consequently, $E\|(\Xi_2 x)(\tau_2) - (\Xi_2 x)(\tau_1)\|^2 \longrightarrow 0$ independently of $x \in \Omega_r$ as $\tau_1 - \tau_2 \longrightarrow 0$, it follows that $\Xi(\Omega_r)$ is equicontinuous on $[0, t_1]$.

Case 2: For any $x \in \Omega_r$ and $s_k < \tau_1 < \tau_2 \leq t_{k+1}, k = 1, 2 \ldots, m$, by Lemma 2.1, (H3)-(H5), we have

$$\begin{aligned} E\Big\|(\Xi_2 x)(\tau_2) - (\Xi_2 x)(\tau_1)\Big\|^2 \leq & 3E\Big\| \Big(R(\tau_2 - s) - R(\tau_1 - s) \Big)\; g(s, x(t_k^-)) \Big\|^2 + 3E\Big\| \int_{\tau_1}^{\tau_2} R(\tau_2 - s)\; f(s, x(s))\; d\mathbb{W}s \Big\|^2 \\ & + 3E\Big\| \int_{s_k}^{\tau_1} \Big(R(\tau_2 - s) - R(\tau_1 - s) \Big)\; f(s, x(s))\; d\mathbb{W}s \Big\|^2 \\ \leq & 3E\Big\| \Big(R(\tau_2 - s) - R(\tau_1 - s) \Big) N_g r \Big\|^2 + 3M^2\; TrQ \int_{\tau_1}^{\tau_2} \varphi(s)\psi_f(r) ds \\ & + 3TrQ \int_{s_k}^{\tau_1} \Big(R(\tau_2 - s) - R(\tau_1 - s) \Big) \varphi(s)\psi_f(r) ds. \end{aligned}$$

Under the same argument as in case 1, and the fact that g_k is compact, we see that $E\Big\|(\Xi x)(\tau_2) - (\Xi x)(\tau_1)\Big\|^2 \longrightarrow 0$ independently of $x \in \Omega_r$ when $\tau_2 - \tau_1 \longrightarrow 0$. Which implies that $\Xi(\Omega_r)$ is equicontinuous on $(s_k, t_{k+1}]$ for $k = 1, 2 \ldots, m$.

As a result, $E\Big\|(\Xi x)(\tau_2) - (\Xi x)(\tau_1)\Big\|^2 \longrightarrow 0$ on each interval on $[0, b]$. For this reason $\Xi(\Omega_r)$ is equicontinuous on each $[0, b]$.

Step 4: Denote $E = \overline{co}\; \Xi(\Omega_r)$. Where $\overline{co}$ is the closure of convex hull, it can be shown that the map $\Xi : E \longrightarrow E$ is equicontinuous on each interval, and $E \subset \Omega_r$ is also equicontinuous.

In what follows we will prove that there exists a constant $0 \leq \delta < 1$ and a positive integer n_0 such that for any bounded and nonprecompact subset $D \subset E$

$$\alpha_{\mathcal{PC}}(\Xi^{n_0}(D)) \leq \delta \alpha_{\mathcal{PC}}(D). \tag{3.10}$$

For any $D \subset E$, by the definition of operator Ξ^n and the equicontinuity of E, we get that $\Xi^n \subset \Omega_r$ is also equicontinuous. It follows by Lemma 2.4, that

$$\alpha_{\mathcal{PC}}(\Xi^n(D)) = \max_{t \in [0,b]} \alpha(\Xi^n(D)(t)), n = 1, 2 \ldots, m. \tag{3.11}$$

And there exists a countable sequence $D_1 = \{x_m^1\} \subset D$ such that

$$\alpha\Big(\Xi(D)(t)\Big) \leq 2\alpha\Big(\Xi(D_1(t))\Big). \tag{3.12}$$

Furthermore, for any bounded set $D_1, D_2 \subset D$, by Lemma 2.1 and H(4) we can deduce that

$$\begin{aligned} &\Big\| \int_{s_k}^t R(t-s) f(s, D_1(s)\; d\mathbb{W}s - \int_{s_k}^t R(t-s) f(s, D_2(s)\; d\mathbb{W}s \Big\| = \left(\Big\| \int_{s_k}^t \Big(R(t-s) \Big[f(s, D_1(s)) - f(s, D_2(s)) \Big]\; d\mathbb{W}s \Big) \Big\|^2 \right)^{\frac{1}{2}} \\ &\leq M\; \left(TrQ \int_{s_k}^t \Big\| f(s, D_1(s)) - f(s, D_2(s)) \Big\|^2\; ds \right)^{\frac{1}{2}}. \end{aligned}$$

Then, by Theorem 2.3-(viii), we get

$$\alpha\Big(\int_{s_k}^t R(t-s) f(s, D(s))\; d\mathbb{W}s \Big) \leq M\; \left(TrQ \int_{s_k}^t \Big[\alpha\Big(f(s, D(s)) \Big) \Big]^2\; ds \right)^{\frac{1}{2}}. \tag{3.13}$$

Therefore, by Lemma 2.1, Theorem 2.6, (3.11), (3.12), (3.13), condition H(3)-H(7), we get for $t \in (0, t_1]$ that

$$\begin{aligned}
\alpha\Big(\Xi^1(D)(t)\Big) &= \alpha\Big(\Xi(D)(t)\Big) \leq 2\ \alpha\Big(\Xi(D_1)(t)\Big) \\
&\leq 2\ \alpha\Big(R(t-s)[x_0 - h(x_m^1)]\Big) + 2\ \alpha\Big(\int_0^t R(t-s) f(s, x_m^1(s))\ d\mathbb{W}s\Big) \\
&\leq 2\ M\bigg(\ TrQ \int_0^t \Big[\alpha\Big(f(s, x_m^1(s))\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \leq 2M\bigg(\ TrQ \int_0^t \Big[L\alpha\Big(D_1(s)\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \\
&\leq 2ML\Big(\ TrQt_1\Big)^{\frac{1}{2}} \alpha_{\mathcal{PC}}(D).
\end{aligned}$$

And similarly, for $t \in (s_k, t_{k+1}], k = 1, 2\ldots, m$, we have

$$\begin{aligned}
\alpha\Big(\Xi^1(D)(t)\Big) &= \alpha\Big(\Xi(D)(t)\Big) \leq 2\ \alpha\Big(\Xi(D_1)(t)\Big) \\
&\leq 2\alpha\Big(R(t-s) g_k(s, x_m^1(t_k^-))\Big) + 2\ \alpha\Big(\int_{s_k}^t R(t-s) f(s, x_m^1(s))\ d\mathbb{W}s\Big) \\
&\leq 2M\bigg(\ TrQ \int_{s_k}^t \Big[\alpha\Big(f(s, x_m^1(s))\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \leq 2M\bigg(\ TrQ \int_{s_k}^t \Big[L\alpha\Big(D_1(s)\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \\
&\leq 2ML\Big(\ TrQ(t_{k+1} - s_k)\Big)^{\frac{1}{2}} \alpha_{\mathcal{PC}}(D).
\end{aligned}$$

Meanwhile, we have

$$\alpha(\Xi^1(D)(t)) = 0,$$

since $g_k(t, x(t_k^-))$ is compact for $t \in (s_k, t_{k+1}]$ and $t \in (t_k, s_k], k = 1, 2\ldots, m$. Furthermore, there exists a countable set $D_2 = \{x_m^2\} \subset \overline{co}\ \Xi^1(D)$ such that

$$\alpha\Big(\Xi(\overline{co}\ \Xi^1(D))(t)\Big) \leq 2\alpha\Big(\Xi(D_2(t))\Big). \tag{3.14}$$

Therefore, by Lemma 2.1, (3.14) and (H_4)-(H_5), (H_7), for $t \in (s_k, t_{k+1}], k = 1, 2\ldots, m$, we obtain

$$\begin{aligned}
\alpha\Big(\Xi^2(D)(t)\Big) &= \alpha\Big(\Xi(\overline{co}\ \Xi^1(D))(t)\Big) \leq 2\alpha\Big(\Xi(D_2(t))\Big) \\
&\leq 2\ \alpha\Big(R(t-s) g_k(s, \{x_m^1\}(t_k^-))\Big) + 2\ \alpha\Big(\int_{s_k}^t R(t-s) f(s, x_m^2(s))\ d\mathbb{W}s\Big) \\
&\leq 2M\bigg(TrQ \int_{s_k}^t \Big[\alpha\Big(f(s, x_m^2(s))\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \leq 2ML\bigg(TrQ \int_{s_k}^t \Big[\alpha\Big(D_2(s)\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \\
&\leq 2ML\bigg(TrQ \int_{s_k}^t \Big[\alpha\Big(\overline{co}\ \Xi^1(D)\Big)\Big]^2\ ds\bigg)^{\frac{1}{2}} \\
&\leq 2ML\bigg(TrQ \int_{s_k}^t \Big[2ML\Big(TrQ(s_{k+1} - \tau_k)\Big)^{\frac{1}{2}}\Big]^2\ ds\bigg)^{\frac{1}{2}} \alpha_{\mathcal{PC}}(D) \\
&\leq \Big(2ML\sqrt{TrQ}\Big)^2 \sqrt{\frac{(t_{k+1} - s_k)^2}{2}} \alpha_{\mathcal{PC}}(D).
\end{aligned}$$

Proceeding with this iterative method, we shall get for a.e $t \in [o, b]$

$$\alpha\Big(\Xi^n(D)(t)\Big) \leq \Big(2M\ L\sqrt{TrQ}\Big)^n \sqrt{\frac{b^n}{n!}} \alpha_{\mathcal{PC}}(D).$$

And from (3.11), we get

$$\alpha\Big(\Xi^n(D)\Big) \leq \Big(2ML\sqrt{TrQ}\Big)^n \sqrt{\frac{b^n}{n!}}\alpha_{\mathcal{PC}}(D).$$

Since

$$\Big(2ML\sqrt{TrQ}\Big)^n \sqrt{\frac{b^n}{n!}} \longrightarrow 0 \ as \ n \longrightarrow +\infty.$$

Then, there exists a large enough positive integer n_0 such that

$$\Big(2ML\sqrt{TrQ}\Big)^{n_0} \sqrt{\frac{(b)^{n_0}}{n_0!}} = \delta < 1.$$

Therefore, we showed that there exists $0 \leq \delta < 1$ and a positive integer n_0 such that (3.10) is satisfied. It follows from Theorem 2.6 that the operator Ξ has at least one fixed point, which is a mild solution of (1.1). □

The second result will be established using the Darbo's fixed point theorem.

Theorem 3.2. Assume that (H_1)-(H_7) are satisfied, then the problem (1.1) has at least one mild solution provided that

$$2M^2\Big(TrQ\ \rho\Lambda + N_g\Big) < 1, \tag{3.15}$$

and

$$2ML\sqrt{TrQb} < 1. \tag{3.16}$$

Proof . We know from the proof of Theorem 2.6 that the operator $\Xi : \Omega_r \to \Omega_r$ is bounded and continuous and that $\{\Xi x : x \in \Omega_r\}$ is a family of equicontinuous functions in $\mathcal{PC}([0,b],\mathbb{H})$.

Using the same method as in the proof of Theorem 2.6, for any bounded set $D \in \Omega_r$, we have for $t \in [0,b]$

$$\alpha_{\mathcal{PC}}(\Xi(D)) \leq 2ML\sqrt{TrQb}\alpha_{\mathcal{PC}}(D). \tag{3.17}$$

It follows from condition (3.16) that the operator $\Xi : \Omega_r \to \Omega_r$ is a α-contraction. Therefore, by Theorem 2.7, the operator Ξ has at least one fixed point $x \in \Omega_r$. □

Remark 3.3. It is clear that in Theorem 3.1 the condition(3.16) can be dropped compared with Theorem 3.2.

Example 3.4. Let us consider the following problem

$$\begin{array}{rcl} \dfrac{\partial}{\partial t}\upsilon(t,z) & = & \dfrac{\partial^2}{\partial z^2}\upsilon(t,z) + \displaystyle\int_0^t \zeta(t-s)\frac{\partial^2}{\partial z^2}\upsilon(s,z)ds + \frac{e^{-t}}{2+|\upsilon(t,z)|}d\mathbb{W}t, \quad (t,z) \in \bigcup_{k=0}^{m}(s_i,t_{i+1}] \times [0,1] \\ \upsilon(t,z) & = & g_i(t,\upsilon(t_i^-,z)), \quad t \in \displaystyle\bigcup_{i=1}^{m}(t_i,s_i], \\ \upsilon(t,0) & + & \displaystyle\int_0^1\int_0^b M_2(r,z)\sin(\upsilon(t,r))dtdr = 0, \\ \upsilon(t,0) & = & \upsilon(t,1) = 0. \end{array} \tag{3.18}$$

Where $\mathbb{W}(t)$ is a standard Brownian motion on a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,1]},P)$.

Let $\mathbb{H}=\mathrm{L}^2([0,1])$ a Hilbert space with the inner product $(u,v)=\displaystyle\int_0^1 u(x)v(x)\ dx$. It is well known that $\mathbb{H}$ is a Banach space. We define $A: D(A)\subset\mathbb{H}\to\mathbb{H}$ by $Au=u^{''}$, with

$$D(A)=\{u\in\mathbb{H}, u, u^{'} \text{ are absolutely continuous }, u^{''}\in\mathbb{H}, u(0)=u(1)=0\}.$$

Then $Au=\Sigma_{n=1}^{+\infty}n^2(u,e_n)e_n, u\in D(A)$, where $e_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns), n\in\mathbb{N}$ is the orthogonal set of eigenvectors. A is the infinitesimal generator of a strongly semigroup $T(t)(t\geq 0)$ in $\mathbb{H}$, which is an equicontinuous analytic semigroup for $t\geq 0$, then $T(t)u=\Sigma_{n=1}^{+\infty}e^{-n^2t}(u,e_n)e_n$.

The corresponding resolvent operator is norm continuous for $t\geq 0$, furthermore, we suppose that $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ is bounded and $\mathcal{C}^1$ continuous function, with $\zeta^{'}$ is bounded and uniformly continuous then (H_1)-(H_3) are satisfied.

To rewrite (3.18) in an abstract form we put

$$x(t)(z)=v(t,z), x'(t)(z)=\frac{\partial v(t,z)}{\partial t}, \quad \text{for } (t,z)\in[0,b]\times[0,1].$$

We introduce the functions $f:[0,b]\times\mathbb{H}\longrightarrow \mathrm{L}(\mathbb{K},\mathbb{H})$, $g_k:(t_k,s_k]\times\mathbb{H}\to\mathbb{H}$ and the nonlocal function $h:\mathcal{PC}([0,b],L^2(\Omega,\mathbb{H}))\longrightarrow\mathbb{H}$ such that

$$\begin{array}{rcl} f(t,x(t))(z) &=& \dfrac{e^{-t}}{2+|v(t,z)|}, \\ g_k(t,v(t_k^-,z)) &=& \displaystyle\int_0^1\int_{t_k}^t M_1(s,z)\frac{1}{\kappa+1}\frac{|v(s,r)|}{(1+|v(s,r)|)}dsdr, \kappa>1, \\ h(v)(z)) &=& \displaystyle\int_0^1\int_0^b M_2(r,z)\sin(v(t,r))dtdr, \end{array}$$

where $M_1, M_2:[0,b]\times[0,1]\to\mathbb{R}^+$ are continuous functions such that $M_1(t,1)=M_2(t,1)=0$.

Lemma 3.5. Let $h: PC([0,b],\mathbb{H})\to\mathbb{H}$ be an operator defined by

$$h: v\to h(v)(\xi)=\int_0^1\int_0^b \bigwedge(r,\xi)g(v(t,r))dtdr$$

where $\bigwedge:[0,b]\times[0,1]\to\mathbb{R}$ and $g:\mathbb{H}\to\mathbb{H}$ are continuous functions where g satisfies

$$\|g(v)\|^2\leq\widetilde{M}(\|v\|^2+1), \quad \text{for all } v\in PC([0,b],\mathbb{H}), \text{for some } \widetilde{M}>0.$$

Then, h is a compact.

Proof . Let $B\in C([0,b],\mathbb{H})$ be a bounded set, then there exists $K>0$ such that

$$\|v\|_\infty=\sup_{t\in[0,b]}\|v(t)\|_{L^2(\Omega,\mathbb{H})}\leq K, \quad v\in B.$$

Let $v\in B$, then

$$\begin{array}{rcl} |h(v)(\xi)|^2 &=& \left|\displaystyle\int_0^1\int_0^b \bigwedge(r,\xi)g(v(t,r))dtdr\right|^2 \\ &\leq& \|\bigwedge\|_\infty^2\left|\displaystyle\int_0^1\int_0^b |g(v(t,r))|dtdr\right|^2. \end{array}$$

By Hölder inequality and Fubini's theorem, we get

$$\begin{aligned}
|h(v)(\xi)|^2 &\leq b\left\|\bigwedge\right\|_\infty^2 \int_0^b \|g(v(t,\cdot))\|_{\mathbb{H}}^2 dt,\\
&\leq b^2(K^2+1)\left\|\bigwedge\right\|_\infty^2 \widetilde{M}.
\end{aligned}$$

Hence,

$$\|h(v)\|_{\mathbb{H}}^2 \leq b^2(K^2+1)\left\|\bigwedge\right\|_\infty^2 \widetilde{M}.$$

Then $h(B)$ is bounded.

Now we show that $h(B)$ satisfied the "integral" equicontinuity condition. Let $l,\xi \in [0,b]$, thus

$$\begin{aligned}
\int_0^1 |h(v)(\xi+l)-h(v)(\xi)|^2 d\xi &= \int_0^1 \left|\int_0^1\int_0^b (\bigwedge(r,\xi+l)-\bigwedge(r,\xi))g(v(t,r))dtdr\right|^2 d\xi\\
&\leq b^2\widetilde{M}(K^2+1)\int_0^1\int_0^b \left|\bigwedge(r,\xi+l)-\bigwedge(r,\xi)\right|^2 d\xi dr.
\end{aligned}$$

As a result, we get

$$\|\tau_l h(v)-h(v))\|_{\mathbb{H}}^2 \to 0 \text{ as } l\to 0,$$

independently of $v\in B$, where, $\tau_l h(v)(\xi)=h(v)(\xi+l)$, $\xi,l\in\mathbb{R}$. We conclude, from Kolmogorov-Riesz-Fréchet theorem [6, Theorem 4.26], that $h(B)$ is relatively compact in $\mathbb{H}$. □

Corollary 3.6. Let $L:[0,b]\times\mathbb{H}\to\mathbb{H}$ be a operator defined by

$$L:(t,v)\to L(t,v)(\xi)=\int_0^1\int_0^t \bigwedge(r,\xi)g(v(s))dsdr$$

where $\bigwedge:[0,b]\times[0,1]\to\mathbb{R}$ and $g:\mathbb{H}\to\mathbb{H}$ are continuous functions where g satisfies

$$\|g(v)\|_{\mathbb{H}}^2 \leq \widetilde{M}(\|v\|_{\mathbb{H}}^2+1), \quad \text{for all } v\in\mathbb{H}, \text{for some } \widetilde{M}>0.$$

Then, for all bounded set $B\subset\mathbb{H}$, $L([0,b]\times B)$ is a relatively compact in $\mathbb{H}$.

Proof . We use the same proof technique used in Lemma 3.5. □

We can verify that the assumptions (H_3)-(H_7) hold with

$$\varphi(t)=\tfrac{e^{-2t}}{4}, \quad \psi_f(x)=1, \quad L=\tfrac{1}{2}.$$

By Lemma 3.5 and Corollary 3.6, h and g_k are compact. Then, (H_6) and (H_5) are satisfied. From the fact that $\rho=0$, one can easily verify that condition (3.1) hold. Therefore, all assumptions of Theorem 3.1 are satisfied. Consequently, the problem (3.18) has a mild solution on $[0,b]$.

4 Application: Controllability Results

In this section and as an application of Theorem 3.1, we consider the controllability of the non-instantaneous impulsive stochastic integro-differential equation with nonlocal initial conditions of the form:

$$\begin{cases}
dx(t)=\left[Ax(t)+\int_0^t \Upsilon(t-s)x(s)ds+Bu(t)\right]dt+f(t,x(t))\ d\mathbb{W}t & t\in\bigcup_{k=0}^m(s_k,t_{k+1}],\\
x(t)=g_k(t,x(t_k^-)), & t\in\bigcup_{k=1}^m(t_k,s_k],\\
x(0)+h(x)=x_0.
\end{cases} \tag{4.1}$$

Definition 4.1. A $\mathcal{F}_t$-adapted stochastic process $x(t):[0,b]\to\mathbb{H}$ is called a mild solution of (4.1) if $x(0)+h(x)=x_0\in\mathbb{H}$, and for each $t\in[0,b]$

$$x(t)=\begin{cases} R(t)\big(x_0-h(x)\big)+\displaystyle\int_0^t R(t-s)Bu(s)ds+\int_0^t R(t-s)f(s,x(s))d\mathbb{W}s, & \text{for } t\in[0,t_1],\\ g_k(t,x(t_k^-)), & t\in\bigcup_{k=1}^m(t_k,s_k],\\ R(t-s_k)g_k(s_k,x(t_k^-))+\displaystyle\int_{s_k}^t R(t-s)Bu(s))ds+\int_{s_k}^t R(t-s)f(s,x(s))\ d\mathbb{W}s, & t\in\bigcup_{k=1}^m(s_k,t_{k+1}]. \end{cases} \tag{4.2}$$

Definition 4.2. The stochastic control system (4.1) is called controllable on the interval $[0,b]$ if for every initial state $x_0,\ x_1\in\mathbb{H}$ there exists a suitable stochastic control $u(\cdot)\in\mathrm{L}^2([0,b],\mathrm{U})$ such that the mild solution of (4.1) satisfies $x(b)+h(x)=x_1$, where x_1 and b are preassigned terminal state and time, respectively.

To prove the controllability result, the following hypotheses are necessary:

(H_8) The linear operator $\mathcal{W}:\mathrm{L}^2([0,b],\mathrm{U})\longrightarrow\mathrm{H}$ defined by

$$\mathcal{W}u=\int_{s_k}^b R(b-s)Bu(s))\ ds,$$

has a bounded invertible operator $\mathcal{W}^{-1}$ which takes values in $\mathrm{L}^2([0,b],\mathrm{U})/\mathrm{Ker}\mathcal{W}$, and

1. There exist two positive constants σ_1,σ_2 such that

$$\|B\|^2\le\sigma_1,\qquad \|\mathcal{W}^{-1}\|^2\le\sigma_2.$$

2. There exists $K_B,\ K_{\mathcal{W}}(t)\in\mathrm{L}^1([0,b],\mathbb{R}^+)$ such that for any bounded set $D_1\subset\mathrm{U},D_2\subset\mathbb{H}$

$$\alpha(B(D_1))\le K_B\alpha(D_1),\qquad \alpha\Big(\mathcal{W}^{-1}(D_2)(t)\Big)\le K_{\mathcal{W}}(t)\alpha\Big(D_2(t)\Big).$$

Theorem 4.3. Assume that the hypotheses (H_1)-(H_8) are satisfied. Then the stochastic integro-differential system (4.1) is controllable on $[0,b]$ provided that

$$3M^2\Big(N_g+\ TrQ\rho\Lambda\Big)\Big(1+3M^2\sigma_1\sigma_2 b\Big)<1. \tag{4.3}$$

Proof . To prove our result, we transform (4.1) into a fixed point problem. Consider the operator $\Xi\in\mathcal{PC}([0,b],\mathbb{H})$ defined by

$$\Xi x(t)=\begin{cases} R(t)\big(x_0-h(x)\big)+\int_0^t R(t-s)f(s,x(s))\ d\mathbb{W}s+\int_{s_k}^t R(t-s)Bu_x(s)\ ds, & t\in[0,t_1],\\ g_k(t,x(t_k^-)), & t\in\bigcup_{k=1}^m(t_k,s_k],\\ R(t-s_k)g_k(s_k,x(t_k^-))+\int_{s_k}^t R(t-s)f(s,x(s))\ d\mathbb{W}s+\int_{s_k}^t R(t-s)Bu_x(s)\ ds, & t\in\bigcup_{k=1}^m(s_k,t_{k+1}]. \end{cases} \tag{4.4}$$

Using (H_8), we define for an arbitrary function $x(\cdot)$ the following control

$$u_x(t)=\mathcal{W}^{-1}\left(x_1-h(x)-R(b-s_k)g_k(s_k,x(t_k^-))-\int_{s_k}^b R(b-s)f(s,x(s))d\mathbb{W}s\right)(t). \tag{4.5}$$

for $u_x\in\Omega_r$, using Lemma 2.1, (H_3)-(H_6) and (H_8), we obtain the following result

$$\begin{aligned}E\|u_x\|^2 &\leq 4\sigma_2\Big(E\|x_1\|^2+E\|h(x)\|^2+M^2\ E\|g_k(s_k,x(t_k^-))\|^2+M^2\ TrQ\int_{s_k}^{b}E\|f(s,x(s))\|^2\ ds\Big)\\ &\leq 4\sigma_2\Big(E\|x_1\|^2+N_h+M^2(N_{g_k}E\|x(t_k^-)\|^2)+M^2\ TrQ\int_{s_k}^{b}\varphi(s)\psi_f(E\|x\|^2)\ ds\Big)\\ &\leq 4\sigma_2\Big(E\|x_1\|^2+N_h+M^2N_gr+M^2\ TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\Big).\end{aligned}$$

Hence

$$E\|u_x\|^2\leq 4\sigma_2\Big(E\|x_1\|^2+N_h+M^2\Big(N_gr+TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\Big)\Big). \tag{4.6}$$

The proof is similar as in problem (1.1). Here, we only prove that there exists a constant $r>0$ such that $\Xi(\Omega_r)\subset\Omega_r$. Suppose that this is not true. Then for each $r>0$, there would exist $x_r\in\Omega_r$ and $t_r\in[0,b]$ such that $E\|\Xi(x_r)(t_r)\|^2>r$.

Case 1: For $t_r\in[0,t_1]$, from Lemma2.1, (4.6) and assumptions (H_3)-(H_6), (H_8), we have

$$\begin{aligned}E\|\Xi(x_r)(t_r)\|^2 &\leq 3M^2E\|x_0-h(x)\|^2+3M^2\ TrQ\int_0^{t_r}E\|f(s,x_r(s))\|^2\ ds+3M^2\ \sigma_1\int_0^{t_r}E\|u_x(s)\|^2\ ds\\ &\leq 3M^2(E\|x_0\|^2+N_h)+3M^2TrQ\int_0^{t_r}\varphi(s)\psi_f(E\|x_r\|^2)\ ds\\ &\quad+12M^2\sigma_1\sigma_2\int_0^{t_r}\Big(E\|x_1\|^2+N_h+M^2N_gr+TrQ\psi_f(r)\|\varphi\|_{\mathrm{L}[\tau_k,b]}\Big)ds\\ &\leq 3M^2(E\|x_0\|^2+N_h)+3M^2TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[0,t_1]}\\ &\quad+12M^2\sigma_1\sigma_2t_1\Big(E\|x_1\|^2+N_h+M^2N_gr+TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\Big).\end{aligned}$$

So we have

$$\begin{aligned}E\|\Xi(x_r)(t_r)\|^2\leq &3M^2\Big(E\|x_0\|^2+N_h+TrQ\psi_f(r)\|\varphi\|_{\mathrm{L}[0,t_1]}\Big)\\ &+12M^2\sigma_1\sigma_2t_1\Big(E\|x_1\|^2+M^2\Big(N_gr+TrQ\psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\Big)\Big).\end{aligned} \tag{4.7}$$

Case 2: For $t_r\in(t_k,s_k], k=1,2,\dots,m$, we have

$$\begin{aligned}E\|\Xi(x_r)(t_r)\|^2 &= E\|g_k(t_r,x_r(t_k^-))\|^2\\ &\leq N_gr.\end{aligned} \tag{4.8}$$

Case 3: For $t_r\in(s_k,t_{k+1}], k=1,2,\dots,m$, we get

$$\begin{aligned}E\|\Xi(x_r)(t_r)\|^2\leq &3M^2\ E\|g_k(s_k,x(t_k^-))\|^2+3M^2\ TrQ\int_{s_k}^{t_r}E\|f(s,x(s))\|^2\ ds+3M^2\int_{s_k}^{t_r}E\|u_x(s)\|^2\ ds\\ \leq&3M^2N_gr+3M^2\ TrQ\int_{s_k}^{t_r}\varphi(s)\psi_f(E\|x_r\|^2)\ ds\\ &+12M^2\ \sigma_1\sigma_2\int_{s_k}^{t_r}\Big(E\|x_1\|^2+N_h+M^2N_gr+TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[\tau_k,b]}\Big)\Big)ds\\ \leq&3M^2\Big(N_gr+\ TrQ\psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,t_{k+1}]}\Big)\\ &+12M^2\ \sigma_1\sigma_2(t_{k+1}-s_k)\Big(E\|x_1\|^2+N_h+M^2\big(N_gr+TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\big)\Big).\end{aligned}$$

Hence

$$E\|\Xi(x_r)(t_r)\|^2 \leq 3M^2\Big(N_g r + \ TrQ\psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,t_{k+1}]}\Big) \\ + 12M^2\ \sigma_1\sigma_2(t_{k+1}-s_k)\Big(E\|x_1\|^2 + N_h + M^2\big(N_g r + TrQ\ \psi_f(r)\|\varphi\|_{\mathrm{L}[s_k,b]}\big)\Big). \tag{4.9}$$

Combining the three cases (4.7), (4.8), (4.9), we obtain

$$r < E\|\Xi(x_r)(t_r)\|^2 \\ \leq 3M^2\Big(E\|x_0\|^2 + N_g r + N_h + \ TrQ\psi_f(r)\Lambda\Big) + 12M^2\ \sigma_1\sigma_2 b\Big(E\|x_1\|^2 + N_h + M^2\big(N_g r + TrQ\ \psi_f(r)\Lambda\big)\Big).$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, we have

$$1 < E\|\Xi(x_r)(t_r)\|^2 \leq 3M^2\Big(N_g + \ TrQ\rho\Lambda\Big) + 12M^4\ \sigma_1\sigma_2 b\Big(N_g + TrQ\ \rho\Lambda\Big)$$

$$1 \leq 3M^2\Big(N_g + \ TrQ\rho\Lambda\Big)\Big(1 + 4M^2\sigma_1\sigma_2 b\Big),$$

which is contradicted with (4.3), hence, there exists a constant $r > 0$ such that $\Xi(\Omega_r) \subset \Omega_r$. Using the same method as in problem (1.1), we shall see that the operator Ξ is continuous in Ω_r and equicontinuous for each $[0, b]$. Also that, there exists a large enough positive integer n_0 such that

$$\Big(2\ M\ L\sqrt{TrQ}\Big)^{n_0}\Big(1 + MK_{\mathcal{B}}\|K_{\mathcal{W}}\|_{\mathrm{L}[0,b]}\Big)^{n_0}\sqrt{\frac{b^{n_0}}{n_0!}} = \delta,$$

where $0 \leq \delta < 1$. Then condition (3.10) is satisfied. It follows from Theorem (2.6), the operator Ξ has at least one fixed point. Hence, the system is controllable on $[0, b]$. □

Example 4.4. Let us consider the following problem

$$\begin{aligned} \frac{\partial}{\partial t}\upsilon(t,z) &= q(z)\upsilon(t,z) + \int_0^t \zeta(t-s)q(z)\upsilon(s,z)ds + \frac{e^{-t}}{2+|\upsilon(t,z)|}d\mathbb{W}t \\ &\quad + m\upsilon(t,z), \quad (t,z) \in [s_i, t_{i+1}] \times [0,1], \\ \upsilon(t,z) &= g_i(t, \upsilon(t_i^-, z)), \quad t \in (t_i, s_i], \\ \upsilon(t,0) &+ \int_0^1\int_0^b M_2(r,z)\sin(\upsilon(t,r))dtdr = 0, \\ \upsilon(t,0) &= \upsilon(t,1) = 0, \end{aligned} \tag{4.10}$$

where $\mathbb{W}(t)$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, P)$. $\upsilon_0(\cdot) \in \mathrm{L}^2([0,1])$. Let $\mathbb{H} = \mathrm{L}^2([0,1], \mathbb{C})$ a space of all integrable complex functions on $[0,1]$, we assume that $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and $\mathcal{C}^1$ continuous function, with $\zeta^{'}$ is bounded and uniformly continuous, and $q : \mathbb{R} \to \mathbb{C}$ is a continuous function, we define the multiplicative operator A as follows

$$D(A) = \{\Delta \in \mathbb{H}, q\Delta \in \mathbb{H}\}, \qquad A\Delta = q\Delta,$$

A generates a norm continuous multiplication semigroup $T_q(t)(t \geq 0)$ on $\mathbb{H}$, given by $T_q(t)\Delta = e^{tq}\Delta$. The corresponding resolvent operator is norm continuous for $t \geq 0$, thus (H_1)-(H_3) are fulfilled. Furthermore, $\Upsilon(t)\Delta = \zeta(t)A\Delta$, for $t \geq 0$, and $\Delta \in D(A)$. To rewrite (4.10) in an abstract form we put

$$x(t)(z) = \upsilon(t,z) \quad (t,z) \in [0,b] \times [0,1],$$
$$x(0) = \upsilon(0,z).$$

We introduce the functions $f : [0, b] \times \mathbb{H} \longrightarrow \mathrm{L}(\mathbb{K}, \mathbb{H})$, $g_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ and the nonlocal function $h : \mathcal{PC}([0, b], \mathbb{H}) \longrightarrow \mathbb{H}$ such that

$$\begin{aligned} f(t, \upsilon(t, z)) &= \frac{e^{-t}}{2 + |\upsilon(t, z)|}, \\ g_k(t, \upsilon(t_k^-, z)) &= \int_0^1 \int_{t_k}^t M_1(s, z) \frac{1}{\kappa + 1} \frac{|\upsilon(s, r)|}{(1 + |\upsilon(s, r)|)} dsdr, \kappa > 1 \\ h(\upsilon)(z)) &= \int_0^1 \int_0^b M_2(r, z) \sin(\upsilon(t, r)) dtdr, \end{aligned}$$

where $M_1, M_2 : [0, b] \times [0, 1] \to \mathbb{R}^+$ are continuous functions such that $M_1(t, 1) = M_2(t, 1) = 0$. The control function $B : U \to \mathbb{H}$ is defined by $Bu(t)(z) = mv(t, z)$, where $z \in [0, 1]$, $u \in \mathrm{L}^2([0, 1], \mathbb{U})$. For $z \in [0, 1]$, the operator $\mathcal{W}$ is given by

$$\mathcal{W}(z)(u) = \int_0^1 R(1 - s)u(s)\ ds.$$

By Lemma 3.5 and Corollary 3.6 we get that h and g_k are compact. Therefore, one can verify that assumptions (H_3)-(H_7) and condition (4.3) hold, and assuming that $\mathcal{W}$ fulfilled (H_8). Then the problem (4.10) is controllable on $[0, b]$.

5 Conclusion

In this article, we gave appropriate conditions to establish the existence of mild solutions and the controllability for a class of non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions in a Hilbert space by using the resolvent operator, a Kuratowskii measure of non-compactness, and a generalized Darbo's fixed point theorem. The use of generalized Darbo's fixed point theorem instead of the famous Darbo's fixed point theorem allows to weak the conditions to ensure the existence of mild solution and controllability of the system. The approximate controllability of a class of non-instantaneous impulsive stochastic integro-differntial equations and inclusions will be the topic of our future work.

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