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Arcwise ρ-connected functions and their generalizations in vector optimization over cones

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Abstract

In this paper, we introduce a new class of arcwise ρ-K-connected, arcwise ρ-K-quasi connected and arcwise ρ-K-pseudo connected functions which encapsulate already known functions. Necessary and sufficient optimality conditions are established for a vector optimization problem over cones by involving these functions. Wolfe type and Mond-Weir type duals are formulated and corresponding duality results are also proved using these functions.

Keywords: Arcwise cone connected functions, Vector optimization problem, Optimality, Duality
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1 Introduction

The concept of convex functions was extended by Ortega and Rheinboldt [7] by introducing the arcwise connected functions defined on arcwise connected sets replacing a line segment joining two points by continuous arcs. Avriel and Zhang [1] call them "arcwise connected" functions. They also extended the same to arcwise Quasi-connected functions and arcwise Pseudo-connected functions and discussed their properties. Later on Singh [8] and Mukherjee and Yadav [6] worked on certain properties of arcwise connected sets and functions. After that by using the directional derivatives, Bhatia and Mehra [2] proposed optimality and duality results for scalar valued non-linear programming problem involving these functions and their generalizations. In 1997 Mukherjee [5] introduced arcwise connected functions over cones. They also discussed optimality conditions and duality theorems for a vector-valued non linear programming problem involving these functions. Authors like Zhang [10] and Suneja et al. [9] studied various aspects of arcwise connected functions in the form of their generalizations.

In this paper, we define arcwise ρ-K-connected, arcwise ρ-K-pseudo connected and arcwise ρ-K-quasi connected functions. Relevant examples are given to illustrate their existence. Using these functions, necessary and sufficient optimality conditions for vector optimization problem over cones are proved. Further, Wolfe type and Mond-Weir type duals are formulated. Weak and strong duality results are established.

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2 Notations and Definitions

Let X be a nonempty subset of $\mathbb{R}^n$ and K be a nonempty subset of $\mathbb{R}^k$. Let K be closed convex pointed cone with nonempty interior. The positive polar cone K^+ and the strict positive polar cone K^{s+} are defined as follows:

$$K^+ = \{y \in \mathbb{R}^k : x^T y \geq 0, \text{ for all } x \in K\}$$

and

$$K^{s+} = \{y \in \mathbb{R}^k : x^T y > 0, \text{ for all } x \in K\}.$$

The interior of K is denoted by $\operatorname{int} K$.

Definition 2.1 ([7]). A subset $X \subseteq \mathbb{R}^n$ is said to be an arcwise connected (AC) set, if for every $x^1 \in X$, $x^2 \in X$, there exists a continuous vector valued function $H_{x^1,x^2} : [0,1] \to X$, called an arc, such that

$$H_{x^1,x^2}(0) = x^1 \quad \text{and} \quad H_{x^1,x^2}(1) = x^2 .$$

Definition 2.2 ([7]). Let f be a real-valued function defined on an AC set $X \subseteq \mathbb{R}^n$. Then f is said to be an arcwise connected function (CN), if for every $x^1 \in X$, $x^2 \in X$, there exists an arc H_{x^1,x^2} such that

$$f(H_{x^1,x^2}(\theta)) \leq (1-\theta) f(x^1) + \theta f(x^2), \text{ for } 0 \leq \theta \leq 1 .$$

The function f will be called CN at x^1 on X if the above inequality holds for all $x^2 \in X$.

Definition 2.3. Let f be a real-valued function defined on an AC set $X \subseteq \mathbb{R}^n$. For every $\bar{x} \in X$, $x \in X$, the directional derivative of f at $\bar{x}$ with respect to an arc $H_{\bar{x},x}$ at $\theta = 0$ is defined as $\lim\limits_{\theta \to 0^+} \frac{f(H_{\bar{x},x}(\theta)) - f(\bar{x})}{\theta}$, provided the limit exists and is denoted by $f^+(H_{\bar{x},x}(0))$.

If f is an arcwise connected function at $\bar{x}$ on x, then $f(x) - f(\bar{x}) \geq f^+(H_{\bar{x},x}(0))$, for all $x \in X$.

Definition 2.4 ([5]). Let $f : X \to \mathbb{R}^k$, $f = (f_1, f_2, \cdots, f_k)^T$ be a vector valued function defined on an AC set $X \subseteq \mathbb{R}^n$. Then f is said to be an arcwise K-connected function (KCN), if for every $\bar{x} \in X$, $x \in X$, these exists an arc $H_{\bar{x},x}$ such that

$$(1-\theta) f(\bar{x}) + \theta f(x) - f(H_{\bar{x},x}(\theta)) \in K, \text{ for } 0 \leq \theta \leq 1 .$$

If f is KCN then

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) \in K,$$

where $f^+(H_{\bar{x},x}(0)) = (f_1^+(H_{\bar{x},x}(0)), f_2^+(H_{\bar{x},x}(0)), \ldots, f_k^+(H_{\bar{x},x}(0)))^T$ and each $f_i^+(H_{\bar{x},x}(0))$ exists for $i = 1, 2, \ldots, k$.

Generalizing the concept of arcwise K-connected function [5], we define new notions of arcwise ρ-K-connected function and its generalizations. For this purpose, we consider $\rho = (\rho_1, \rho_2, \ldots, \rho_k)^T \in \mathbb{R}^k$, a nonempty subset X of $\mathbb{R}^n$, a pseudometric function $d : X \times X \to \mathbb{R}_+$, $f = (f_1, f_2, \ldots, f_k)^T : X \to \mathbb{R}^k$ and a closed convex pointed cone K in $\mathbb{R}^k$ having nonempty interior.

Definition 2.5. The function f is said to be an arcwise ρ-K-connected function (ρKCN), if for every $\bar{x} \in X$, $x \in X$, there exists an arc $H_{\bar{x},x}$ such that

$$(1-\theta) f(\bar{x}) + \theta f(x) - f(H_{\bar{z},x}(\theta)) - \rho\theta d(\bar{x}, x) \in K, \text{ for } 0 \leq \theta \leq 1 .$$

If f is ρKCN then

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x}, x) \in K,$$

where $f^+(H_{\bar{x},x}(0)) = (f_1^+(H_{\bar{x},x}(0)), f_2^+(H_{\bar{x},x}(0)), \ldots, f_k^+(H_{\bar{x},x}(0))))^T$ and each $f_i^+(H_{\bar{x},x}(0))$ exists for $i = 1, 2, \ldots, k$.

The function f will be called ρKCN at $\bar{x}$ on X if the above result $f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x}, x) \in K$ holds for all $x \in X$.

Definition 2.6 ([3]). The function $f : X \to \mathbb{R}^k$ is said to be K-convex like if there exists $\theta \in (0,1)$ such that, for each $x^1 \in X$, $x^2 \in X$, there exists $x^3 \in X$ with

$$\theta f(x^1) + (1-\theta) f(x^2) - f(x^3) \in K\,.$$

We now give an example of a function which is arcwise ρ-K-connected function.

Example 2.7. Let $n = 2$, $k = 2$. Define $X \subseteq \mathbb{R}^2$ as

$$X = \{(x_1, x_2)^T : x_1^2 + x_2^2 \geq 1,\, x_1 > 0,\, x_2 > 0\}\,.$$

Then X is an AC set with respect to $H_{\bar{x},x} : [0,1] \to X$ given by

$$H_{\bar{x},x}(\theta) = (((1-\theta)\bar{x}_1^2 + \theta x_1^2)^{1/2}, ((1-\theta)\bar{x}_2^2 + \theta x_2^2)^{1/2})^T, \text{for all } \theta \in [0,1],$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2)^T$, $x = (x_1, x_2)^T \in X$. Let $\bar{x} = (1,1)^T$, $K = \{(x_1, x_2) : x_1 \geq x_2, x_1 \leq 0\}$, $\rho = (0,1)^T \in \mathbb{R}^2$ and $d(\bar{x}, x) = (\bar{x}_1 - x_1)^2 + (\bar{x}_2 - x_2)^2$. Define $f : X \to \mathbb{R}^2$, $f(x) = (f_1(x), f_2(x))$ as

$$f_1(x) = \begin{cases} x_1^2 + x_2^2, & \text{if } x_1 > 1, x_2 > 1 \\ 2, & \text{otherwise} \end{cases} \quad ; \quad f_2(x) = \begin{cases} -x_1^2, & \text{if } x_1 > 1, x_2 > 1 \\ -1, & \text{otherwise} \end{cases}.$$

Clearly,

$$f_1^+(H_{\bar{x},x}(0)) = \begin{cases} x_1^2 + x_2^2 - 2, & \text{if both the components of } H_{\bar{x},x} > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_2^+(H\bar{x}, x^{(0)}) = \begin{cases} -x_1^2 + 1, & \text{if both the components of } H_{\bar{x},x} > 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f^+(H_{\bar{x},x}(0)) = \begin{cases} (x_1^2 + x_2^2 - 2, -x_1^2 + 1)^T, & \text{if both the components } of \ H_{\bar{x},x} > 1 \\ (0,0)^T, & \text{otherwise} \end{cases}$$

Now,

$$f(x) - f(\bar{x}) = \begin{cases} (x_1^2 + x_2^2 - 2, -x_1^2 + 1), & \text{if } x_1 > 1, x_2 > 1 \\ (0,0)^T, & \text{otherwise} \end{cases}$$

then

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x}, x) = (0, -\{(x_1 - 1)^2 + (x_2 - 1)^2\}) \in K\,.$$

Thus, f is arcwise ρ-K-connected function (ρKCN).

Remark 2.8. If $\rho \in K$, then definition of arcwise ρKCN reduces to that of KCN function (Definition 2.4).

We now give an example of a function which is arcwise ρ-K-connected function at $\bar{x}$ but fails to be arcwise K-connected function at the same point when $\rho \notin K$.

Example 2.9. Let X be an AC set with respect to $H_{\bar{x},x}$ as defined in Example 2.7. Let us consider $\bar{x} = (1,1)^T$, $K = \{(x_1, x_2) : -x_1 \leq x_2, x_2 \geq 0\}$, $\rho = (-1,-1)^T \notin K$ and $d(\bar{x}, x) = \begin{cases} x_1^2 x_2^2 - 1, & \text{if } x_1 > 1, x_2 > 1 \\ (x_1 + \bar{x}_1)^2 + (x_2 + \bar{x}_2)^2, & \text{otherwise} \end{cases}$.

Let us define $f : X \to \mathbb{R}^2$ as

$$f(x_1, x_2) = \begin{cases} (-x_1^2 x_2^2, x_1^2 - x_2^2), & \text{if } x_1 > 1, x_2 > 1 \\ (-1, 0), & \text{otherwise} \end{cases}$$

So,

$$f^+(H_{\bar{x},x}(0)) = \begin{cases} (2 - x_1^2 - x_2^2, x_1^2 - x_2^2), & \text{if both the components of } H_{\bar{x},x} > 1 \\ (0,0), & \text{otherwise} \end{cases}$$

Then,

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x},x) = \begin{cases} (x_1^2 + x_2^2 - 2, x_1^2 x_2^2 - 1), & \text{if } x_1 > 1, x_2 > 1 \text{ and both} \\ & \text{components of } H_{\bar{x},x} > 1 \\ ((x_1+1)^2 + (x_2+1)^2, (x_1+1)^2 + (x_2+1)^2), & \text{otherwise} \end{cases}$$

$\in K$

Therefore, f is arcwise ρ-K-connected function at $\bar{x}$ on X. Since,

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) = \begin{cases} (-x_1^2 x_2^2 + x_1^2 + x_2^2 - 1, 0), & \text{if } x_1 > 1, x_2 > 1 \text{ and both} \\ & \text{the components of } H_{\bar{x},x} > 1 \\ (0,0), & \text{otherwise} \end{cases}$$

However, the function f fails to be arcwise K-connected function, because for $x = (2,2)^T \in X$, $f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) = (-9,0)^T \notin K$.

We now introduce the notions of arcwise ρ-K-pseudo connected and arcwise ρ-K-quasi connected functions.

Definition 2.10. f is said to be an arcwise ρ-K-pseudo connected function at $\bar{x} \in X$ on X with respect to $H_{\bar{x},x}$, if for every $x \in X$,

$$f(\bar{x}) - f(x) \in \operatorname{int} K \Rightarrow -(f^+(H_{\bar{x},x}(0)) + \rho d(\bar{x},x)) \in \operatorname{int} K\,.$$

Remark 2.11. Every arcwise ρ-K-connected function at $\bar{x} \in X$ is arcwise ρ-K-pseudo connected but the converse is not true as can be seen from the following example.

Example 2.12. Let us consider the set X, an arc $H_{\bar{x},x}$, K, f and $\bar{x}$ as defined in Example 2.9. Let $\rho = (1,-1)^T$ and $d(\bar{x},x) = \begin{cases} x_1^2 x_2^2 - 1, & \text{if } x_1 > 1, x_2 > 1 \\ (x_1 + \bar{x}_1)^2 + (x_2 + \bar{x}_2)^2 + 2, & \text{otherwise} \end{cases}$

Then

$$f(\bar{x}) - f(x) = \begin{cases} (x_1^2 x_2^2 - 1, -x_1^2 + x_2^2), & \text{if } x_1 > 1, x_2 > 1 \\ (0,0), & \text{otherwise} \end{cases}$$

If $f(\bar{x}) - f(x) \in \operatorname{int} K$ which implies that

$$x_1 > 1,\, x_2 > 1,\, 1 - x_1^2 x_2^2 < -x_1^2 + x_2^2 \;\text{ and }\; x_2^2 - x_1^2 > 0\,.$$

Then

$$-f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x},x) = \begin{cases} (x_1^2 + x_2^2 - x_1^2 x_2^2 - 1, x_2^2 - x_1^2 + x_1^2 x_2^2 - 1), & \text{if } x_1 > 1, x_2 > 1 \text{ and both} \\ & \text{the components of } H_{\bar{x},x} > 1 \\ (-(x_1+1)^2 - (x_2+1)^2 - 2, (x_1+1)^2 + (x_2+1)^2 + 2), & \text{otherwise} \end{cases}$$

$\in \operatorname{int} K$.

Therefore, the function is arcwise ρ-K-pseudoconnected at $\bar{x}$. However, the function fails to be arcwise ρ-K-connected function at $\bar{x}$ because for $x = (2,2)^T$,

$$f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x},x) = (-24,15)^T \notin K\,.$$

Definition 2.13. f is said to be an arcwise ρ-K-quasi connected function at $\bar{x} \in X$ on X with respect to $H_{\bar{x},x}$, if for every $x \in X$,

$$f(x) - f(\bar{x}) \notin \operatorname{int} K \Rightarrow -f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x},x) \in K\,.$$

We now give an example of a function which is arcwise ρ-K-quasi connected function at $\bar{x}$.

Example 2.14. Let X be an AC set with respect to $H_{\bar{x},x}$ as defined in Example 2.7. Let us consider $\bar{x} = (1,1)^T$, $K = \{(x_1, x_2) : x_1 \leq x_2, x_1 \leq 0\}$, $\rho = (1,0)^T \notin K$ and $d(\bar{x}, x) = (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2$.

Let us define $f : X \to \mathbb{R}^2$ as

$$f(x_1, x_2) = \begin{cases} (x_1^2 x_2^2, x_2^2), & \text{if } x_1 > 1, x_2 > 1 \\ (1,1), & \text{otherwise.} \end{cases}$$

So,

$$f^+(H_{\bar{x},x}(0)) = \begin{cases} (x_1^2 + x_2^2 - 2, x_2^2 - 1), & \text{if both the components} \\ & \text{of } H_{\bar{x},x} > 1 \\ (0,0), & \text{otherwise.} \end{cases}$$

Now, $f(x) - f(\bar{x}) \notin \operatorname{int} K$ which implies that both the components of x cannot be greater than 1 which further implies that both the components of $H_{\bar{x},x}$ cannot be greater than 1. So,

$$-f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x}, x) = (-(x_1 - 1)^2 - (x_2 - 1)^2, 0) \in K\,.$$

Therefore, f is arcwise ρ-K-quasi connected function at $\bar{x}$ on X.

3 Optimality Conditions

Consider the following vector optimization problem:

$$\begin{array}{ll} \text{(VP)} & \text{K-Minimize } f(x) \\ & \text{subject to } -g(x) \in Q \end{array}$$

where $f : X \to \mathbb{R}^k$, $g : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ is an AC set, with respect to an arc $H_{\bar{x},x} : [0,1] \to X$, where $\bar{x} \in X$, $x \in X$, K and Q are closed convex pointed cones with nonempty interiors in $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively. We denote the set of feasible solutions of (VP) by X_0, that is,

$$X_0 = \{x \in X : -g(x) \in Q\}\,.$$

We now recall optimality notion of weak minimum for the problem (VP) involving cones.

Definition 3.1. A feasible point $\bar{x}$ of the problem (VP) is called a weak minimum of (VP), if

$$f(\bar{x}) - f(x) \notin \operatorname{int} K, \text{ for all } x \in X_0\,.$$

We shall be obtaining the necessary optimality conditions for a feasible point to be a weak minimum of (VP) using the following generalized Slater's type cone constraint qualification.

Definition 3.2. The problem (VP) is said to satisfy the generalized Slater's type cone constraint qualification at $\bar{x} \in X_0$, if g is arcwise σ-Q-connected at $\bar{x}$ on X with respect to the same arc $H_{\bar{x},x}$ and there exists $\hat{x} \in X$ such that $-g(\hat{x}) + \sigma d(\bar{x}, \hat{x}) \in \operatorname{int} Q$.

We now establish the following necessary optimality conditions.

Theorem 3.3. Suppose that $\bar{x} \in X_0$ is a weak minimum of the problem (VP) and problem (VP) satisfies the generalized Slater's type cone constraint qualification at $\bar{x}$. Let $F(x) = (f^+(H_{\bar{x},x}(0)), g^+(H_{\bar{x},x}(0)))$, for all $x \in X$ and let $F(X)+(K \times Q)$ have nonempty interior. Also assume $f^+(H_{\bar{x},x}(0))$ is K-convexlike and $g^+(H_{\bar{x},x}(0))$ is Q-convexlike, then there exist $0 \neq \alpha^* \in K^+, \beta^* \in Q^+$ such that

$$(\alpha^{*T} f)^+(H_{\bar{x},x}(0)) + (\beta^{*T} g)^+(H_{\bar{x},x}(0)) \geq 0, \quad \text{for all } x \in X \tag{3.1}$$

$$(\beta^{*T} g)(\bar{x}) = 0 \tag{3.2}$$

Proof . First, we claim that there is no $x \in X$ such that

$$f^+(H_{\bar{x},x}(0)) \in -\operatorname{int} K, \quad g^+(H_{\bar{x},x}(0)) + g(\bar{x}) \in -Q\,. \tag{3.3}$$

Let if possible, there exist $\hat{x} \in S$ satisfying (3.3). Then there exists $\theta_0 > 0$ such that $0 < \theta < \theta_0$,

$$\begin{aligned} &-(f(H_{\bar{x},\hat{x}}(\theta)) - f(\bar{x})) \in \operatorname{int} K \\ &-(g(H_{\bar{x},\hat{x}}(\theta)) - g(\bar{x})) - g(\bar{x}) \in Q, \text{ that is } -g(H_{\bar{x},\hat{x}}(\theta)) \in Q \end{aligned} \tag{3.4}$$

On using (3.4), we get a contradiction to the fact that $\bar{x}$ is weak minimum of (VP). Hence, the system (3.3) has no solution. We are given that $f^+(H_{\bar{x},x}(0))$ is K-convexlike, $g^+(H_{\bar{x},x}(0))$ is Q-convexlike, therefore by Alternative Theorem given by Illes and Kassay in [4], there exist $\alpha^* \in K^+$, $\beta^* \in Q^+$, not all zero, such that for all $x \in X$

$$(\alpha^{*T} f)^+(H_{\bar{x},x}(0)) + (\beta^{*T} g)^+(H_{\bar{x},x}(0)) + (\beta^{*T} g)(\bar{x}) > 0\,. \tag{3.5}$$

Substituting $x = \bar{x}$ in the above equation, we get $(\beta^{*T} g)(\bar{x}) \geq 0$. Using the fact that $\beta^* \in Q^+$ and $-g(\bar{x}) \in Q$ we get $(\beta^{*T} g)(\bar{x}) \leq 0$. It follows that $(\beta^* g)(\bar{x}) = 0$.

Using the above equation in (3.5), we get

$$(\alpha^{*T} f)^+(H_{\bar{x},x}(0)) + (\beta^{*T} g)^+(H_{\bar{x},x}(0)) \geq 0, \text{ for all } x \in X\,. \tag{3.6}$$

Now we proceed to show that $\alpha^* \neq 0$. Let if possible, $\alpha^* = 0$. Then (3.6) reduces to

$$(\beta^{*T} g)^+(H_{\bar{x},x}(0)) \geq 0, \text{ for all } x \in X\,. \tag{3.7}$$

Since g is arcwise σ-Q-connected function at $\bar{x}$ on X with respect to $H_{\bar{x},x}$, it follows that

$$g(x) - g(\bar{x}) - g^+(H_{\bar{x},x}(0)) - \sigma d(\bar{x}, x) \in Q\,.$$

As $\beta^* \in Q^+$, we get

$$(\beta^{*T} g)(x) - (\beta^{*T} g)(\bar{x}) - (\beta^{*T} g)^+(H_{\bar{x},x}(0)) - (\beta^{*T} \sigma) d(\bar{x}, x) \geq 0\,. \tag{3.8}$$

On adding (3.7) and (3.8), we get

$$(\beta^{*T} g)(x) - (\beta^{*T} g)(\bar{x}) - (\beta^{*T} \sigma) d(\bar{x}, x) \geq 0, \quad \text{for all } x \in X\,.$$

Using $(\beta^{*T} g)(\bar{x}) = 0$ in the above equation, we conclude that

$$(\beta^{*T} g)(x) - (\beta^{*T} \sigma) d(\bar{x}, x) \geq 0, \text{ for all } x \in X\,. \tag{3.9}$$

Now by the generalized Slater's type cone constraint qualification at $\bar{x} \in X_0$ there exists $\hat{x} \in X$ such that

$$-g(\hat{x}) + \sigma d(\bar{x}, \hat{x}) \in \operatorname{int} Q$$

which gives that

$$(\beta^{*T} g)(\hat{x}) - (\beta^{*T} \sigma) d(\bar{x}, \hat{x}) < 0$$

which is a contradiction to (3.9). Hence $\alpha^* \neq 0$. □

In proving the sufficient optimality conditions, we will be using the following condition

$$\alpha^{*T} \rho + \beta^{*T} \sigma \geq 0\,. \tag{3.10}$$

We give a sufficient optimality theorem for a weak minimum of (VP).

Theorem 3.4. Let $\bar{x} \in X_0$, f be arcwise ρ-K-connected and g be arcwise σ-Q-connected at $\bar{x}$ on X_0 with respect to same arc $H_{\bar{x},x}$ for every $x \in X$. If there exist $0 \neq \alpha^* \in K^+$ and $\beta^* \in Q^+$ such that (3.1), (3.2) and (3.10) hold, then $\bar{x}$ is a weak minimum of (VP).

Proof . Let if possible $\bar{x}$ be not a weak minimum of (VP) then there exists $x \in X_0$ such that

$$\begin{aligned} & f(\bar{x}) - f(x) \in \operatorname{int} K \\ \Rightarrow \quad & \alpha^{*T}(f(\bar{x}) - f(x)) > 0 \end{aligned} \tag{3.11}$$

Since f is arcwise ρ-K connected at $\bar{x}$ on X_0, we get

$$\begin{aligned} & f(x) - f(\bar{x}) - f^+(H_{\bar{x},x}(0)) - \rho d(\bar{x}, x) \in K \\ \Rightarrow \quad & \alpha^{*T}(f(x) - f(\bar{x})) - (\alpha^{*T} f)^+(H_{\bar{x},x}(0)) - (\alpha^{*T}\rho) d(\bar{x}, x) \geq 0 \end{aligned} \tag{3.12}$$

Similarly,

$$\beta^{*T}(g(x) - g(\bar{x})) - (\beta^{*T} g)^+(H_{\bar{x},x}(0)) - (\beta^{*T}\sigma) d(\bar{x}, x) \geq 0\,. \tag{3.13}$$

Adding (3.12) and (3.13), and using (3.1) and (3.2), we get

$$\alpha^{*T}(f(x) - f(\bar{x})) - (\alpha^{*T}\rho + \beta^{*T}\sigma) d(\bar{x}, x) + \beta^{*T} g(x) \geq 0.$$

For all $x \in X_0$, $\beta^{*T} g(x) \leq 0$, as $\beta^* \in Q^+$ and $-g(x) \in Q$ and on using (3.10), we have

$$\alpha^{*T}(f(x) - f(\bar{x})) \geq 0$$

which is a contradiction to (3.11). Therefore, $\bar{x}$ is a weak minimum of (VP).□

We now give another sufficient optimality theorem for a weak minimum of (VP).

Theorem 3.5. Let $\bar{x} \in X_0$, f be arcwise ρ-K-pseudo connected and g be arcwise σ-Q-quasi connected at $\bar{x}$ on X_0 with respect to same arc $H_{\bar{x},x}$ for every $x \in X$. If there exist $0 \neq \alpha^* \in K^+$ and $\beta^* \in Q^+$ such that (3.1), (3.2) and (3.10) hold, then $\bar{x}$ is a weak minimum of (VP).

Proof . For all $x \in X_0$, $\beta^{*T} g(x) \leq 0$, as $\beta^* \in Q^+$ and $-g(x) \in Q$. On using (3.2), we get $\beta^{*T}(g(x) - g(\bar{x})) \leq 0$ for all $x \in X_0$. If $\beta^* \neq 0$, then we have,

$$g(x) - g(\bar{x}) \notin \operatorname{int} Q, \quad \text{for all } x \in X_0.$$

Since g is arcwise σ-Q-quasi connected at $\bar{x}$ on X_0,

$$-(g^+(H_{\bar{x},x}(0)) + \sigma d(\bar{x}, x)) \in Q, \quad \text{for all } x \in X_0$$

which implies that

$$-((\beta^{*T} g)^+(H_{\bar{x},x}(0)) + (\beta^{*T}\sigma) d(\bar{x}, x)) \geq 0, \quad \text{for all } x \in X_0\,. \tag{3.14}$$

If $\beta^* = 0$, then also (3.14) holds. Now on using (3.1) and (3.10) in (3.14)

$$((\alpha^{*T} f)^+(H_{\bar{x},x}(0)) + (\alpha^{*T}\rho) d(\bar{x}, x) \geq 0, \quad \text{for all } x \in X_0$$

which implies that

$$-(f^+(H_{\bar{x},x}(0)) + \rho d(\bar{x}, x)) \notin \operatorname{int} K, \ \text{for all } x \in X_0\,.$$

Since f is arcwise ρ-K-pseudo connected at $\bar{x}$ on X_0, we get

$$f(\bar{x}) - f(x) \notin \operatorname{int} K, \quad \text{for all } x \in X_0.$$

Therefore, $\bar{x}$ is a weak minimum of (VP). □

4 Wolfe Type Dual

We first consider the following Wolfe type dual associated with the vector optimization problem (VP):

$$\begin{aligned} \text{(WD)}\quad & \text{K-Maximize } f(u) + (\beta^T g)(u)k \\ & \text{subject to } (\alpha^T f)^+(H_{u,x}(0)) + (\beta^T g)^+(H_{u,x}(0)), \text{ for all } x \in X_0 \\ & 0 \neq \alpha \in K^+,\ \alpha^T k = 1,\ \beta \in Q^+,\ u \in X, \end{aligned} \tag{4.1}$$

where $k \in \operatorname{int} K$ is a fixed vector. Now we prove weak duality and strong duality results.

Theorem 4.1 (Weak Duality). Let x and (u, α, β) be feasible for (VP) and (WD), respectively. If f is arcwise ρ-K-connected and g is arcwise σ-Q-connected at $u \in X$ on X_0 with respect to same arc $H_{u,x}$ for every $x \in X$ and

$$\alpha^T \rho + \beta^T \sigma \geq 0 \tag{4.2}$$

then, $f(u) + (\beta^T g)(u)k - f(x) \notin \operatorname{int} K$.

Proof . Let if possible,

$$f(u) + (\beta^T g)(u)k - f(x) \in \operatorname{int} K. \tag{4.3}$$

Since f is arcwise ρ-K connected at $u \in S$ on X_0, therefore,

$$f(x) - f(u) - f^+(H_{u,x}(0)) - \rho d(u, x) \in K\,. \tag{4.4}$$

Adding (4.3) and (4.4), we get

$$-f^+(H_{u,x}(0)) + (\beta^T g)(u)k - \rho d(u, x) \in \operatorname{int} K$$

which implies on using $0 \neq \alpha \in K^+$,

$$-(\alpha^T f)^+(H_{u,x}(0)) + (\beta^T g)(u)(\alpha^T k) - (\alpha^T \rho)d(u, x) > 0\,.$$

Since (u, α, β) is feasible for (WD), (4.1) holds and $\alpha^T k = 1$. Adding (4.1) and using (4.2) in the above inequality, we have,

$$(\beta^T g)^+(H_{u,x}(0)) + (\beta^T g)(u) + (\beta^T \sigma)d(u, x) > 0\,. \tag{4.5}$$

Since g is arcwise σ-Q-connected at $u \in X$ on X_0 and using $\beta \in Q^+$, we get

$$(\beta^T g)(x) - (\beta^T g)(u) - (\beta^T g)^+(H_{u,x}(0)) - (\beta^T \sigma)d(u, x) \geq 0 \tag{4.6}$$

Adding (4.5) and (4.6), we get

$$(\beta^T g)(x) > 0. \tag{4.7}$$

Since x is feasible for (VP) and $\beta \in Q^+$, we have $(\beta^T g)(x) \leq 0$ which is a contradiction to (4.7) and hence the result. □

Theorem 4.2 (Strong Duality). Let $\bar{x} \in X_0$ be a weak minimum of the problem (VP) and suppose hypothesis of Theorem 3.3 are satisfied. Then there exists $0 \neq \alpha^* \in K^+$, $\beta^* \in Q^+$ such that $(\bar{x}, \alpha^*, \beta^*)$ is a feasible solution of (WD). Further if conditions of Weak Duality Theorem 4.1 hold for all feasible solutions of (VP) and (WD) then $(\bar{x}, \alpha^*, \beta^*)$ is a weak maximum of (WD).

Proof . Since $\bar{x}$ is a weak minimum of (VP), by Theorem 3.3, there exist $0 \neq \alpha^* \in K^+$, $\beta^* \in Q^+$ such that (3.1) and (3.2) hold, which gives that $(\bar{x}, \alpha^*, \beta^*)$ is a feasible solution of (WD) and the values of two objective functions are equal. Further if $(\bar{x}, \alpha^*, \beta^*)$ is not a weak maximum of (WD), then there exists a feasible solution (u, α, β) of (WD) such that

$$f(u) + (\beta^T g)(u) - f(\bar{x}) - (\beta^{*T} g)(\bar{x}) \in \operatorname{int} K\,.$$

On using (3.2), we get

$$f(u) + (\beta^T g)(u) - f(\bar{x}) \in \operatorname{int} K$$

which is a contradiction to the Weak Duality Theorem 4.1 for the feasible solution $\bar{x}$ of (VP) and (u, α, β) of (WD). Therefore, $(\bar{x}, \alpha^*, \beta^*)$ is a weak maximum of (WD).□

5 Mond-Weir Type Dual

Next, we associate Mond-Weir type dual to (VP) and prove duality results using arcwise ρ-K-pseudo connected, arcwise σ-Q-quasi connected functions

$$\begin{aligned}
\text{(MD)} \quad & \text{K-Maximize } f(u) \\
& \text{subject to } (\alpha^T f)^+(H_{u,x}(0)) + (\beta^T g)^+(H_{u,x}(0)) \geq 0, \quad \text{for all } x \in X_0, && (5.1)\\
& \beta^T g(u) \geq 0, && (5.2)\\
& 0 \neq \alpha \in K^+, \; \beta \in Q^+, \; u \in X.
\end{aligned}$$

Theorem 5.1 (Weak Duality). Let x be feasible for (VP) and (u, α, β) be feasible for (MD). Let f be arcwise ρ-K-pseudo connected and g be arcwise σ-Q-quasi connected at $u \in X$ on X_0 and (4.2) holds then, $f(u) - f(x) \notin \operatorname{int} K$.

Proof . For $x \in X_0$, $\beta^T g(x) \leq 0$, as $\beta \in Q^+$ and $-g(x) \in Q$. Since (u, α, β) is feasible for (MD) therefore (5.1) and (5.2) hold. Adding $\beta^T g(x) \leq 0$ and (5.2), we get

$$\beta^T(g(x) - g(u)) \leq 0\,.$$

If $\beta \neq 0$, then we have,

$$g(x) - g(u) \notin \operatorname{int} Q.$$

Since g is arcwise σ-Q-quasi connected at $u \in X$ on X_0, therefore

$$-(g^+(H_{u,x}(0)) + \sigma d(u, x)) \in Q\,.$$

Since $\beta \in Q^+$, we get

$$-(\beta^T g)^+(H_{u,x}(0)) + (\beta^T \sigma) d(u, x) \geq 0. \tag{5.3}$$

If $\beta = 0$, then also (5.3) holds. Using (5.1) and (5.2), we get

$$(\alpha^T f)^+(H_{u,x}(0)) + (\alpha^T \rho) d(u, x) \geq 0$$

which implies that

$$(f^+(H_{u,x}(0)) + \rho d(u, x)) \notin \operatorname{int} K\,.$$

Further, f is arcwise ρ-K-pseudo connected at $u \in X$ on X_0, therefore $f(u) - f(x) \notin \operatorname{int} K$. □

Theorem 5.2 (Strong Duality). Let $\bar{x} \in X_0$ be a weak minimum of the problem (VP) and suppose that hypotheses of Theorem 3.3 are satisfied. Then there exist $0 \neq \alpha^* \in K^+$, $\beta^* \in Q^+$ such that $(\bar{x}, \alpha^*, \beta^*)$ is feasible solution of (MD). Further if conditions of Weak Duality Theorem 5.1 are satisfied for each feasible solution x of (VP) and (u, α, β) of (MD), then $(\bar{x}, \alpha^*, \beta^*)$ is a weak maximum of (MD).

Proof . Since $\bar{x}$ is a weak minimum of (VP), therefore proceeding on the same lines as in the proof of Theorem 4.2, we get $(\bar{x}, \alpha^*, \beta^*)$ is feasible for (MD).

Now, let if possible, $(\bar{x}, \alpha^*, \beta^*)$ be not a weak maximum of (MD), then there exists a feasible solution (u, α, β) of (MD) such that $f(u) - f(x) \in \operatorname{int} K$ which is a contradiction to the Weak Duality Theorem 5.1 for the feasible solution $\bar{x}$ for (VP) and (u, α, β) of (MD). Therefore, $(\bar{x}, \alpha^*, \beta^*)$ is a weak maximum of (MD). □

References

[1] M. Avriel and I. Zhang, *Generalized arc-wise connected functions and characterizations of local-global minimum properties*, J. Optim. Theory Appl. **32** (1980), no. 4, 407–425.

[2] D. Bhatia and A. Mehra, *Optimality and duality involving arcwise connected and generalized arcwise connected functions*, J. Optim. Theory Appl. **100** (1999), no. 1, 181–194.

[3] M. Hayashi and H. Komiya, *Perfect duality for convexlike programs*, J. Optim. Theory Appl. **38** (1982), no. 2, 179–189.

[4] T. Illés and G. Kassay, *Theorems of the alternative and optimality conditions for convexlike and general convexlike pogramming*, J. Optim. Theory Appl. **101** (1999), no. 2, 243–257.

[5] R.N. Mukherjee, *Arcwise connected functions and applications in multiobjective optimization*, Optimization **39** (1997), 151–163.

[6] R.N. Mukherjee and S.R. Yadav, *A note on arcwise connected set and functions*, Bull. Aust. Math. Soc. **31** (1985), no. 3, 369–375.

[7] J. M. Ortega and W.C. Rheinboldt, *Iterative solutions of non linear equations in several variables*, Academic Press, New York, 1970.

[8] C. Singh, *Elementary properties of arcwise connected set and functions*, J. Optim. Theory Appl. **41** (1983), no. 2, 377–387.

[9] S.K. Suneja and M. Sharma, *Optimality and duality in vector optimization problem involving arcwise connected d-type-I functions over cone*, Opsearch **52** (2015), no. 4, 884–902.

[10] Q. Zhang, *Optimality conditions and duality for semi-infinite programming involving B-arcwise connected functions*, J. Glob. Optim. **45** (2009), no. 4, 615–629.