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Characterization of multipliers on $\star$-algebras acting on orthogonal elements

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Abstract

Let A be a Banach $\star$-algebra, X be a Banach $\star$-A-bimodule and $T : A \longrightarrow X$ be a continuous linear map. In this paper, by using orthogonality conditions on A, we characterize the map T on certain Banach algebra including $C^{\star}$-algebras, group algebras, standard operator algebras and Banach algebras that is generated by idempotents. We also characterize a continuous linear map from zero Jordan product determined Banach algebra A into a Banach A-bimodule X, and give some applications of this result.

Keywords: $\star$-algebra, Multiplier, Orthogonality, Commuting map
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1 Introduction and Preliminaries

Throughout this paper all algebras and linear spaces will be over the complex field $\mathbb{C}$. Let A be an algebra and X be an A-bimodule. A linear map $T : A \longrightarrow X$ is called *left multiplier* $\big($*right multiplier*$\big)$ if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad \big(T(ab) = aT(b)\big),$$

and T is called a *multiplier* if it is both left and right multiplier. It should be remark that some authors used the term centralizer instead of multiplier, such as Johnson, who developed the general theory of centralizers [13]. One may refer to the monograph [14] for the additional fundamental results in the theory of multipliers.

The following conditions can be expressed for maps behaving like multipliers at zero product elements.

(1) $a, b \in A, \quad ab = 0 \implies aT(b) = 0,$
(2) $a, b \in A, \quad ab = ba = 0 \implies aT(b) + bT(a) = 0,$
(3) $a, b \in A, \quad a \circ b = 0 \implies aT(b) + bT(a) = 0.$

A result of Brešar shows that if A is a prime ring and $T : A \longrightarrow A$ is an additive map, then T satisfies in condition (1) if and only if T is a right multiplier. Recently, the first author in [16], investigated the above conditions for a unital $C^{\star}$-algebra A and a unital Banach A-bimodule X, and proved that each of conditions (1), (2) and (3) imply that $T : A \longrightarrow X$ is a right multiplier. Another type of condition (1) is

$$a, b \in A, \quad ab = e_A \implies T(ab) = aT(b). \tag{1.1}$$

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It is proved in [15, Corollary 2.10] that each continuous linear map T from unital Banach algebra A into unital Banach A-bimodule X satisfying the condition (1.1) is a right multiplier.

The Banach algebra A is said to be *zero product determined* if every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, satisfying

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \tag{1.2}$$

can be written as

$$\phi(a, b) = T(ab), \quad a, b \in A,$$

for some continuous linear map $T : A \longrightarrow X$. This concept appeared as a byproduct of the so-called property $(\mathbb{B})$ introduced in [1].

The Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear map $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (1.2) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

Recall that a *bounded approximate identity* for A is a bounded net $\{e_\lambda\}_{\lambda \in I}$ in A such that $e_\lambda a \longrightarrow a$ and $ae_\lambda \longrightarrow a$ for every $a \in A$. It is known that the group algebra $L^1(G)$ for a locally compact group G and every $C^\star$-algebra has a bounded approximate identity bounded by one, see [7].

It should be pointed out that if A has a bounded approximate identity, then the condition $\phi(ab, c) = \phi(a, bc)$ for every $a, b, c \in A$, implies that A is zero product determined. Indeed, we take $c = e_\lambda$ and note that T can be defined according to $T(a) = \phi(a, e_\lambda)$.

The set of idempotents of given Banach algebra A is denoted by $\mathcal{I}(A)$ and let $\mathfrak{J}(A)$ be the subalgebra of A generated by idempotents. We say that the Banach algebra A is generated by idempotents, if $A = \overline{\mathfrak{J}(A)}$.

It is known that the linear span of projections is norm dense in every von-Neumann algebra A, therefore $A = \overline{\mathfrak{J}(A)}$. Also, by [1, Example 1.3] topologically simple Banach algebras containing a non-trivial idempotent and the Banach algebras $L^p(G)$ with $1 \leq p < \infty$ and $C(G)$ (with convolution product) for any compact group G are generated by idempotens. For more examples of Banach algebra A with the property that $A = \overline{\mathfrak{J}(A)}$, see [1].

Let A be a $\star$-algebra and X be a $\star$-A-bimodule. A linear map $\delta : A \longrightarrow X$ is called $\star$-derivation if

$$\delta(ab) = \delta(a)b + a\delta(b), \quad \delta(a^\star) = \delta(a)^\star, \quad a, b \in A, \tag{1.3}$$

and it is called $\star$-Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ and the second equality in (1.3) satisfied.

For $\star$-derivations and $\star$-Jordan derivations, in [8, 10], the authors characterize the following two conditions on a linear map $\delta : A \longrightarrow X$;

$(\mathbb{D}1)$ $a, b \in A, \quad ab^\star = 0 \implies a\delta(b)^\star + \delta(a)b^\star = 0,$
$(\mathbb{D}2)$ $a, b \in A, \quad ab^\star = b^\star a = 0 \implies a\delta(b)^\star + \delta(a)b^\star = \delta(b)^\star a + b^\star \delta(a) = 0,$

where A is a $C^\star$-algebra or a zero product determined algebra. Moreover, the same problems for the group algebras solved by Ghahramani in [11]. In [9], the author considered the problem of characterizing linear maps on special $\star$-algebras behaving like left or right multipliers at orthogonal elements. Since then many authors have been studied linear maps on algebras ($\star$-algebra) through zero products and different results have been obtained; see for example [5, 8, 10, 11, 16, 17] and the references therein.

In this paper we consider the problem of characterizing linear maps behaving like multipliers at orthogonal elements for certain orthogonality conditions. In particular, we investigate the subsequent conditions on a linear map T from $\star$-algebra A into $\star$-A-bimodule X:

$(\mathbb{J}1)$ $a, b \in A, \quad ab^\star = 0 \implies aT(b)^\star = 0,$
$(\mathbb{J}2)$ $a, b \in A, \quad ab^\star = b^\star a = 0 \implies aT(b)^\star + b^\star T(a^\star)^\star = 0,$
$(\mathbb{J}3)$ $a, b \in A, \quad a \circ b^\star = 0 \implies aT(b)^\star + b^\star T(a^\star)^\star = 0,$
$(\mathbb{J}4)$ $a, b \in A, \quad ab^\star = b^\star a = 0 \implies a \diamond T(b)^\star + b^\star \diamond T(a^\star)^\star = 0,$

where $a \circ b = ab + ba$ is a Jordan product in A and "$\diamond$" denotes the Jordan product on X defined through

$$a \diamond x = x \diamond a = ax + xa, \quad a \in A, \ x \in X.$$

We investigate whether these conditions characterizes multipliers on $C^\star$-algebras, group algebra, standard operator algebras, and Banach $\star$-algebras that is generated by idempotents.

2 Characterizing multipliers on Banach $\star$-algebras

In this section, by using zero products preserving bilinear maps, we prove that each linear map T from a Banach $\star$-algebra A into a Banach $\star$-A-bimodule X which satisfies one of the conditions ($\mathbb{J}1$)-($\mathbb{J}4$) is a multiplier.

The following results presented by the first author in [16].

Theorem 2.1 ([16, Theorem 2.6]). Suppose that T is a linear map from unital Banach algebra A into an essential Banach A-bimodule X such that

$$a, b \in A, \quad ab = ba = 0 \implies aT(b) + bT(a) = 0. \tag{2.1}$$

Then $T(xa) = xT(a)$ for all $a \in A$ and every $x \in \mathfrak{J}(A)$.

Corollary 2.2 ([16, Corollary 2.7]). Let A be a Banach algebra with a bounded approximate identity and X be an essential Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map such that the condition (2.1) holds. If $A = \overline{\mathfrak{J}(A)}$, then T is a right multiplier.

Recall that an A-bimodule X is called left (right) *faithful* if the condition $ax = 0$ ($xa = 0$) for $a \in A$ implies that $x = 0$. For example, if A is unital or has a bounded approximate identity, then the n-th dual A-bimodule A^n, with the usual structures, is faithful.

The following result improve [9, Theorem 2.1] for any faithful Banach $\star$-A-bimodule.

Theorem 2.3. Let A be a Banach $\star$-algebra with property ($\mathbb{B}$) and X be a faithful $\star$-A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{J}1$). Then T is a left multiplier.

Proof . Define a continuous bilinear map $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = aT(b^\star)^\star, \qquad a, b \in A.$$

Then $\phi(a, b) = 0$ whether $ab = 0$. Now the property ($\mathbb{B}$) gives

$$cT(a^\star b^\star)^\star = \phi(c, ba) = \phi(cb, a) = cbT(a^\star)^\star, \tag{2.2}$$

for all $a, b \in A$. Since X is faithful, from (2.2) we get

$$T(a^\star b^\star) = T(a^\star)b^\star \qquad a, b \in A.$$

Consequently, $T(ab) = T(a)b$ for all $a, b \in A$ and the result follows. □

It turned out in [1] that many important class of Banach algebra including $C^\star$-algebras and group algebra $L^1(G)$, where G is any locally compact group have the property ($\mathbb{B}$). It follows from [7, Theorem 3.3.15] that $M(G)$ with respect to convolution product is the dual of $C_0(G)$ as a Banach $M(G)$-bimodule and $C_0(G)$ is faithful. Thus, we get the following results.

Corollary 2.4. Let A be a $C^\star$-algebra and let $T : A \longrightarrow A^*$ be a continuous linear map satisfying ($\mathbb{J}1$). Then T is a left multiplier.

Corollary 2.5. Let $L^1(G)$ be a group algebra and let $T : L^1(G) \longrightarrow C_0(G)$ be a continuous linear map such that the condition ($\mathbb{J}1$) holds. Then T is a left multiplier.

Since every Banach algebra that is generated by idempotents satisfies in the property ($\mathbb{B}$), we have

Corollary 2.6. Let A be a Banach $\star$-algebra such that $A = \overline{\mathfrak{J}(A)}$ and let X be a faithful $\star$-A-bimodule. Then every continuous linear map $T : A \longrightarrow X$ satisfying ($\mathbb{J}1$) is a left multiplier.

It should be pointed out that every zero product determined Banach algebra has the property ($\mathbb{B}$). Consequently, Theorem 2.3 and their corollaries also true if the property ($\mathbb{B}$) replaced by zero product determined Banach algebras.

Theorem 2.7. [5, Theorem 4.1] If ϕ is a bilinear map from $A \times A$ into a vector space X such that

$$a, b \in A, \quad ab = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) = \phi(ax, e_A), \quad and \;\; \phi(x, a) = \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Lemma 2.8. Suppose that T is a linear map from unital $\star$-algebra A into $\star$-A-bimodule X such that the condition ($\mathbb{J}1$) holds. Then $T(ap) = T(a)p$ for all $a \in A$ and every $p \in \mathfrak{J}(A)$.

Proof . Let $p \in \mathcal{I}(A)$ be arbitrary. Since $p(e_A - p) = 0$, therefore we have $(e_A - p)^\star p^\star = 0$ and hence $(e_A - p^\star)p^\star = 0$. Thus, $(e_A - p^\star)T(p)^\star = 0$ and so $T(p) = T(p)p$. Define $\Delta : A \longrightarrow X$ via $\Delta(a) = T(a) - T(e_A)a$. Then Δ is a linear map, $\Delta(e_A) = 0$ and

$$a\Delta(b)^\star = a(T(b) - T(e_A)b)^\star = aT(b)^\star - ab^\star T(e_A)^\star = 0,$$

whenever $ab^\star = 0$. Define a bilinear map $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = a\Delta(b^\star)^\star, \qquad a, b \in A.$$

If $ab = 0$, then $a(b^\star)^\star = 0$ and thus we get $\phi(a, b) = 0$. Hence by Theorem 2.7,

$$\phi(a, p) = \phi(ap, e_A), \quad and \;\; \phi(p, a) = \phi(e_A, pa), \tag{2.3}$$

for all $a \in A$ and each $p \in \mathcal{I}(A)$. Now by (2.3) we obtain $\Delta(ap) = \Delta(a)p$, for all $a \in A$. Consequently, $T(ap) = T(a)p$ for each $p \in \mathcal{I}(A)$. Now from definition of $\mathfrak{J}(A)$ we get $T(ap) = T(a)p$ for all $a \in A$ and $p \in \mathfrak{J}(A)$. This finishes the proof. □

An ideal I of A is called right faithful if for every $a \in I$, the equality $xa = 0$ for $x \in X$ impolite that $x = 0$

Theorem 2.9. Suppose that T is a linear map from unital $\star$-algebra A into $\star$-A-bimodule X such that the condition

$$a, b \in A, \;\; ab^\star = 0 \implies aT(b)^\star = 0,$$

holds. If $I \subseteq \mathfrak{J}(A)$ is a right faithful ideal, then T is a left multiplier.

Proof . It follows from Lemma 2.8 that $T(abw) = T(ab)w$ and

$$T(abw) = T(a(bw)) = T(a)bw, \quad a, b \in A, \;\; w \in I.$$

Thus, $\big(T(ab) - T(a)b\big)w = 0$ for all $a, b \in A$ and every $w \in I$. Since I is a right faithful, $T(ab) = T(a)b$ for all $a, b \in A$. Therefore, T is a left multiplier. □

Theorem 2.10. Let A be a von-Neumann algebra and X be an essential Banach $\star$-A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map such that the condition ($\mathbb{J}2$) holds. Then T is a left multiplier.

Proof . Define a continuous linear map $\delta : A \longrightarrow X$ by $\delta(a) = T(a^\star)^\star$. Let $a, b \in A$ such that $ab = ba = 0$. Then we have $a(b^\star)^\star = (b^\star)^\star a = 0$ and hence condition ($\mathbb{J}2$) implies that

$$a\delta(b) + b\delta(a) = a\big(T(b^\star)\big)^\star + bT(a^\star)^\star = 0,$$

for all $a, b \in A$ with $ab = ba = 0$. Applying Theorem 2.1, we obtain $\delta(pa) = p\delta(a)$ for all $a \in A$ and $p \in \mathcal{I}(A)$. Since $\delta(a) = T(a^\star)^\star$, form the above equality we get $T(ap) = T(a)p$ for all $a \in A$ and $p \in \mathcal{I}(A)$. Let A_{sa} denote the set of self-adjoint elements of A and let $x \in A_{sa}$. Then x is the limit of a sequence of linear combinations of projections in A, i.e., self-adjoint idempotents, therefore

$$x = \lim_n \sum_{k=1}^{n} \lambda_k p_k.$$

Hence for all $a \in A$,

$$T(ax) = \lim_n T(a \sum_{k=1}^{n} \lambda_k p_k) = \lim_n \sum_{k=1}^{n} \lambda_k T(ap_k) = \lim_n \sum_{k=1}^{n} \lambda_k T(a)p_k = T(a)x.$$

Now, let $b \in A$ be arbitrary. Then $b = x + iy$ for $x, y \in A_{sa}$ and thus we arrive at

$$T(ab) = T\big(a(x+iy)\big) = T(a)x + iT(a)y = T(a)b.$$

Consequently, $T(ab) = T(a)b$ for all $a, b \in A$ and hence T is a left multiplier. □

It is well-known that on the second dual space A^{**} of a Banach algebra A there are two multiplications, called the first and second Arens products which make A^{**} into a Banach algebra [7]. If these products coincide on A^{*}, then A is said to be Arens regular. It is shown [7] that every $C^{\star}$-algebra A is Arens regular.

For each Banach A-bimodule X, the second dual X^{**} turns into a Banach A^{**}-bimodule where A^{**} equipped with the first Arens product. One may refer to the monograph of Dales [7] for a full account of Arens product and $w^{\star}$-continuity of module structures.

Corollary 2.11. Let A be a $C^{\star}$-algebra and X be an essential Banach $\star$-A-bimodule. Then every continuous linear map $T : A \longrightarrow X$ satisfying ($\mathbb{J}2$) is a left multiplier.

Proof . Since the second dual of each $C^{\star}$-algebra is a von-Neumann algebra [7], by extending $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and applying Theorem 2.10, the result follows. □

A linear map $T : A \longrightarrow X$ is called *left anti-multiplier*, $\big($*right anti-multiplier*$\big)$ if for all $a, b \in A$,

$$T(ab) = bT(a), \qquad \big(T(ab) = T(b)a\big),$$

and T is called *anti-multiplier* if it is both left and right anti-multiplier. It is proved in [16, Theorem 3.8] that every continuous right anti-multiplier T from a $C^{\star}$-algebra A into a Banach A-bimodule X is a left multiplier.

Remark 2.12. Let A be a $C^{\star}$-algebra, X be a essential Banach $\star$-A-bimodule and let $T : A \longrightarrow X$ be a continuous linear map such that

$$a, b \in A, \quad ab^{\star} = b^{\star}a = 0 \implies aT(b^{\star})^{\star} + b^{\star}T(a)^{\star} = 0.$$

Then T is a left multiplier. To see this, define a continuous linear map $\delta : A \longrightarrow X$ by $\delta(a) = T(a)^{\star}$. Then by similar argument as in the proof of Theorem 2.10, we conclude that $T(ab) = T(b)a$ for all $a, b \in A$ and hence T is a right anti-multiplier. Now it follows from [16, Theorem 3.8] that T is a left multiplier.

Let $B(\mathcal{H})$ be the operator algebra of all bounded linear operators on Hilbert space $\mathcal{H}$, and let $F(\mathcal{H})$ denotes the algebra of all finite rank operators in $B(\mathcal{H})$. Recall that a standard operator algebra is any subalgebra of $B(\mathcal{H})$ which contains the identity and the ideal $F(\mathcal{H})$.

In the following result we characterize the multipliers at orthognal element on standard operator algebra on Hilbert space $\mathcal{H}$.

Theorem 2.13. Let A be a standard operator algebra on Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 2$ such that A is closed under adjoint operation. Let X be an essential Banach $\star$-A-bimodule and $T : A \longrightarrow X$ be a continuous linear map such that the condition ($\mathbb{J}2$) holds. Then T is a left multiplier.

Proof . Let us define a linear map $\delta : A \longrightarrow X$ by $\delta(u) = T(u^{\star})^{\star}$. Let $u, v \in A$ be two operator such that $uv = vu = 0$. Then $u(v^{\star})^{\star} = (v^{\star})^{\star}u = 0$ and hence

$$u\delta(v) + v\delta(u) = u\big(T(v^{\star})\big)^{\star} + (v^{\star})^{\star}T(u^{\star})^{\star} = 0,$$

for all $u, v \in A$ with $uv = vu = 0$. Suppose that p be an idempotent operator of rank-one in A. By Theorem 2.1, $\delta(pu) = p\delta(u)$ for all $u \in A$. From definition of δ, we have $T(up) = T(u)p$ for all $u \in A$. By [6, Lemma 1.1], every element $v \in F(\mathcal{H})$ is a linear combinations of rank-one idempotents. Thus,

$$T(uv) = T(u)v, \qquad v \in F(\mathcal{H}). \tag{2.4}$$

Replacing u by the identity operator I in (2.4) we get

$$T(Iv) = T(I)v \quad v \in F(\mathcal{H}).$$

Since $F(\mathcal{H})$ is ideal in A, it follows that

$$T(uv) = T(Iuv) = T(I)uv. \tag{2.5}$$

By (2.4) and (2.5), we arrive at

$$T(u)v = T(I)uv.$$

On the other hand, $F(\mathcal{H})$ is an essential ideal in A. Consequently, $T(u) = T(I)u$ for all $u \in A$ and hence T is a left multiplier. □

Next under special hypothesis we prove the analogue result for the group algebra $L^1(G)$.

Theorem 2.14. Let $A = L^1(G)$ for locally compact group G and X be a Banach $\star$-A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{J}2$). Then T is a left multiplier with each of the following condition.

(i) G is compact;
(ii) G is abelian and X is symmetric.

Proof . Define $\delta : A \longrightarrow X$ by $\delta(a) = T(a^\star)^\star$. Then $a\delta(b) + b\delta(a) = 0$ for all $a, b \in A$ with $ab = ba = 0$. Now

(i) if G is compact, then A is generated by idempotents and by Corollary 2.2, δ is a right multiplier. Consequently, T is a left multiplier.

(ii) follows from [17, Theorem 2.10]. □

It should be note that the condition ($\mathbb{J}3$) implies ($\mathbb{J}2$) and therefore Theorem 2.10, Corollary 2.11, Theorem 2.13 and Theorem 2.14 still works with condition ($\mathbb{J}2$) replaced by ($\mathbb{J}3$).

In [10], Ghahramani and Pan prove that if A is a unital zero product determined $\star$-algebra and a linear map δ from A into itself satisfies the condition

$$a, b \in A, \quad ab^\star = 0 \implies a\delta(b)^\star + \delta(a)b^\star = 0, \tag{2.6}$$

then $\delta(a) = \Delta(a) + \delta(e_A)a$ for every $a \in A$, where Δ is a $\star$-derivation. Note that (2.6) implies condition ($\mathbb{J}4$).

Theorem 2.15. Let A be a $C^\star$-algebra and X be a essential Banach $\star$-A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map such that the condition ($\mathbb{J}4$) holds. Then there exist a linear map $\Delta : A \longrightarrow X$ and a multiplier $\psi : A \longrightarrow X$ such that $T = \Delta + \psi$, $\Delta(a) = D(a^\star)^\star$ for all $a \in A$, where $D : A \longrightarrow X$ is a derivation and it is a linear combination of two $\star$-derivations. Moreover, if A is commutative, then T is a multiplier.

Proof . Define $\delta : A \longrightarrow X$ by $\delta(a) = T(a^\star)^\star$. Then $a \diamond \delta(b) + b \diamond \delta(a) = 0$ for all $a, b \in A$ with $ab = ba = 0$. Define the bilinear map $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = a \diamond \delta(b) + b \diamond \delta(a).$$

Then $\phi(a, b) = 0$ whether $ab = ba = 0$. By applying [2, Theorem 3.1], there exist a derivarion $D : A \longrightarrow X$ and a multiplier $h : A \longrightarrow X$ such that $\delta = D + h$. Note that $h(a) = \xi \cdot a = a \cdot \xi$ for all $a \in A$, where $\xi = \delta^{**}(e_{A^{**}})$. Therefore

$$T(a) = D(a^\star)^\star + \xi^\star \cdot a, \quad a \in A.$$

For all $a \in A$, define $\Delta : A \longrightarrow X$ via $\Delta(a) = D(a^\star)^\star$ and $\psi : A \longrightarrow X$ by $\psi(a) = \xi^\star \cdot a$. Then Δ is a linear map, ψ is a multiplier and $T = \Delta + \psi$.

Now, consider $d_1, d_2 : A \longrightarrow X$ defined by $d_1(a) = \frac{1}{2}(D + \Delta)$ and $d_2(a) = \frac{1}{2i}(D - \Delta)$ for all $a \in A$. Then d_1 and d_2 are $\star$-derivation and $D = d_1 + id_2$. If A is commutative, then by [7, Theorem 2.8.63], D is zero, and hence $T = \psi$ is a multiplier. □

3 A result on commuting maps

The Banach algebra A is called *zero Jordan product determined* if every continuous bilinear map $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, satisfying

$$a, b \in A, \quad a \circ b = 0 \quad \Longrightarrow \quad \phi(a, b) = 0,$$

can be written as

$$\phi(a, b) = T(a \circ b), \qquad a, b \in A,$$

for some continuous linear map $T : A \longrightarrow X$. It is shown in [3, Theorem 3.6] that every $C^\star$-algebra, as well as every group algebra $L^1(G)$ of an amenable locally compact group G, is a zero Jordan product determined Banach algebra.

Recall that a linear map T from algebra A into A-bimodule X is called *Lie multiplier* if

$$T([a, b]) = [T(a), b], \qquad a, b \in A,$$

and it is called a *commuting map* if $[T(a), a] = 0$ for all $a \in A$, where $[a, b] = ab - ba$ is the Lie product. Obviously, each Lie multiplier is a commuting map, but the converse is, in general, not true, see [12, Example 2.3]. Ghahramani in [12], prove that the linear map $T : B(X) \longrightarrow B(X)$, where X is a Banach space, is a commuting map, whether it satisfies

$$a, b \in A, \quad ab = 0 \implies T([a, b]) = [T(a), b].$$

We now turns continuous linear maps which are necessary commuting maps.

Theorem 3.1. Let A be a zero Jordan product determined Banach algebra with a bounded approximate identity and X be an essential Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying

$$a, b \in A, \quad a \circ b = 0 \implies T([a, b]) = [T(a), b], \tag{3.1}$$

then T is a commuting map.

Proof . Define a continuous bilinear map $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = T([a, b]) - [T(a), b], \quad a, b \in A.$$

Then $a \circ b = 0$ implies that $\phi(a, b) = 0$. Since A is a zero Jordan product determined, there exists a continuous linear map $\psi : A \longrightarrow X$ such that $\psi(a \circ b) = \phi(a, b)$ for all $a, b \in A$. Hence,

$$\psi(a \circ b) = \phi(a, b) = T([a, b]) - [T(a), b], \quad a, b \in A.$$

Letting $b = e_\alpha$ in the above equality, we get $\psi(a) = 0$ for all $a \in A$ and hence

$$T([a, b]) = [T(a), b], \quad a, b \in A.$$

This means that T is a Lie multiplier. Thus, $[T(a), a] = 0$ for all $a \in A$. $\square$

Finally, we give an application of the above theorem to zero Jordan product determined Banach algebras.

Corollary 3.2. Let A be a $C^\star$-algebra and X be an essential Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying (3.1). Then T is a commuting map.

Corollary 3.3. Let $A = L^1(G)$ of an amenable locally compact group G, and X be an essential Banach A-bimodule. If $T : A \longrightarrow X$ is a continuous linear map such that the condition (3.1) holds. Then T is a commuting map.

From Theorem 3.1 and [4, Theorem 2.1] we get the next result.

Corollary 3.4. Let A be a unital Banach algebra that is generated by idempotents and let X be an essential Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying (3.1). Then T is a commuting map.

It is known that the matrix algebra $M_n(B)$ of $n \times n$ matrices over a unital Banach algebra B is zero Jordan product determined [4]. Now as a consequence of Theorem 3.1 we have

Corollary 3.5. Let $A = M_n(B)$ and X be an essential Banach A-bimodule. Let $T : A \longrightarrow X$ be a continuous linear map satisfying (3.1). Then T is a commuting map.

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