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A meshfree regularization method for recovering a time-dependent Robin coefficient in one-dimensional transient heat conduction

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Abstract

In the current paper, we numerically investigate the approximation of a timewise-dependent heat transfer coefficient (HTC) along with the temperature in the one-dimensional heat equation with the third-type boundary conditions and an integral measurement. We utilize the integral overdetermination condition to reformulate the third-type boundary conditions and seek the solution to the converted problem in the form of the linear combination of the method of fundamental solutions and the heat polynomials. By applying the collocation method, the problem is reduced to the solution of a linear system of algebraic equations. The method takes advantage of the combination of the natural cubic spline technique and the Tikhonov regularization method to provide a stable approximation of the derivative of the perturbed boundary data. We provide several numerical tests to show the effectiveness of the proposed method.

Keywords: Parabolic equation, Method of fundamental solutions, Heat polynomials, Heat transfer coefficient, Tikhonov regularization
2020 MSC: 35R30, 35K20, 65M30, 65M32

1 Introduction

Consider the problem of simultaneously determination of the functions $\Big(u(x,t),\sigma(t)\Big)$ satisfying the following system of equations

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad \text{in} \quad \Omega := (0,1)\times(0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad 0<x<1, \tag{1.2}$$

$$-u_x(0,t) + \sigma(t)u(0,t) = h_0(t), \quad 0<t<T, \tag{1.3}$$

$$u_x(1,t) + \sigma(t)u(1,t) = h_1(t), \quad 0<t<T, \tag{1.4}$$

$$\int_0^1 \omega(x)u(x,t)dx = E(t), \quad t\in[0,T], \tag{1.5}$$

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where

$$u_0(x) \in C^1([0,1]),\ h_0(t)\ \&\ h_1(t) \in C([0,T]),\ \omega(x) \in L^1(0,1),\ E(t), \tag{1.6}$$

are given functions and further assume that $\omega(x) > 0$ and the following compatibility conditions hold

$$-u_0'(0) + \sigma(0)u_0(0) = h_0(0),\ u_0'(1) + \sigma(0)u_0(1) = h_1(0),\ \int_0^1 \omega(x)u_0(x)dx = E(0). \tag{1.7}$$

Theoretical results concerning the existence and uniqueness of the solution for the nonlinear inverse problem (1.1)-(1.5) were presented in [8]. It has been shown that with the properties (1.6)-(1.7) if $|E(t)| > 0$ on $[0,T]$, then the solution to the inverse problem given by Eqs. (1.1)-(1.5) is unique.

Considering an electrical device being capable of heat exchange with the surrounding environment, according to Newton's law of cooling, the linear boundary conditions of the third kind given by Eqs. (1.3)-(1.4) relate the heat flux at the ends of a rod to the boundary temperature and technically state that this convective heat exchange takes place at the boundary of the space domain [19]. However, in the theory of electrical contacts and in general cases the heat exchange is modelled as a nonlinear boundary condition of the third kind and the fourth-order Stefan-Boltzmann power law is applied.

Typically, integral boundary conditions are used to reconstruct the problem in the form of an integro-differential equation, and then the necessary conditions for the existence and uniqueness of the solution of the problem as well as the continuous dependence upon the solution of the problem are distinguished. The kernel function $\omega(x)$ in the integral condition can be effective in the solvability of the problem, but in particular it can affect the accuracy of numerical methods [20].

From practical point of view, the inverse problem (1.1)-(1.5) and other similar problems usually appear in hostile environments or in heat transfer situations at high temperatures such as combustion chambers and gas turbines. In fact, in the process of cooling of hot steel or glass in fluids or gases and also in the quenching heat treatment of metals in a liquid medium, the HTC $\sigma(t)$ characterizes the contribution that an interface makes to the overall thermal resistance to the system [19, 24] which can be used for recommendations on the choice of optimal parameters for the electrical contact systems. Estimation of HTC also can be important in other industrial applications such as airconditioning systems, food processing and nuclear power production [3].

The problem of recovering the time-dependent Robin coefficient in the heat equation using various boundary conditions have been discussed in several articles and useful theoretical and numerical results have been obtained [8, 19, 24]. In [27], the authors presented a Bayesian inference approach for identifying a Robin coefficient from boundary temperature measurement and a hierarchical Bayesian method for automatic selection of the regularization parameter in the function estimation inverse problem was discussed. In [3], the authors presented a scheme including the combination of the truncated generalized singular value decomposition, Tikhonov regularization, and a filtering technique to estimate the local convective HTC in coiled tubes. In [1], the authors studied the simultaneous identification problem for recovering the Robin coefficient and heat flux and adopted a constrained minimization problem using the output least squares method with Tikhonov regularization. In [28, 29], the collocation procedures by means of Sinc method for determining the time-dependent function in the boundary condition of the parabolic equations were proposed.

The method of fundamental solutions (MFS) is a meshless boundary interpolation technique which belongs to the class of the Trefftz method [12, 14, 25]. The following pure features have made this technique a popular method among those interested in applied and computational mathematics. First, being efficient in solving various problems centered on partial differential equations defined in multi-dimensional domains, provided that the governing equation has a fundamental solution. Second, MFS is easy to use in numerical implementations and obtain appropriate accuracy, and third, MFS is inexpensive in numerical simulations because unlike conventional methods such as the finite difference method and the finite element method, we do not need to discretize the domain or boundary of the problem [21]. As a proof of concept, the authors stated the features and application of the MFS in solving problems related to solid mechanics [15]. Later, this method was developed for solving various problems in the fields of direct and inverse problems. For more background on the MFS, we refer the interested reader to [2, 4, 6, 9, 10, 11, 17, 18, 22, 23, 26] and the references therein.

In this article, we propose a computational method based on the combination of the mathod of fundamental solutions and the heat polynomials method to solve the problem given by Eqs. (1.1)-(1.5). Our main goal is to provide an easy method to implement with low computational cost and at the same time with appropriate accuracy. In addition, when dealing with perturbed input boundary data, the cubic spline technique will be adopted to cope with noisy data and obtain stable numerical derivatives.

The organization of the paper is as follows. In Section 2, we discuss the reconstruction algorithm. Section 3 includes the numerical implementation of the proposed technique. In Section 4, we discuss some concluding remarks.

2 Approximation method

In the first place, we use the integral boundary condition (1.5) to reduces the problem (1.1)–(1.5) to a PDE containing only the unknown function $u(x,t)$. In this respect, we differentiate Eq. (1.5) with respect to t to get

$$\int_0^1 \omega(x)u_t(x,t)dx = E'(t), \quad t \in [0,T], \tag{2.1}$$

then multiplying (1.1) by $w(x)$ and integrating it with respect to x we find

$$\int_0^1 \omega(x)u_t(x,t)dx = \omega(1)u_x(1,t) - \omega(0)u_x(0,t) - \int_0^1 \omega'(x)u_x(x,t)dx. \tag{2.2}$$

From (2.1) and (2.2) we conclude

$$E'(t) = \omega(1)u_x(1,t) - \omega(0)u_x(0,t) - \int_0^1 \omega'(x)u_x(x,t)dx. \tag{2.3}$$

Now by multiplying Eqs. (1.3) and (1.4) by $\omega(0)$ and $\omega(1)$, respectively and paying attention to the Eq. (2.3) we achieve to the following equations

$$\sigma(t)u(0,t) = \frac{\omega(0)h_0(t) + \omega(1)u_x(1,t) - \int_0^1 \omega'(x)u_x(x,t)dx - E'(t)}{\omega(0)}, \tag{2.4}$$

$$\sigma(t)u(1,t) = \frac{-\omega(1)h_1(t) + \omega(0)u_x(0,t) + \int_0^1 \omega'(x)u_x(x,t)dx + E'(t)}{-\omega(1)}. \tag{2.5}$$

Thus, the main problem is reformulated as the following PDE

$$u_t(x,t) = u_{xx}(x,t), \quad in \quad (0,1)\times(0,T), \tag{2.6}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{2.7}$$

$$-u_x(0,t) + \frac{\omega(0)h_0(t) + \omega(1)u_x(1,t) - \int_0^1 \omega'(x)u_x(x,t)dx - E'(t)}{\omega(0)} = h_0(t), \quad 0 < t < T, \tag{2.8}$$

$$u_x(1,t) + \frac{\omega(1)h_1(t) - \omega(0)u_x(0,t) - \int_0^1 \omega'(x)u_x(x,t)dx - E'(t)}{\omega(1)} = h_1(t), \quad 0 < t < T. \tag{2.9}$$

Next, we present the source points which are scattered external to the domain $(0,1)\times(0,T)$ as

$$x_1^{(i)} = 1 + r,\ t_1^{(i)} = \frac{iT}{N_1+1}, \quad x_2^{(i)} = \frac{i}{N_2+1},\ t_2^{(i)} = -\tau, \quad x_3^{(i)} = -1 - r,\ t_3^{(i)} = \frac{iT}{N_3+1}, \tag{2.10}$$

where r and τ are fixed positive constants. Then, considering a non-singular solution to the one-dimensional heat Eq. (1.1) as

$$\phi(x,t) = \frac{Heavisde(t+T_0)\exp(-\frac{x^2}{4(t+T_0)})}{\sqrt{4\pi(t+T_0)}},\ T_0 > T, \tag{2.11}$$

and taking the heat polynomial of degree n given by explicit formula [5, 13, 22]

$$\psi_n(x,t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k}t^k}{k!(n-2k)!},\ n = 0,1,2,..., \tag{2.12}$$

where $[\frac{n}{2}]$ denotes the largest integer less than or equal to $\frac{n}{2}$, the approximate solution of the problem (2.6)-(2.9) is proposed via the following expression:

$$u_{approx}(x,t) = \sum_{k=1}^{N_1} c_k^{(1)} \phi(x - x_1^{(k)}, t - t_1^{(k)}) + \sum_{k=1}^{N_2} c_k^{(2)} \phi(x - x_2^{(k)}, t - t_2^{(k)}) \tag{2.13}$$

$$+ \sum_{k=1}^{N_3} c_k^{(3)} \phi(x - x_3^{(k)}, t - t_3^{(k)}) + \sum_{k=0}^{N_4-1} c_k^{(4)} \psi_k(x,t). \tag{2.14}$$

Following, we use the collocation method to impose the initial and boundary conditions (2.7)-(2.9) and then construct a linear system of algebraic equation. In this respect, we define

$$R_1(x) := u_{approx}(x,0) - u_0(x), \tag{2.15}$$

$$R_2(t) := -u^*(0,t) + \frac{\omega(0)h_0(t) + \omega(1)u^*(1,t) - \int_0^1 \omega'(x)u^*(x,t)dx - E'(t)}{\omega(0)} - h_0(t), \tag{2.16}$$

$$R_3(t) := u^*(1,t) + \frac{\omega(1)h_1(t) - \omega(0)u^*(0,t) - \int_0^1 \omega'(x)u^*(x,t)dx - E'(t)}{\omega(1)} - h_1(t), \tag{2.17}$$

where

$$\begin{aligned} u^*(x,t) =& \sum_{k=1}^{N_1} c_k^{(1)} \frac{x_1^{(k)} - x}{2(t - t_1^{(k)} + T_0)} \frac{Heavisde(t - t_1^{(k)} + T_0)\exp(-\frac{(x - x_1^{(k)})^2}{4(t - t_1^{(k)} + T_0)})}{\sqrt{4\pi(t - t_1^{(k)} + T_0)}} \\ &+ \sum_{k=1}^{N_2} c_k^{(2)} \frac{x_2^{(k)} - x}{2(t - t_2^{(k)} + T_0)} \frac{Heavisde(t - t_2^{(k)} + T_0)\exp(-\frac{(x - x_2^{(k)})^2}{4(t - t_2^{(k)} + T_0)})}{\sqrt{4\pi(t - t_2^{(k)} + T_0)}} \\ &+ \sum_{k=1}^{N_3} c_k^{(3)} \frac{x_3^{(k)} - x}{2(t - t_3^{(k)} + T_0)} \frac{Heavisde(t - t_3^{(k)} + T_0)\exp(-\frac{(x - x_3^{(k)})^2}{4(t - t_3^{(k)} + T_0)})}{\sqrt{4\pi(t - t_3^{(k)} + T_0)}} + \sum_{z=0}^{N_4-1} \sum_{k=0}^{[\frac{z}{2}]} c_z^{(4)} (z - 2k) \frac{x^{z-2k-1} t^k}{k!(z-2k)!}. \end{aligned} \tag{2.18}$$

Assuming that $M_1 + 2M_2 = \sum_{i=1}^{4} N_i$ and collocating the residual functions

$$R_1(y_i) = 0,\ i = \overline{0, M_1 - 1},\ R_2(z_i) = 0,\ i = \overline{0, M_2 - 1},\ R_3(z_i) = 0,\ i = \overline{0, M_2 - 1}, \tag{2.19}$$

at the points

$$y_i = \frac{i}{M_1 - 1},\ i = \overline{0, M_1 - 1}, \quad z_i = \frac{iT}{M_2 - 1},\ i = \overline{0, M_2 - 1},$$

we form a system of linear equations such as

$$Ac = g, \tag{2.20}$$

where c is the vector of unknown constants

$$\left(c_1^{(1)}, ..., c_{N_1}^{(1)}, c_1^{(2)}, ..., c_{N_2}^{(2)}, c_1^{(3)}, ..., c_{N_3}^{(3)}, c_0^{(4)}, ..., c_{N_4-1}^{(4)} \right).$$

Typically, A is an ill-conditioned matrix, therefore we need to apply the regularization techniques to obtain a stable solution. Hence, instead of (2.20), according to the Tikhonov regularization method [16, 20], we solve the following modified system of equations

$$(A^{tr}A + \lambda_1 I)c = A^{tr}g, \tag{2.21}$$

where I is the unit matrix, A^{tr} denotes the transpose of the matrix A and $\lambda_1 > 0$ is the regularization parameter. For obtaining the appropriate regularization parameter we use the $L - Curve$ criterion developed by [7]. Once the approximation of $u(x,t)$ is drived, from Eqs. (1.3) and (2.18) we get

$$\sigma_{approx}(t) = \frac{u^*(0,t) + h_0(t)}{u_{approx}(0,t)}, \tag{2.22}$$

provided that $u_{approx}(0,t) \neq 0, \forall t \in [0,T]$. It should be noted that the approximations constructed by Eqs. (2.1)-(2.22) are valid as long as the input initial and boundary data of the problem are free of errors. Otherwise, appropriate instructions should be adopted so that the errors in the input data are controlled and stable numerical derivatives are obtained. Thus, in the case of inaccurate boundary data, assume $E^\sigma(t)$ be perturbation subject to $\|E(t)-E^\sigma(t)\|_\infty \leq \sigma$. Then, we fix the constant N and consider the approximations of $E^\sigma(t)$ and $(E^\sigma)'(t)$ based on the following spline functions

$$E^\sigma(t) \simeq S(t) = \begin{cases} S_1(t), & 0 \leq t < \frac{T}{N} \\ S_2(t), & \frac{T}{N} \leq t < \frac{2T}{N} \\ \vdots & \\ S_N(t), & \frac{(N-1)T}{N} \leq t < T, \end{cases} \qquad (E^\sigma)'(t) \simeq S'(t) = \begin{cases} S'_1(t), & 0 \leq t < \frac{T}{N} \\ S'_2(t), & \frac{T}{N} \leq t < \frac{2T}{N} \\ \vdots & \\ S'_N(t), & \frac{(N-1)T}{N} \leq t < T, \end{cases} \tag{2.23}$$

where

$$S_i(t) = \alpha_{i,1}t^3 + \alpha_{i,2}t^2 + \alpha_{i,3}t + \alpha_{i4}, \quad S'_i(t) = 3\alpha_{i,1}t^2 + 2\alpha_{i,2}t + \alpha_{i,3}.$$

We wish to find the unknown coefficients $\alpha_{i,j},\ i = \overline{1,N},\ j = \overline{1,4}$ such that the function $S(t)$ be a natural spline approximation for $E^\sigma(t)$. In this repect we form the following equations

$$\begin{cases} \alpha_{i,1}(\frac{jT}{N})^3 + \alpha_{i,2}(\frac{jT}{N})^2 + \alpha_{i,3}(\frac{jT}{N}) + \alpha_{i,4} = E^\sigma(\frac{jT}{N}), \quad j \in \{i-1, i\},\ i = \overline{1,N}, \\ \alpha_{1,2} = 0, \quad 6\alpha_{N,1}T + 2\alpha_{N,2} = 0, \\ 3\alpha_{i,1}(\frac{iT}{N})^2 + 2\alpha_{i,2}(\frac{iT}{N}) + \alpha_{i,3} - 3\alpha_{i+1,1}(\frac{iT}{N})^2 - 2\alpha_{i+1,2}(\frac{iT}{N}) - \alpha_{i,3} = 0, \quad i = \overline{1,N-1}, \\ 6\alpha_{i,1}(\frac{iT}{N}) + 2\alpha_{i,2} - 6\alpha_{i+1,1}(\frac{iT}{N}) - 2\alpha_{i+1,2} = 0, \quad i = \overline{1,N-1}, \end{cases} \tag{2.24}$$

and the result of which will be a system represented generically by

$$B\alpha = e,$$

where α is a vector including the unknowns $\alpha_{ij},\ i = \overline{1,N},\ j = \overline{1,4}$ and similar to Eq. (2.21), the Tikhonov regularization method solves the modified system

$$(B^{tr}B + \lambda_2 I)\alpha = B^{tr}e, \quad \lambda_2 > 0,$$

to get the unknown vector α and then $(E^\sigma)'(t)$ is specified which is contributed through Eqs. (2.1)-(2.22) instead of $E'(t)$.

3 Numerical experiments

In this section, we solve three examples to test the applicability of the proposed technique and use the notations

$$Abs(u) = |u_{exact}(x,t) - u_{approx}(x,t)|, \quad Abs(\sigma) = |\sigma_{exact}(t) - \sigma_{approx}(t)|,$$

to present the absolute errors of the functions $u(x,t)$ and $\sigma(t)$. Numerical computations are implemented by the MATHEMATICA software version 12.3. We use routine commands such as, LinearSolve (for solving the linear system of algebraic equations) and RandomReal[-1,1] (for generating random real numbers belonging to the interval $[-1,1]$).

3.0.1 Example 1

As the first example [19], consider the problem given by Eqs. (1.1)-(1.5) with the following properties:

$$u_0(x) = x^2+1,\ h_0(t) = 2t^2+t,\ h_1(t) = 2(1+t+t^2),\ \omega(x) = 1,\ E(t) = 2t + \frac{4}{3},\ x \in (0,1),\ t \in (0,1). \tag{3.1}$$

The exact solutions of this problem are

$$u(x,t) = x^2 + 2t + 1, \quad \sigma(t) = t.$$

We solve the problem by applying the numerical scheme presented in Section 2 in the presence of exact initial and boundary data with

$$N_1 = N_2 = N_3 = N_4 = 6,\ M_1 = M_2 = 8,\ T_0 = 1,\ \tau = r = 0.6, \tag{3.2}$$

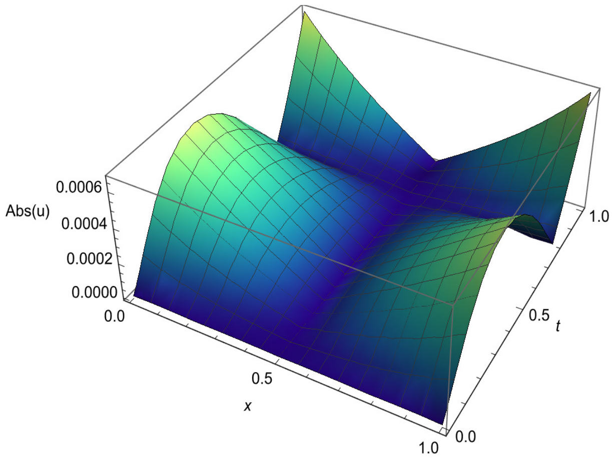

Figure 1: Graph of the absolute error of the approximate solution for $u(x,t)$ obtained by employing the proposed method in the presence of presice boundary data with $\lambda_1 = 6 \times 10^{-9}$, discussed in Example 3.0.1.

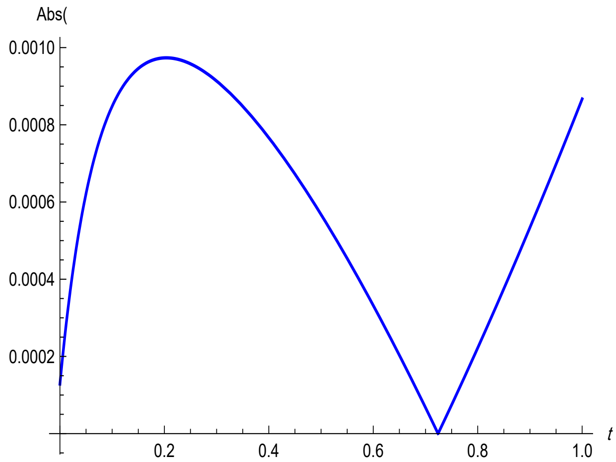

Figure 2: Graph of the absolute error of the approximate solution for $\sigma(t)$ obtained by employing the proposed method in the presence of presice boundary data with $\lambda_1 = 6 \times 10^{-9}$, discussed in Example 3.0.1.

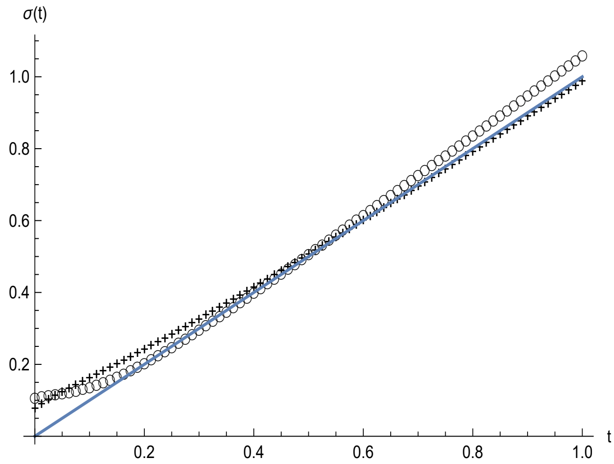

Figure 3: Representation of the exact (blue line) and approximate solutions for $\sigma(t)$ obtained by applying the proposed method in the presence of the perturbed boundary data subject to different values of η, i.e. $+++$: corresponding to $\eta = 10^{-2},\ \lambda_1 = 8 \times 10^{-5}$, $\circ\circ\circ$: corresponding to $\eta = 0.03,\ \lambda_1 = 6 \times 10^{-5}$, discussed in Example 3.0.1.

Table 1: Comparison between the infinity norm of absolute errors for the unknown functions u and σ, in the presence of the perturbed boundary data discussed in Example 3.0.1.

η	$\Vert Abs(u)\Vert_\infty$	$\Vert Abs(\sigma)\Vert_\infty$
0	0.00067	0.00097
10^{-2}	0.035	0.082
3×10^{-2}	0.096	0.1

and derive the outcomes illustrated in Figures 1-2.

To study the numerical stability of the approximate solution with respect to the small perturbations of the input data, we contaminate the extra condition (1.5) with artificial errors using the following rule

$$E^{\sigma}(t_i) = E(t_i) + \eta \times RandomReal[-1,1], \quad t_i \in [0,1], \tag{3.3}$$

where $\eta \in \{10^{-2}, 3 \times 10^{-2}\}$ is the percentage of noise. By solving the problem using the presented method with the following parameters

$$N_1 = N_2 = N_3 = N_4 = 6,\ M_1 = M_2 = 8,\ N = 10,\ T_0 = 1,\ \tau = r = 0.6,\ N = 10, \tag{3.4}$$

and $\lambda_2 = 10^{-7}$, we get the results depicted by Figure 3 and Table 1. We observe that by applying the regularization method, the errors introduced into the boundary measurement (1.5) are restrained and acceptable approximations are obtained.

3.1 Example 2

Consider the inverse problem

$$u_t(x,t) = u_{xx}(x,t), \quad in \quad (0,1) \times (0,1), \tag{3.5}$$

with initial condition

$$u_0(x) = \cos(x),\ 0 < x < 1, \tag{3.6}$$

and boundary conditions

$$-u_x(0,t) + \sigma(t)u(0,t) = 1, \quad u_x(1,t) + \sigma(t)u(1,t) = \cos(1) - e^{-t}\sin(1),\ 0 < t < 1, \tag{3.7}$$

$$\int_0^1 (1+x^2)u(x,t)dx = 2e^{-t}\cos(1),\ 0 \le t \le 1. \tag{3.8}$$

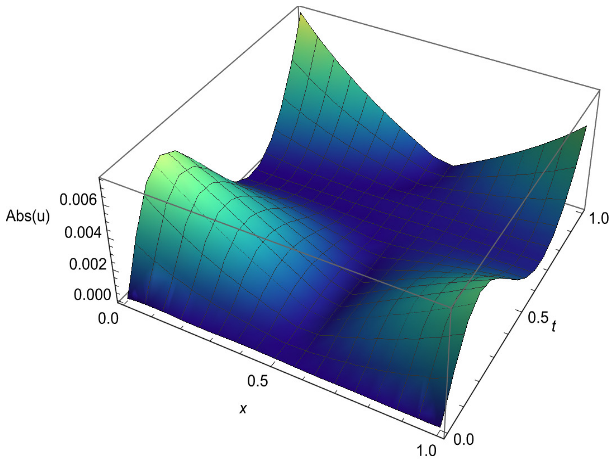

Figure 4: Graph of the absolute error of the approximate solution for $u(x,t)$ obtained by employing the proposed method in the presence of presice boundary data with $\lambda_1 = 10^{-5}$, discussed in Example 3.1.

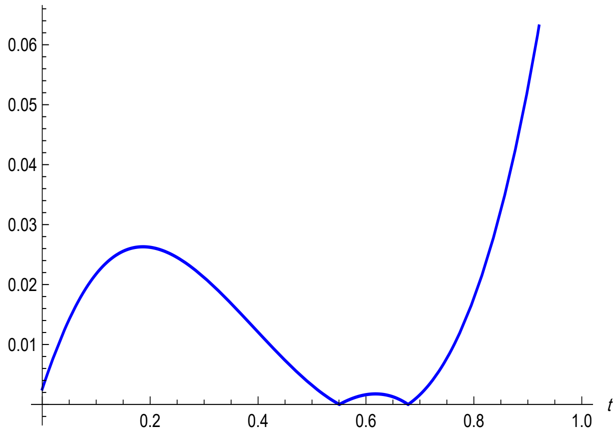

Figure 5: Graph of the absolute error of the approximate solution for $\sigma(t)$ obtained by employing the proposed method in the presence of presice boundary data with $\lambda_1 = 10^{-5}$, discussed in Example 3.1.

The exact solutions of this problem are $u(x,t) = e^{-t}\cos(x)$, $\sigma(t) = e^t$. We solve the problem by applying the numerical scheme presented in Section 2, with the properties (3.2) and $\lambda_1 = 10^{-5}$ in the presense of exact boundary data. Figures 4-5 show the good agreement between the exact and approximate solutions.

We aim to examine the performance of the proposed method in the presence of perturbed additional specification (1.5) and employ the rule (3.3) to generate noisy data such that $\eta \in \{10^{-2}, 3\times 10^{-2}\}$. The problem is solved using the presented technique with the properties (3.4) and $\lambda_2 = 10^{-5}$ and the results are shown in Figure 6 and Table 2. It can be observed that the unknown HTC is accurately retrieved proportional with the amount of noise η.

3.1.1 Example 3

Consider the inverse problem (1.1)-(1.5) with the following properties

$$u_0(x) = \omega(x) = 1, \quad h_0(t) = 1, \quad h_1(t) = 1+t, \quad x \in (0,1),\ t \in (0,1). \tag{3.9}$$

Since the exact solution of this problem is not available, in the first step the solution of the direct problem including Eqs. (1.1)-(1.4) with $\sigma(t) = 1+t$ is sought of the form (2.13)-(2.14) and the following boundary conditions are imposed using the collocation method

$$-u_x(0,t) + (1+t)u(0,t) = 1, \quad u_x(1,t) + (1+t)u(1,t) = 1+t, \quad t \in (0,1). \tag{3.10}$$

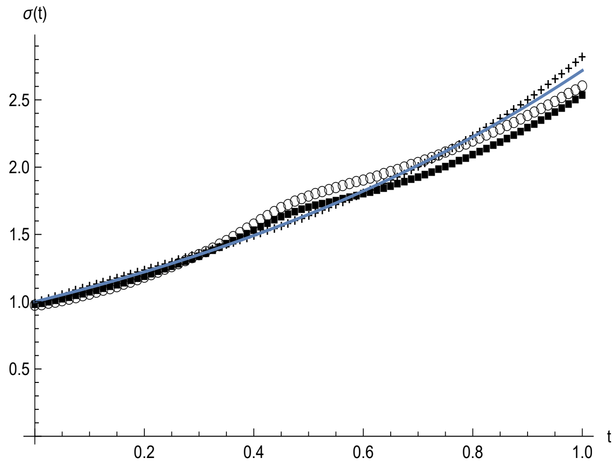

Figure 6: Representation of the exact (blue line) and approximate solutions for $\sigma(t)$ obtained by applying the proposed method in the presence of the perturbed boundary data subject to different values of η, i.e. $+++$: corresponding to $\eta = 0,\ \lambda_1 = 8 \times 10^{-6}$, $\circ\circ\circ$: corresponding to $\eta = 0.01,\ \lambda_1 = 4 \times 10^{-5}$, $\blacksquare\blacksquare\blacksquare$: corresponding to $\eta = 0.03,\ \lambda_1 = 10^{-4}$, discussed in Example 3.1.

Table 2: Comparison between the infinity norm of absolute errors for the unknown functions u and σ, in the presence of the perturbed boundary data discussed in Example 3.1.

η	$\Vert Abs(u)\Vert_\infty$	$\Vert Abs(\sigma)\Vert_\infty$
0	0.0054	0.11
10^{-2}	0.027	0.14
3×10^{-2}	0.04	0.18

For solving the direct problem we use the parameters (3.2) where the appropriate regularization parameter is chosen as $\lambda_2 = 10^{-10}$ and the approximations of $u(x,t)$ and $E(t) = \int_0^1 u(x,t)dx$ are calculated.
Taking the approximation of $E(t)$ along with the conditions (3.9) into account and applying the approximation method for solving the inverse problem with $\lambda_1 = 8 \times 10^{-5}$ we find the results shown in Figures 7-8.

4 Concluding remarks

This paper proposes an approximation method for recovering a time dependent HTC along with the temperature in the one-dimensional heat equation with the third-type boundary conditions and an integral measurement. By employing the integral condition the unknown HTC is disapeared and the problem is reformulated as a certain PDE and the solution of the converted problem is sought in the form of linear combination of the method of fundamental solutions and the heat polynomials. By applying the collocation method, the problem is reduced to the solution of a linear system of algebraic equations. The method employs the combination of the natural cubic spline technique and the Tikhonov regularization method to provide stable approximation of the derivative of the perturbed boundary data. We provide several numerical tests and the issue of numerical stability is discussed. It can be seen that by employing the meshless regularization technique, satisfactory approximate solutions are obtained such that in the presence of the precise initial and boundary data the unknown fuctions are excellently retrieved and regarding the noisy boundary data the obtained approximations deviate from the analytical solution almost proportional to the amount of introduced noise.

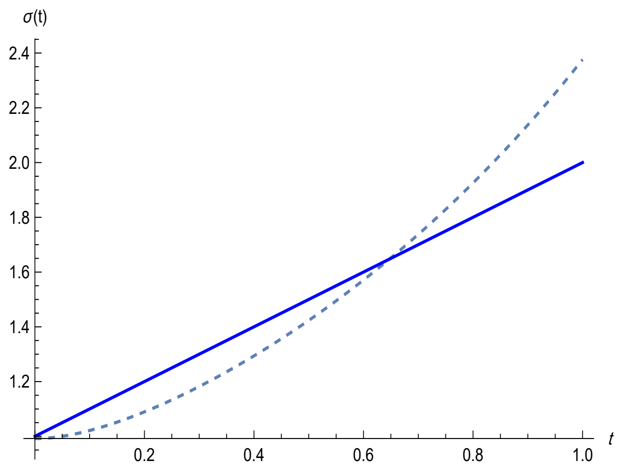

Figure 7: Graphs of the exact (thick line) and approximate (dashed line) solutions corresponding to the unknown function $\sigma(t)$, discussed in Example 3.1.1.

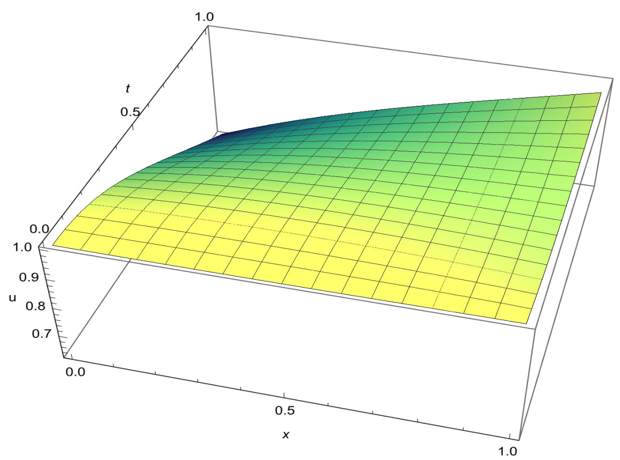

Figure 8: Approximate solution of $u(x, t)$, discussed in Example 3.1.1.

References

[1] T. Abdelhamid, A.H. Elsheikh, A. Elazab, S.W. Sharshir, E.S. Selima and D. Jiang, *Simultaneous reconstruction of the timedependent Robin coefficient and heat flux in heat conduction problems*, Inverse Probl. Sci. Eng. **26** (2018), no. 9, 1231–1248.

[2] G. Airweather and A. Karageorghis, *The method of fundamental solutions for numerical solutions of the biharmonic equation*, J. Comput. Phys. **69** (1987), no. 2, 434–459.

[3] F.S. V. Bazan, L. Bedin and F. Bozzoli, *Numerical estimation of convective heat transfer coefficient through linearization*, Int. J. Heat Mass Tran. **102** (2016), 1230–1244.

[4] A. Bogomolny, *Fundamental solutions method for elliptic boundary value problems*, SIAM. J. Numer. Anal. **22** (1985), no. 4, 644–669.

[5] J.R. Cannon, *The One-Dimensional Heat Equation*, Reading (MA): Addison Wesley, 1984.

[6] G. Fairweather and A. Karageorghis, *The method of fundamental solutions for elliptic boundary value problems*, Adv. Comput. Math. **9** (1998), 69–95.

[7] P.C. Hansen, *Analysis of discrete ill-posed problems by means of the L-curve*, SIAM Rev. **34** (1992), no. 4, 561–580.

[8] D.N. Hao, P.X. Thanh and D. Lesnic, *Determination of the heat transfer coefficients in transient heat conduction*, Inverse Probl. **29** (2013), no. 9, 095020.

[9] B.T. Johansson, D. Lesnic and T. Reeve, *A method of fundamental solutions for the one-dimensional inverse Stefan problem*, Appl. Math. Model. **35** (2011), no. 9, 4367–4378.

[10] B.T. Johansson, D. Lesnic and T. Reeve, *A meshless method for an inverse two-phase onedimensional linear Stefan problem*, Inverse Prob. Sci. Eng. **21** (2013), no. 1, 17–33.

[11] B.T. Johansson, D. Lesnic and T. Reeve, *A meshless method for an inverse two-phase one-dimensional nonlinear Stefan problem*, Math. Comput. Simul. **101** (2014), 61–77.

[12] A. Karageorghis, D. Lesnic and L. Marin, *A survey of applications of the MFS to inverse problems*, Inverse Probl. Sci. Eng. **19** (2011), no. 3, 309–336.

[13] S. A. Kassabek and D. Suragan, *Numerical approximation of the one-dimensional inverse Cauchy–Stefan problem using heat polynomials methods*, Comput. Appl. Math. **41** (2022), no. 189.

[14] E. Kita and N. Kamiya, *Trefftz method: An overview*, Adv. Eng. Softw. **24** (1995), no. 1-3, 3–12.

[15] V.D. Kupradze, *Potential Methods in Elasticity Theory*, Fizmatgiz: Moscow. Zbl.115,187. English transl., New York, 1965.

[16] D. Lesnic, *Inverse Problems with Applications in Science and Engineering*, Chapman and Hall/CRC, 2021.

[17] L. Marin, *A meshless method for solving the Cauchy problem in threedimensional elastostatics*, Comput. Math. Appl. **50** (2005), no. 1-2, 73–92.

[18] L. Marin and D. Lesnic, *The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations*, Comput. Struct. **83** (2005), no. 4-5, 267–278.

[19] T.T. M. Onyango, D. B. Ingham, D. Lesnic and M. Slodicka, *Determination of a time-dependent heat transfer coefficient from non-standard boundary measurements*, Math. Comput. Simul. **79** (2009), no. 5, 1577–1584.

[20] K. Rashedi, *Recovery of coefficients of a heat equation by Ritz collocation method*, Kuwait J. Sci. **50** (2023), no. 2A, 1–12.

[21] K. Rashedi and A. Sarraf, *Heat source identification of some parabolic equations based on the method of fundamental solutions*, Eur. Phys. J. Plus **133** (2018), no. 403.

[22] M. Rostamian and A. Shahrezaee, *A meshless method for solving 1D time-dependent heat source problem*, Inverse Probl. Sci. Eng. **26** (2017), no. 1, 51–82.

[23] T. Shigeta and D. L. Young, *Method of fundamental solutions with optimal regularization techniques for Cauchy problem of the Laplace equation with singular points*, J. Comput. Phys. **228** (2009), no. 6, 1903–1915.

[24] M. Slodicka, D. Lesnic and T.T.M. Onyango, *Determination of a time-dependent heat transfer coefficient in a nonlinear inverse heat conduction problem*, Inverse Prob. Sci. Eng. **18** (2010), no. 1, 65–81.

[25] E. Trefftz, *Ein Gegenstuck zum Ritzschen Verfahren*, Proc. 2nd Ind. Congr. Appl. Mech., Zurich **24** (1926), 131–137.

[26] T. Ushijima and A. Chiba, *A fundamental solution method for the reduced wave problem in a domain exterior to a disc*, J. Comput. Appl. Math. **152** (2003), no. (1-2), 545–557.

[27] L. Yan, F. Yang and C. Fu, *A Bayesian inference approach to identify a Robin coefficient in one-dimensional parabolic problem*, J. Comput. Appl. Math. **231** (2009), no. 2, 840–850.

[28] R. Zolfaghari and A. Shidfar, *Reconstructing an unknown time-dependent function in the boundary conditions of a parabolic PDE*, Appl. Math. Comput. **226** (2014), 238–249.

[29] R. Zolfaghari and A. Shidfar, *Restoration of the heat transfer coefficient from boundary measurements using the Sinc method*, Comput. Appl. Math. **34** (2015), 29–44.