Int. J. Nonlinear Anal. Appl. In Press, 1–13
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.30076.4325

Perturbed absolute value variational inequalities

Tirth Ram*, Mohd Iqbal, Ravdeep Kour

Department of Mathematics, University of Jammu, Jammu-180006, India

(Communicated by Abasalt Bodaghi)

Abstract

In this paper, we examine the perturbed absolute value variational inequalities (PAVVI), a new class of variational inequalities. For the (PAVVI), some new merit functions are established. We develop the error bounds for (PAVVI) using these merit functions. The results presented here recapture a number of previously established findings in the relevant fields because (PAVVI) include variational inequalities, the absolute value complementarity problem, and systems of absolute value equations as special cases.

Keywords: merit functions, error bounds, projection operator, fixed point problem
2010 MSC: 26A33, 26A51, 26D07, 26D10, 26D15

1 Introduction

Variational inequalities (VI) theory was introduced and studied by Stampacchia [30], and now it is developed and widely applied in the areas of management, economics, finance, transportation, optimization, pure and applied sciences, see [1, 2, 3, 11, 14, 17, 20]. Since (VI) is an important tool to solve mathematical and scientific problems and a number of numerical methods including the projection method, Wiener-Hopf equations, auxiliary principle, and dynamical systems have been established for solving the (VI) and related optimization problems, see for example [1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 17, 20] and the references therein.

Over the past few decades, the concept of convexity has been crucial to the generalizations and extensions of inequality. Convexity and inequality theories are closely connected to one another. The integral inequalities have applications in information technology, statistics, stochastic processes, probability, integral operator theory, optimization theory, and numerical integration. Over the past several decades, a large number of mathematicians and researchers have concentrated their enormous efforts and contributions on the study of inequalities. The following articles on various forms of inequality are available for interested readers to read. The Hermite-Hadamard inequalities and their improvements for modified p-convex function utilizing a new identity with the help of power mean and Hölder inequalities were explored by the authors in [29]. Using the Jensen-Mercer inequality, the authors in [26] construct several enhanced generalizations of H-H-M type inequalities relevant to the Caputo-Fabrizio fractional integrals. They also establish several new bounds for differentiable convex mappings using a recently established identity and various well-known inequalities, including Holder's, Young's, Holder-Iscan, and Power-mean inequality. Some Hermite-Hadamard type inequalities for integrals emerging in conformable fractional calculus are provided in [6].

*Corresponding author
Email addresses: tir1ram2@yahoo.com (Tirth Ram), iqbalmohd486@gmail.com (Mohd Iqbal), ravdeepkour2011@gmail.com (Ravdeep Kour)

Received: March 2023 *Accepted:* June 2023

Absolute value variational inequalities (AVVI) include the (VI) as a special case. It is proved that if the underlying set is the whole space, then (AVVI) transform into the absolute system of equations which are introduced and studied by Mangasarian [15]. The (AVVI) is equivalent to the complementarity problem studied by Rohn [25] and further examined by Mangasarian and Meyer [16] by applying a different approach. It has been shown through the projection lemma that the (AVVI) and fixed point theorem are equivalent and by making use of the equivalence relation between (AVVI) and fixed point problem, several iterative methods are established for solving (AVVI) and the related optimization problems, see [3, 20].

A new outlook in the study of (VI) analyzes merit function through which the (VI) are reformulated into an optimization problem. Merit functions play an important role in developing convergent iterative methods and evaluating the rate of convergence for some iterative methods, see for example [9, 10, 12, 13, 15]. Various merit functions are investigated and recommended for variational inequalities, absolute value variational inequalities, and complementarity problems, see [4, 18, 19, 21, 23, 27, 28]. Error bounds also play an important role in (VI) as error bounds are the functions that estimates the closeness of the arbitrary point to the solution set is an approximate computation of iterates for solving (VI), see [19, 21, 22, 23]

Motivated and inspired by the aforementioned work, in this paper, we introduce a new class of absolute value variational inequalities with two perturbed operators known as perturbed absolute value variational inequalities (PAVVI). Next, we consider merit functions for (PAVVI) under suitable conditions and we also analyze the error bounds for the solution of (PAVVI).

The rest of the paper is organized as follows: In section 2, we present some definitions which will be used later. In section 3, we prove Lemmas and developed some new merit functions for (PAVVI). Further, we develop the error bounds for (PAVVI) using these merit functions.

2 Preliminaries

Let $\mathcal{H}$ be a Hilbert space, whose norm and inner product are denoted by $\|.\|$ and $\langle ., . \rangle$, respectively. Let K be a closed and convex set in $\mathcal{H}$. For given operators $\mathcal{T}, \mathcal{B} : \mathcal{H} \to \mathcal{H}$, consider the problem of finding $x \in K$ such that

$$\langle \mathcal{T}_\epsilon x + \mathcal{B}_\epsilon |x|, y - x \rangle \geq 0, \text{ for all } y \in \mathcal{H}, \tag{2.1}$$

where $\mathcal{T}_\epsilon = \mathcal{T} + \epsilon I$ and $\mathcal{B}_\epsilon = \mathcal{B} + \epsilon I$ are perturbed operators, I is the identity mapping and $|x|$ contains the absolute value of components of $x \in \mathcal{H}$. The inequality (2.1) is called perturbed absolute value variational inequality (PAVVI). This inequality (2.1) can be seen as a difference of two operators and contains previously known classes of (VI) as special cases.

In order to derive the main results of this paper, we recall some standard definitions and results.

Definition 2.1. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{T}x - \mathcal{T}y, x - y \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in \mathcal{H}.$$

Definition 2.2. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \beta \|x - y\|, \text{ for all } x, y \in \mathcal{H}.$$

If $\mathcal{T}$ is strongly monotone and Lipschitz continuous operator, then from Definitions 2.1 and 2.2, we have $\alpha \leq \beta$.

Definition 2.3. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be monotone, if

$$\langle \mathcal{T}x - \mathcal{T}y, x - y \rangle \geq 0, \text{ for all } x, y \in \mathcal{H}.$$

Definition 2.4. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be pseudomonotone, if

$$\langle \mathcal{T}x, y - x \rangle \geq 0,$$

$$\text{implies } \langle \mathcal{T}y, y - x \rangle \geq 0, \text{ for all } x, y \in \mathcal{H}.$$

Definition 2.5. [28] A function $\mathcal{M}:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ is called a merit(gap) function for the inequality (2.1), if and only if

(i) $\mathcal{M}(x)\geq 0,$ for all $x\in\mathcal{H}$, and

(ii) $\mathcal{M}(x)=0,$ if and only if, $x\in\mathcal{H}$ solves inequality (2.1).

We now consider the well known projection lemma due to [11]. The variational inequalities can be reformulated into a fixed point problem using this lemma.

Lemma 2.6. [11] Let K be a closed and convex set in $\mathcal{H}$. Then for a given $z\in\mathcal{H},\ x\in K$ satisfies

$$\langle x-z, y-x\rangle\geq 0, \quad \text{for all } y\in K,$$

if and only if

$$x=P_K z,$$

where P_K is the projection of $\mathcal{H}$ onto a closed and convex set K in $\mathcal{H}$.

It is remarkable that the projection operator P_K, is non-expansive operator, that is

$$\|P_K[x]-P_K[y]\|\leq\|x-y\|, \quad \text{for all } x,y\in\mathcal{H}.$$

3 Main Results

In this section, we propose some merit functions associated with (PAVVI) and get some error bounds for (PAVVI) using these merit functions. To obtain this, we show that the (VI) is equivalent to the fixed point problem.

Lemma 3.1. [3, 17] Let K be a convex set in $\mathcal{H}$. The function $x\in K$ is a solution of perturbed absolute value variational inequality (2.1), if and only if, $x\in K$ satisfies the relation

$$x=P_K[x-\rho\mathcal{T}_\epsilon x-\rho\mathcal{B}_\epsilon\,|x|], \tag{3.1}$$

where $\rho>0$ is a constant.

It follows from the above lemma that (PAVVI) (2.1) and the fixed point problem (3.1) are equivalent. This alternative equivalent formulation is very useful from the theoretical as well as from the numerical point of view and is obtained by using projection technique. The projection methods are due to Lions and Stampacchia [14] which provide several effective schemes to approximate the solution of (VI). The equivalence between (VI) and the fixed point problem (FPP) plays a significant role in establishing the various results from problem (2.1) and its related formulations.

Lemma 3.2. For all $x,y\in\mathcal{H}$, we have

$$\|x\|^2+\langle x,y\rangle\geq-\frac{1}{4}\|y\|^2.$$

Now we define the residue vector $\mathcal{R}(x)$ by the following relation

$$\mathcal{R}_\rho(x)\equiv\mathcal{R}(x)=x-P_K[x-\rho\mathcal{T}_\epsilon x-\rho\mathcal{B}_\epsilon\,|x|]. \tag{3.2}$$

From Lemma 2.6, it can also be concluded that $x\in K$ is a solution of the perturbed absolute value variational inequality (2.1) if and only if $x\in K$ is a zero of the equation

$$\mathcal{R}_\rho(x)\equiv\mathcal{R}(x)=0.$$

Next, we show that the residue vector $\mathcal{R}_\rho(x)$ is strongly monotone and Lipschitz continuous.

Lemma 3.3. Let the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ be Lipschitz continuous with constants $(\beta+\epsilon)_\mathcal{T}>0$ and $(\beta+\epsilon)_\mathcal{B}>0$, respectively and $\mathcal{T}_\epsilon$ be strongly monotone with constant $(\alpha+\epsilon)_\mathcal{T}>0$. Then the residue vector $\mathcal{R}_\rho(x)$ defined by (3.2) is strongly monotone on $\mathcal{H}$.

Proof . For all $x, y \in \mathcal{H}$, consider

$$\begin{aligned}
\langle \mathcal{R}_\rho(x) - \mathcal{R}_\rho(y), x-y\rangle &= \langle x - P_k\left[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon\left|x\right|\right] - y + P_k\left[y - \rho\mathcal{T}_\epsilon y - \rho\mathcal{B}_\epsilon\left|y\right|\right], x-y\rangle \\
&= \langle x - P_k\left[x - \rho\left(\mathcal{T}+\epsilon I\right)x - \rho\left(\mathcal{B}+\epsilon I\right)\left|x\right|\right] - y + P_k\left[y - \rho\left(\mathcal{T}+\epsilon I\right)y - \rho\left(\mathcal{B}+\epsilon I\right)\left|y\right|\right], x-y\rangle \\
&= \langle x - y - P_k\left[x - \rho\left(\mathcal{T}+\epsilon I\right)x - \rho\left(\mathcal{B}+\epsilon I\right)\left|x\right|\right] + P_k\left[y - \rho\left(\mathcal{T}+\epsilon I\right)y - \rho\left(\mathcal{B}+\epsilon I\right)\left|y\right|\right], x-y\rangle \\
&= \langle x-y, x-y\rangle - \langle P_k\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right] \\
&\quad -P_k\left[y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right], x-y\rangle \\
&\geq \left\|x-y\right\|^2 - \left\|P_k\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right]\right. \\
&\quad \left.-P_k\left[y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right]\right\|\left\|x-y\right\| \\
&\geq \left\|x-y\right\|^2 - \left\|\left(x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right)\right. \\
&\quad \left.-\left(y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right)\right\|\left\|x-y\right\| \\
&\geq \left\|x-y\right\|^2 - \left\|x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right| - y\right. \\
&\quad \left.+\rho\mathcal{T}y + \rho\epsilon Iy + \rho\mathcal{B}\left|y\right| + \rho\epsilon I\left|y\right|\right\|\left\|x-y\right\| \\
&= \left\|x-y\right\|^2 - \left\|(x-y) - \rho\left(\mathcal{T}x - \mathcal{T}y\right) - \rho\left(\mathcal{B}\left|x\right| - \mathcal{B}\left|y\right|\right)\right. \\
&\quad \left.-\rho\epsilon\left(Ix - Iy\right) - \rho\epsilon\left(I\left|x\right| - I\left|y\right|\right)\right\|\left\|x-y\right\| \\
&\geq \left\|x-y\right\|^2 - \left(\left\|x-y\right\| + \rho\left\|\mathcal{T}x - \mathcal{T}y\right\| + \rho\left\|\mathcal{B}\left|x\right| - \mathcal{B}\left|y\right|\right\|\right. \\
&\quad \left.+\rho\epsilon\left\|Ix - Iy\right\| + \rho\epsilon\left\|I\left|x\right| - \left|I\left|y\right|\right|\right\|\right)\left\|x-y\right\| \\
&\geq \left\|x-y\right\|^2 - \left(\left\|x-y\right\|^2 + \rho\left(\beta+\epsilon\right)_\mathcal{T}\left\|x-y\right\|^2 + \rho\left(\beta+\epsilon\right)_\mathcal{B}\left\|x-y\right\|^2 + 2\rho\epsilon\left\|x-y\right\|^2\right) \\
&\geq \left\|x-y\right\|^2 - \left\{\sqrt{(1+\rho(\beta+\epsilon)_\mathcal{T})^2} + \rho\left(\beta+\epsilon\right)_\mathcal{B} + 2\rho\epsilon\right\}\left\|x-y\right\|^2 \\
&= \left\|x-y\right\|^2 - \left(\sqrt{1 + 2\rho(\beta+\epsilon)_\mathcal{T} + \rho^2(\beta+\epsilon)^2_\mathcal{T}} + \rho\left(\beta+\epsilon\right)_\mathcal{B} + 2\rho\epsilon\right) \times \left\|x-y\right\|^2 \\
&= \left(1 - \sqrt{1 + 2\rho(\beta+\epsilon)_\mathcal{T} + \rho^2(\beta+\epsilon)^2_\mathcal{T}} + \rho\left(\beta+\epsilon\right)_\mathcal{B} + 2\rho\epsilon\right)\left\|x-y\right\|^2
\end{aligned}$$

which implies that

$$\langle \mathcal{R}_\rho(x) - \mathcal{R}_\rho(y), x-y\rangle \geq \mathcal{V}\left|\left\|x-y\right\|^2\right.$$

where $\mathcal{V} = \left(1 - \sqrt{1 + 2\rho(\beta+\epsilon)_\mathcal{T} + \rho^2(\beta+\epsilon)^2_\mathcal{T}} + \rho\left(\beta+\epsilon\right)_\mathcal{B} + 2\rho\epsilon\right) > 0$, which proves that the residue vector $\mathcal{R}_\rho(x)$ is strongly monotone with constant $\mathcal{V} > 0$. □

Lemma 3.4. Let the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ be Lipschitz continuous with constants $(\beta+\epsilon)_\mathcal{T} > 0$ and $(\beta+\epsilon)_\mathcal{B} > 0$, respectively and $\mathcal{T}_\epsilon$ be strongly monotone with constant $(\alpha+\epsilon)_\mathcal{T} > 0$. Then the residue vector $\mathcal{R}_\rho(x)$ defined by (3.2) is Lipschitz continuous on $\mathcal{H}$.

Proof . For all $x, y \in \mathcal{H}$, consider

$$\begin{aligned}
\left\|\mathcal{R}_\rho(x) - \mathcal{R}_\rho(y)\right\| &= \left\|x - P_k\left[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon\left|x\right|\right] - y + P_k\left[y - \rho\mathcal{T}_\epsilon y - \rho\mathcal{B}_\epsilon\left|y\right|\right]\right\| \\
&= \left\|x - P_k\left[x - \rho\left(\mathcal{T}+\epsilon I\right)x - \rho\left(\mathcal{B}+\epsilon I\right)\left|x\right|\right] - y + P_k\left[y - \rho\left(\mathcal{T}+\epsilon I\right)y - \rho\left(\mathcal{B}+\epsilon I\right)\left|y\right|\right]\right\| \\
&= \left\|x - P_k\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right] - y + P_k\left[y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right]\right\| \\
&= \left\|x - y - P_k\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right] + P_k\left[y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right]\right\| \\
&\leq \left\|x-y\right\| + \left\|P_k\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right|\right] - P_k\left[y - \rho\mathcal{T}y - \rho\epsilon Iy - \rho\mathcal{B}\left|y\right| - \rho\epsilon I\left|y\right|\right]\right\| \\
&\leq \left\|x-y\right\| + \left\|x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}\left|x\right| - \rho\epsilon I\left|x\right| - y + \rho\mathcal{T}y + \rho\epsilon Iy + \rho\mathcal{B}\left|y\right| + \rho\epsilon I\left|y\right|\right\| \\
&\leq \left\|x-y\right\| + \left\|(x-y) - \rho\left(\mathcal{T}x - \mathcal{T}y\right) - \rho\left(\mathcal{B}\left|x\right| - \mathcal{B}\left|y\right|\right) - \rho\epsilon\left(Ix - Iy\right) - \rho\epsilon\left(I\left|x\right| - \left|I\left|y\right|\right)\right)\right\| \\
&\leq \left\|x-y\right\| + \left(\left\|x-y\right\| + \rho\left\|\mathcal{T}x - \mathcal{T}y\right\| + \rho\left\|\mathcal{B}\left|x\right| - \mathcal{B}\left|y\right|\right\| + \rho\epsilon\left\|Ix - Iy\right\| + \rho\epsilon\left\|I\left|x\right| - \left|I\left|y\right|\right|\right\|\right) \\
&\leq \left\|x-y\right\| + \left(\left\|x-y\right\| + \rho(\beta+\epsilon)_\mathcal{T}\left\|x-y\right\| + \rho(\beta+\epsilon)_\mathcal{B}\left\|x-y\right\| + \rho\epsilon\left\|x-y\right\| + \rho\epsilon\left\|x-y\right\|\right) \\
&\leq \left\|x-y\right\| + \left(1 + \rho(\beta+\epsilon)_\mathcal{T} + \rho(\beta+\epsilon)_\mathcal{B} + 2\rho\epsilon\right)\left\|x-y\right\| \\
&= \left(1 + \sqrt{(1-\rho(\beta+\epsilon)_\mathcal{T})^2} + \rho(\beta+\epsilon)_\mathcal{B} + 2\rho\epsilon\right)\left\|x-y\right\| \\
&= \theta\left\|x-y\right\|,
\end{aligned}$$

where $\theta = 1 + \sqrt{(1+\rho(\beta+\epsilon)_{\mathcal{T}})^2} + \rho(\beta+\epsilon)_{\mathcal{B}} + 2\rho\epsilon$, which proves that the residue vector $\mathcal{R}_\rho(x)$ is Lipschitz continuous with constant $\theta > 0$. □

We now use the residue vector $\mathcal{R}_\rho(u)$ defined by (3.2) to derive the error bound for the solution of the problem (2.1).

Theorem 3.5. Let $x^* \in \mathcal{H}$ be a solution of the perturbed absolute variational inequality (2.1). If the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ are Lipschitz continuous with constants $(\beta+\epsilon)_{\mathcal{T}} > 0$ and $(\beta+\epsilon)_{\mathcal{B}} > 0$, and strongly monotone with constants $(\alpha+\epsilon)_{\mathcal{T}} > 0$ and $(\alpha+\epsilon)_{\mathcal{B}} > 0$, respectively, then

$$\frac{1}{s_1}\|\mathcal{R}_\rho(x)\| \leq \|x^* - x\| \leq s_2\|\mathcal{R}_\rho(x)\|, \quad \text{for all } x \in \mathcal{H}.$$

Proof . Let $x^* \in \mathcal{H}$ solves the perturbed absolute value variational inequality (2.1). Then, we have

$$\langle \rho\mathcal{T}_\epsilon x^* + \rho\mathcal{B}_\epsilon |x^*|, y - x^*\rangle \geq 0, \quad \text{for all } y \in \mathcal{H}$$
$$\text{or } \langle \rho(\mathcal{T}+\epsilon I)x^* + \rho(\mathcal{B}+\epsilon I)|x^*|, y - x^*\rangle \geq 0, \ \text{for all } y \in \mathcal{H}. \tag{3.3}$$

Take $y = P_K[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon |x|]$ in (3.3), we have

$$\langle \rho(\mathcal{T}+\epsilon I)x^* + \rho(\mathcal{B}+\epsilon I)|x^*|, P_K[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon |x|] - x^*\rangle \geq 0$$

$$\begin{aligned}&\text{or } \langle \rho(\mathcal{T}+\epsilon I)x^* + \rho(\mathcal{B}+\epsilon I)|x^*|, P_K[x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|] - x^*\rangle \geq 0.\\ &\text{or } \langle \rho\mathcal{T}x^* + \rho\epsilon I x^* + \rho\mathcal{B}|x^*| + \rho\epsilon I|x^*|, P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle \geq 0.\end{aligned} \tag{3.4}$$

Take $x = P_K[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon |x|]$, $z = x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon |x|$ and $y = x^*$ in Lemma 2.6, we have

$$\begin{aligned}&\langle P_K[x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|] - [x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|],\\ &x^* - P_K[x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|]\rangle \geq 0\end{aligned}$$

$$\begin{aligned}\text{or} \quad &\langle P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x + \rho\mathcal{T}x + \rho\epsilon I x\\ &+\rho\mathcal{B}|x| + \rho\epsilon I|x|, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle \geq 0\end{aligned}$$

which shows that

$$\begin{aligned}&\langle -\rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x| + x - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x\\ &-\rho\mathcal{B}|x| - \rho\epsilon I|x|], P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle \geq 0.\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\begin{aligned}&\langle \rho(\mathcal{T}x^* - \mathcal{T}x) + \rho(\mathcal{B}|x^*| - \mathcal{B}|x|) + \rho\epsilon(I|x^*| - I|x|) + \rho\epsilon(Ix^* - Ix) + x - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x|\\ &-\rho\epsilon I|x|], P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle \geq 0.\end{aligned}$$

From (3.2), we have

$$\begin{aligned}&\langle \mathcal{T}x^* - \mathcal{T}x, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle + \langle \mathcal{B}|x^*| - \mathcal{B}|x|, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle\\ &+ \epsilon\langle I|x^*| - I|x|, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle\\ &+ \epsilon\langle Ix^* - Ix, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle\\ \leq&\frac{1}{\rho}\langle \mathcal{R}(x), P_K[x - \rho\mathcal{T}x - \rho\epsilon I x - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle\end{aligned}$$

or

$$\begin{aligned}&\langle \mathcal{T}x^* - \mathcal{T}x, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle \\ &+ \langle \mathcal{B}|x^*| - \mathcal{B}|x|, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle + \epsilon\|x^* - x\|^2 + \epsilon\|x^* - x\|^2 \\ &\leq \frac{1}{\rho}\langle \mathcal{R}(x), P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle .\end{aligned} \tag{3.6}$$

By the strong monotonicity of the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ with constants $(\alpha+\epsilon)_\mathcal{T} > 0$ and $(\alpha+\epsilon)_\mathcal{B} > 0$, respectively, we have

$$\begin{aligned}(\alpha+\epsilon)_\mathcal{T}\|x^* - x\|^2 &\leq \langle \mathcal{T}_\epsilon x^* - \mathcal{T}_\epsilon x, x^* - x\rangle \\ &= \langle (\mathcal{T}+\epsilon I)x^* - (\mathcal{T}+\epsilon I)x, x^* - x\rangle \\ &\leq \langle \mathcal{T}x^* - \mathcal{T}x, x^* - x\rangle + \epsilon\langle Ix^* - Ix, x^* - x\rangle \\ &\leq \langle \mathcal{T}x^* - \mathcal{T}x, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle \\ &\quad + \langle \mathcal{T}x^* - \mathcal{T}x, P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x\rangle + \epsilon\|x^* - x\|^2 ,\end{aligned}$$

and

$$\begin{aligned}(\alpha+\epsilon)_\mathcal{B}\|x^* - x\|^2 &\leq \langle \mathcal{B}x^* - \mathcal{B}x, x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\rangle \\ &+ \langle \mathcal{B}x^* - \mathcal{B}x, P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x\rangle + \epsilon\|x^* - x\|^2 .\end{aligned}$$

From (3.2) and (3.6), we have

$$\begin{aligned}((\alpha+\epsilon)_\mathcal{T} + (\alpha+\epsilon)_\mathcal{B})\|x^* - x\|^2 &\leq \frac{1}{\rho}\langle \mathcal{R}(x), P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle \\ &\quad + \langle \mathcal{T}x^* - \mathcal{T}x, -\mathcal{R}(x)\rangle + \langle \mathcal{B}x^* - \mathcal{B}x, -\mathcal{R}(x)\rangle .\end{aligned}$$

Using the Lipschitz continuity of operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ with constants $(\beta+\epsilon)_\mathcal{T} > 0$ and $(\beta+\epsilon)_\mathcal{B} > 0$, respectively, we have

$$\begin{aligned}\rho((\alpha+\epsilon)_\mathcal{T} + (\alpha+\epsilon)_\mathcal{B})\|x^* - x\|^2 &\leq \langle \mathcal{R}(x), P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|] - x^*\rangle \\ &\quad + \rho\langle \mathcal{T}x^* - \mathcal{T}x, -\mathcal{R}(x)\rangle + \rho\langle \mathcal{B}x^* - \mathcal{B}x, -\mathcal{R}(x)\rangle \\ &\leq \langle \mathcal{R}(x), -\mathcal{R}(x)\rangle - \langle \mathcal{R}(x), x^* - x\rangle + \rho\langle \mathcal{T}x^* - \mathcal{T}x, -\mathcal{R}(x)\rangle + \rho\langle \mathcal{B}x^* - \mathcal{B}x, -\mathcal{R}(x)\rangle \\ &\leq -\|\mathcal{R}(x)\|^2 + \|x^* - x\|\|\mathcal{R}(x)\| + \rho(\beta+\epsilon)_\mathcal{T}\|x^* - x\|\|\mathcal{R}(x)\| \\ &\quad + \rho(\beta+\epsilon)_\mathcal{B}\|x^* - x\|\|\mathcal{R}(x)\| \\ &= -\|\mathcal{R}(x)\|^2 + (1 + \rho((\beta+\epsilon)_\mathcal{T} + (\beta+\epsilon)_\mathcal{B}))\|x^* - x\|\|\mathcal{R}(x)\| \\ &\leq (1 + \rho((\beta+\epsilon)_\mathcal{T} + (\beta+\epsilon)_\mathcal{B}))\|x^* - x\|\|\mathcal{R}(x)\| ,\end{aligned}$$

which implies that

$$\|x^* - x\| \leq \frac{(1 + \rho((\beta+\epsilon)_\mathcal{T} + (\beta+\epsilon)_\mathcal{B}))}{\rho((\alpha+\epsilon)_\mathcal{T} + (\alpha+\epsilon)_\mathcal{B}))}\|\mathcal{R}(x)\| = s_2\|\mathcal{R}(x)\| , \tag{3.7}$$

where $s_2 = \frac{(1+\rho((\beta+\epsilon)_\mathcal{T}+(\beta+\epsilon)_\mathcal{B}))}{\rho((\alpha+\epsilon)_\mathcal{T}+(\alpha+\epsilon)_\mathcal{B}))}$. Now using the relation (3.2), we have

$$\begin{aligned}\|\mathcal{R}(x)\| &= \|x - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\| \\ &\leq \|x^* - x\| + \|x^* - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\| \\ &= \|x^* - x\| + \|P_K[x^* - \rho\mathcal{T}x^* - \rho\epsilon Ix^* - \rho\mathcal{B}|x^*| - \rho\epsilon I|x^*|] - P_K[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|]\| \\ &\leq \|x^* - x\| + \|x^* - \rho\mathcal{T}x^* - \rho\epsilon Ix^* - \rho\mathcal{B}|x^*| - \rho\epsilon I|x^*| - x + \rho\mathcal{T}x + \rho\mathcal{B}|x| + \rho\epsilon Ix + \rho\epsilon I|x|\| \\ &\leq \|x^* - x\| + \|x^* - x\| + \rho\|\mathcal{T}x^* - \mathcal{T}x\| + \rho\|\mathcal{B}|x^*| - \mathcal{B}|x|\| + \rho\epsilon\|I|x^*| - I|x|\| + \rho\epsilon\|Ix^* - \mathcal{I}x\| \\ &\leq 2\|x^* - x\| + \rho(\beta+\epsilon)_\mathcal{T}\|x^* - x\| + \rho(\beta+\epsilon)_\mathcal{B}\|x^* - x\| + \rho\epsilon\|x^* - x\| + \rho\epsilon\|x^* - x\| \\ &= 2\|x^* - x\| + \rho(\beta+\epsilon)_\mathcal{T}\|x^* - x\| + \rho(\beta+\epsilon)_\mathcal{B}\|x^* - x\| + 2\rho\epsilon\|x^* - x\| \\ &= (2 + 2\rho\epsilon + \rho((\beta+\epsilon)_\mathcal{T} + (\beta+\epsilon)_\mathcal{B}))\|x^* - x\| \\ &= (2(1+\rho\epsilon) + \rho((\beta+\epsilon)_\mathcal{T} + (\beta+\epsilon)_\mathcal{B}))\|x^* - x\| \\ &= s_1\|x^* - x\| ,\end{aligned}$$

which shows that

$$\frac{1}{s_1}\left\|\mathcal{R}(x)\right\| \leq \left\|x^* - x\right\|, \tag{3.8}$$

where $s_1 = (2(1+\rho\epsilon) + \rho\left((\beta+\epsilon)_{\mathcal{T}} + (\beta+\epsilon)_{\mathcal{B}}\right))$. From (3.7) and (3.8), we have

$$\frac{1}{s_1}\left\|\mathcal{R}(x)\right\| \leq \left\|x^* - x\right\| \leq s_2\left\|\mathcal{R}(x)\right\|, \quad \text{for all } x \in \mathcal{H}. \tag{3.9}$$

Now substituting $x = 0$ in (3.9), we have

$$\frac{1}{s_1}\left\|\mathcal{R}(0)\right\| \leq \left\|x^*\right\| \leq s_2\left\|\mathcal{R}(0)\right\|, \quad \text{for all } x \in \mathcal{H}. \tag{3.10}$$

Combining (3.9) and (3.10), we get a relative error bound for any $x \in \mathcal{H}$. □

Theorem 3.6. Suppose all the conditions of Theorem 3.5 hold. If $0 \neq x \in \mathcal{H}$ is a solution of the perturbed absolute value variational inequality (2.1), then

$$t_1\frac{\|\mathcal{R}(x)\|}{\|\mathcal{R}(0)\|} \leq \frac{\|x - x^*\|}{\|x^*\|} \leq t_2\frac{\|\mathcal{R}(x)\|}{\|\mathcal{R}(0)\|}.$$

It is noted that the normal residue vector $\mathcal{R}(x)$, defined in (3.2), is non differentiable. To resolve the non differentiability which is a significant limitation of the regularized merit function, we examine another merit function associated with the perturbed absolute value variational inequality (2.1). This merit function can be regarded as a regularized merit function. For all $x \in \mathcal{H}$, consider the function

$$\mathcal{M}_\rho(x) = \langle \mathcal{T}_\epsilon x + \mathcal{B}_\epsilon|x|, x - P_K[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon|x|]\rangle - \frac{1}{2\rho}\left\|x - P_K[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon|x|]\right\|^2. \tag{3.11}$$

It is clear from the above equation that $\mathcal{M}_\rho(x) \geq 0$, for all $x \in \mathcal{H}$.

Now we prove that the function defined in (3.11) is a merit function, which is the primary goal of our following results.

Theorem 3.7. For all $x \in \mathcal{H}$, we have

$$\mathcal{M}_\rho(x) \geq \frac{1}{2\rho}\left\|R_\rho(x)\right\|^2.$$

In particular, we have $\mathcal{M}_\rho(x) = 0$, if and only if $x \in \mathcal{H}$ is a solution of the perturbed absolute value variational inequality (2.1).

Proof . By substituting $x = P_K\left[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon|x|\right]$, $z = x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon|x|$ and $y = x$ in Lemma 2.6, we have

$$\langle P_K\left[x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|\right] - \left[x - \rho(\mathcal{T}+\epsilon I)x - \rho(\mathcal{B}+\epsilon I)|x|\right], x - x\rangle \geq 0,$$

which implies that

$$\begin{aligned} & \langle P_K\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|\right] - x + \rho\mathcal{T}x + \rho\epsilon Ix + \rho\mathcal{B}|x| + \rho\epsilon I|x|, x - x\rangle \geq 0 \\ \text{or} \quad & \langle \rho\mathcal{T}x + \rho\epsilon Ix + \rho\mathcal{B}|x| + \rho\epsilon I|x| + P_K\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|\right] \\ & -x, x - P_K\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|\right]\rangle \geq 0. \end{aligned}$$

From (3.11) and Lemma 3.2, we have

$$\begin{aligned} 0 &\leq \langle \rho\mathcal{T}x + \rho\epsilon Ix + \rho\mathcal{B}|x| + \rho\epsilon I|x| - (x - P_K\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|\right]), x \\ &\quad -P_K\left[x - \rho\mathcal{T}x - \rho\epsilon Ix - \rho\mathcal{B}|x| - \rho\epsilon I|x|\right]\rangle \\ &= \langle \rho\mathcal{T}x + \rho\epsilon Ix + \rho\mathcal{B}|x| + \rho\epsilon I|x| - \mathcal{R}_\rho(x), \mathcal{R}_\rho(x)\rangle \\ &= \langle \mathcal{T}x + \epsilon Ix + \mathcal{B}|x| + \epsilon I|x|, \mathcal{R}_\rho(x)\rangle - \frac{1}{\rho}\langle \mathcal{R}_\rho(x), \mathcal{R}_\rho(x)\rangle \\ &= \mathcal{M}_\rho(x) + \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 \\ &= \mathcal{M}_\rho(x) - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 \end{aligned}$$

which shows that $\mathcal{M}_\rho(x) \geq \frac{1}{2\rho}\|\mathcal{R}_\rho(x)\|^2$. □

It is clear from the above inequality that $\mathcal{M}_\rho(x) \geq 0$, for all $x \in \mathcal{H}$. Also, if $\mathcal{M}_\rho(x) = 0$, then from the above inequality, we obtain $\mathcal{R}_\rho(x) = 0$. Hence, according to Lemma 3.1, it is clear that $x \in \mathcal{H}$ is the solution of the perturbed absolute value variational inequality (2.1). Therefore from (3.11), we obtain $\mathcal{M}_\rho(x) = 0$, which is the required result. It is observed from the Theorem 3.7 that $\mathcal{M}_\rho(x)$ defined by (3.11), is a merit function for the perturbed absolute value variational inequality (2.1). We also notice that the regularized merit function is differentiable, if the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ are differentiable.

Now we obtain the error bounds for the perturbed absolute value variational inequality if both the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$ are not Lipschitz continuous.

Theorem 3.8. Let $x^* \in \mathcal{H}$ be a solution of the perturbed absolute value variational inequality (2.1). Let the operators T_ϵ and B_ϵ be strongly monotone with the constants $(\alpha+\epsilon)_T > 0$ and $(\alpha+\epsilon)_B > 0$, respectively. Then

$$\|x - x^*\|^2 \leq \frac{4\rho}{4\rho\left((\alpha+\epsilon)_{\mathcal{T}} + (\alpha+\epsilon)_{\mathcal{B}}\right) - 3 + 8\rho\epsilon}\left[\mathcal{M}_\rho(x) + \frac{1}{\rho}\|\rho\mathcal{T}x^* + \rho\mathcal{B}|x^*| + \rho 2\epsilon x^*\|^2\right],$$

for all $x \in \mathcal{H}$.

Proof . Let $x^* \in \mathcal{H}$ be a solution of the perturbed absolute value variational inequality (2.1) and by taking $y = x$, we have

$$\langle \rho\mathcal{T}_\epsilon x^* + \rho\mathcal{B}_\epsilon|x^*|, x - x^*\rangle \geq 0,$$

which implies that

$$\langle \rho(\mathcal{T}+\epsilon I)x^* + \rho(\mathcal{B}+\epsilon I)|x^*|, x - x^*\rangle \geq 0.$$

Applying Lemma 3.2, we have

$$\langle \mathcal{T}x^* + \mathcal{B}|x^*| + 2\epsilon x^*, x - x^*\rangle \geq -\frac{1}{4\rho}\|x - x^*\|^2 - \frac{1}{\rho}\|Tx^* + \mathcal{B}|x^*| + 2\epsilon x^*\|^2.$$

From (3.11) and strong monotonicity of the operators $\mathcal{T}_\epsilon$ and $\mathcal{B}_\epsilon$, we have

$$\begin{aligned}
\mathcal{M}_\rho(x) &= \langle \mathcal{T}_\epsilon x + \mathcal{B}_\epsilon|x|, x - x^*\rangle - \frac{1}{2\rho}\|x - x^*\|^2 \\
&= \langle \mathcal{T}_\epsilon x - \mathcal{T}_\epsilon x^* + \mathcal{T}_\epsilon x^* + \mathcal{B}_\epsilon|x| - \mathcal{B}_\epsilon|x^*| + \mathcal{B}_\epsilon|x^*|, x - x^*\rangle - \frac{1}{2\rho}\|x - x^*\|^2 \\
&= \langle \mathcal{T}_\epsilon x - \mathcal{T}_\epsilon x^*, x - x^*\rangle + \langle \mathcal{B}_\epsilon|x| - \mathcal{B}_\epsilon|x^*|, x - x^*\rangle + \langle \mathcal{T}_\epsilon x^* + \mathcal{B}_\epsilon|x^*|, x - x^*\rangle - \frac{1}{2\rho}\|x - x^*\|^2 \\
&= \langle (\mathcal{T}+\epsilon I)x - (\mathcal{T}+\epsilon I)x^*, x - x^*\rangle + \langle (B+\epsilon I)|x| - (B+\epsilon I)|x^*|, x - x^*\rangle \\
&\quad + \langle (\mathcal{T}+\epsilon I)x^* + (\mathcal{B}+\epsilon I)|x^*|, x - x^*\rangle - \frac{1}{2\rho}\|x - x^*\|^2 \\
&= \langle \mathcal{T}x - \mathcal{T}x^*, x - x^*\rangle + \langle \mathcal{B}|x| - \mathcal{B}|x^*|, x - x^*\rangle + 2\epsilon\|x - x^*\|^2 \\
&\quad + \langle \mathcal{T}x^* + \mathcal{B}|x^*| + 2\epsilon x^*, x - x^*\rangle - \frac{1}{2\rho}\|x - x^*\|^2 \\
&\geq (\alpha+\epsilon)_{\mathcal{T}}\|x - x^*\|^2 + (\alpha+\epsilon)_{\mathcal{B}}\|x - x^*\|^2 + 2\epsilon\|x - x^*\|^2 \\
&\quad - \frac{1}{4\rho}\|x - x^*\|^2 - \frac{1}{\rho}\|\mathcal{T}x^* + \mathcal{B}|x^*| + 2\epsilon x^*\|^2 - \frac{1}{2\rho}\|x - x^*\|^2 \\
&= \left((\alpha+\epsilon)_{\mathcal{T}} + (\alpha+\epsilon)_{\mathcal{B}} - \frac{1}{2\rho} - \frac{1}{4\rho} + 2\epsilon\right)\|x - x^*\|^2 - \frac{1}{\rho}\|\mathcal{T}x^* + \mathcal{B}|x^*| + 2\epsilon x^*\|^2,
\end{aligned}$$

which shows that

$$\|x - x^*\|^2 \leq \frac{4\rho}{4\rho\left((\alpha+\epsilon)_{\mathcal{T}} + (\alpha+\epsilon)_{\mathcal{B}}\right) - 3 + 8\rho\epsilon}\left[\mathcal{M}_\rho(x) + \frac{1}{\rho}\|\rho\mathcal{T}x^* + \rho\mathcal{B}|x^*| + \rho 2\epsilon x^*\|^2\right].$$

□

Next, we define the D-merit function for the (PAVVI), which is the difference of regularized merit functions (3.11). We consider the following function

$$\begin{aligned}
\mathcal{D}_{\rho,\delta}(x) &= \mathcal{M}_{\rho}(x) - \mathcal{M}_{\delta}(x) \\
&= \langle \mathcal{T}_{\epsilon}x + \mathcal{B}_{\epsilon}|x|, x - P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]\rangle - \frac{1}{2\rho}\|x - P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]\|^2 \\
&\quad - \langle \mathcal{T}_{\epsilon}x + \mathcal{B}_{\epsilon}|x|, x - P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|]\rangle + \frac{1}{2\delta}\|x - P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|]\|^2 \\
&= \langle (\mathcal{T} + \epsilon I)x + (\mathcal{B} + \epsilon I)|x|, x - P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]\rangle - \frac{1}{2\rho}\|x - P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]\|^2 \\
&\quad - \langle (\mathcal{T} + \epsilon I)x + (\mathcal{B} + \epsilon I)|x|, x - P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|]\rangle + \frac{1}{2\delta}\|x - P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|]\|^2 \\
&= \langle \mathcal{T}x + \mathcal{B}|x| + 2\epsilon x, \mathcal{R}_{\rho}(x)\rangle - \frac{1}{2\rho}\|\mathcal{R}_{\rho}(x)\|^2 - \langle \mathcal{T}x + \mathcal{B}|x| + 2\epsilon x, \mathcal{R}_{\delta}(x)\rangle + \frac{1}{2\rho}\|\mathcal{R}_{\delta}(x)\|^2 \\
&= \langle \mathcal{T}x + \mathcal{B}|x| + 2\epsilon x, \mathcal{R}_{\rho}(x) - \mathcal{R}_{\delta}(x)\rangle - \frac{1}{2\rho}\|\mathcal{R}_{\rho}(x)\|^2 + \frac{1}{2\delta}\|\mathcal{R}_{\delta}(x)\|^2. \qquad (3.12)
\end{aligned}$$

It is clear from (3.12) that $\mathcal{D}_{\rho,\delta}(x)$ is finite everywhere. We will now prove that $\mathcal{D}_{\rho,\delta}(x)$ is in fact a merit function for the perturbed absolute value variational inequality which is the prime inspiration for the following result.

Theorem 3.9. For all $x \in \mathcal{H}$ and $\rho \geq \delta$, we have

$$(\rho - \delta)\|\mathcal{R}_{\rho}(x)\|^2 \geq 2\rho\delta\mathcal{D}_{\rho,\delta}(x) \geq (\rho - \delta)\|\mathcal{R}_{\delta}(x)\|^2.$$

Particularly, $\mathcal{D}_{\rho,\delta}(x) = 0$, if and only if $x \in \mathcal{H}$ is the solution of the perturbed absolute value variational inequality (2.1).

Proof . Take $x = P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]$, $y = P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|]$ and $z = x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|$ in Lemma 2.6, we have

$$\langle P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|] - x + \rho\mathcal{T}_{\epsilon}x + \rho\mathcal{B}_{\epsilon}|x|, P_K[x - \delta\mathcal{T}_{\epsilon}x - \delta\mathcal{B}_{\epsilon}|x|] - P_K[x - \rho\mathcal{T}_{\epsilon}x - \rho\mathcal{B}_{\epsilon}|x|]\rangle \geq 0$$

or

$$\begin{aligned}
&\langle P_K[x - \rho(\mathcal{T} + \epsilon I)x - \rho(\mathcal{B} + \epsilon I)|x|] - x + \rho(\mathcal{T} + \epsilon I)x + \rho(\mathcal{B} + \epsilon I)|x|, P_K[x - \delta(\mathcal{T} + \epsilon I)x - \delta(\mathcal{B} + \epsilon I)|x|] \\
&- P_K[x - \rho(\mathcal{T} + \epsilon I)x - \rho(\mathcal{B} + \epsilon I)|x|]\rangle \geq 0.
\end{aligned}$$

That is,

$$\begin{aligned}
&\langle P_K[x - \rho\mathcal{T}x - \rho\mathcal{B}|x| - 2\rho\epsilon x] - x + \rho\mathcal{T}x + \rho\mathcal{B}|x| + 2\rho\epsilon x, P_K[x - \delta\mathcal{T}x - \delta\mathcal{B}|x| - 2\delta\epsilon x] \\
&- P_K[x - \rho\mathcal{T}x - \rho\mathcal{B}|x| - 2\rho\epsilon x]\rangle \geq 0,
\end{aligned}$$

which shows that

$$\langle \mathcal{T}x + \mathcal{B}|x| + 2\epsilon x, \mathcal{R}_{\rho}(x) - \mathcal{R}_{\delta}(x)\rangle \geq \frac{1}{\rho}\langle \mathcal{R}_{\rho}(x), \mathcal{R}_{\rho}(x) - \mathcal{R}_{\delta}(x)\rangle. \qquad (3.13)$$

From (3.12) and (3.13), we have

$$
\begin{aligned}
\mathcal{D}_{\rho,\delta}(x) &\geq \frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{\rho}\left\langle \mathcal{R}_\rho(x), \mathcal{R}_\delta(x)\right\rangle \\
&= \frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{2\rho}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{\rho}\left\langle \mathcal{R}_\rho(x), \mathcal{R}_\delta(x)\right\rangle \\
&= \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{2\rho}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{\rho}\left\langle \mathcal{R}_\rho(x), \mathcal{R}_\delta(x)\right\rangle \\
&= \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{2\rho}\left\|\mathcal{R}_\delta(x) - \mathcal{R}_\rho(x)\right\|^2 \\
&\geq \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\delta(x)\right\|^2,
\end{aligned}
$$

which clearly shows that

$$2\rho\delta\mathcal{D}_{\rho,\delta}(x) \geq (\rho - \delta)\left\|\mathcal{R}_\delta(x)\right\|^2. \tag{3.14}$$

Similarly by substituting $x = P_K\left[x - \delta\mathcal{T}_\epsilon x - \delta\mathcal{B}_\epsilon\left|x\right|\right]$, $y = P_K\left[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon\left|x\right|\right]$ and $z = x - \delta\mathcal{T}_\epsilon x - \delta\mathcal{B}_\epsilon\left|x\right|$ in Lemma 2.6, we have

$$\left\langle P_K\left[x - \delta\mathcal{T}_\epsilon x - \delta\mathcal{B}_\epsilon\left|x\right|\right] - x + \delta\mathcal{T}_\epsilon x + \delta\mathcal{B}_\epsilon\left|x\right|, P_K\left[x - \rho\mathcal{T}_\epsilon x - \rho\mathcal{B}_\epsilon\left|x\right|\right] - P_K\left[x - \delta\mathcal{T}_\epsilon x - \delta\mathcal{B}_\epsilon\left|x\right|\right]\right\rangle \geq 0$$

or

$$
\begin{aligned}
&\left\langle P_K\left[x - \delta(\mathcal{T} + \epsilon I)x - \delta(\mathcal{B} + \epsilon I)\left|x\right|\right] - x + \delta(\mathcal{T} + \epsilon I)x + \delta(\mathcal{B} + \epsilon I)\left|x\right|, P_K\left[x - \rho(\mathcal{T} + \epsilon I)x - \rho(\mathcal{B} + \epsilon I)\left|x\right|\right]\right. \\
&\left. - P_K\left[x - \delta(\mathcal{T} + \epsilon I)x - \delta(\mathcal{B} + \epsilon I)\left|x\right|\right]\right\rangle \geq 0.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\left\langle P_K\left[x - \delta\mathcal{T}x - \delta\mathcal{B}\left|x\right| - 2\delta\epsilon x\right] - x + \delta\mathcal{T}x + \delta\mathcal{B}\left|x\right| + 2\delta\epsilon x, P_K\left[x - \rho\mathcal{T}x - \rho\mathcal{B}\left|x\right| - 2\rho\epsilon x\right]\right. \\
&\left. - P_K\left[x - \delta\mathcal{T}x - \delta\mathcal{B}\left|x\right| - 2\delta\epsilon x\right]\right\rangle \geq 0,
\end{aligned}
$$

which shows that

$$\left\langle \mathcal{T}x + \mathcal{B}\left|x\right| + 2\epsilon x, \mathcal{R}_\rho(x) - \mathcal{R}_\delta(x)\right\rangle \leq \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x) - \mathcal{R}_\delta(x)\right\rangle. \tag{3.15}$$

From (3.12) and (3.15), we have

$$
\begin{aligned}
\mathcal{D}_{\rho,\delta}(x) &\leq -\frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x) - \mathcal{R}_\delta(x)\right\rangle \\
&= \frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x)\right\rangle \\
&= -\frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x)\right\rangle \\
&= -\frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 - \frac{1}{2\rho}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{2\delta}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{2\delta}\left\|\mathcal{R}_\rho(x)\right\|^2 + \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x)\right\rangle \\
&= \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{2\delta}\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{2\delta}\left\|\mathcal{R}_\delta(x)\right\|^2 + \frac{1}{\delta}\left\langle \mathcal{R}_\delta(x), \mathcal{R}_\rho(x)\right\rangle \\
&= \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\rho(x)\right\|^2 - \frac{1}{2\delta}\left\|\mathcal{R}_\rho(x) - \mathcal{R}_\delta(x)\right\|^2 \\
&\leq \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\rho}\right)\left\|\mathcal{R}_\rho(x)\right\|^2,
\end{aligned}
$$

which proves the left most inequality of the required result.

That is, $(\rho - \delta)\left\|\mathcal{R}_\rho(x)\right\|^2 \geq 2\rho\delta\mathcal{D}_{\rho,\delta}(x)$. (3.16)

From (3.14) and (3.16), we have

$$(\rho-\delta)\left\|\mathcal{R}_{\rho}(x)\right\|^{2}\geq 2\rho\delta\mathcal{D}_{\rho,\delta}(x)\geq(\rho-\delta)\left\|\mathcal{R}_{\delta}(x)\right\|^{2}.$$

□

Theorem 3.10. Suppose $x^{*}\in\mathcal{H}$ be a solution of the perturbed absolute value variational inequality (2.1). If the operators $\mathcal{T}_{\epsilon}$ and $\mathcal{B}_{\epsilon}$ are strongly monotone with constants $(\alpha+\epsilon)_{\mathcal{T}}>0$ and $(\alpha+\epsilon)_{\mathcal{B}}>0$, respectively, then

$$\|x-x^{*}\|^{2}\leq\frac{4\rho\delta}{4\rho\delta\left((\alpha+\epsilon)_{\mathcal{T}}+(\alpha+\epsilon)_{\mathcal{B}}\right)-3\delta+2\rho+8\rho\delta\epsilon}[\mathcal{D}_{\rho,\delta(x)}+\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}].$$

Proof . Since $x^{*}\in\mathcal{H}$ be a solution of the perturbed absolute value variational inequality (2.1) and by substituting $y=x$ in (2.1), we have

$$\left\langle\rho\mathcal{T}_{\epsilon}x^{*}+\rho\mathcal{B}_{\epsilon}\left|x^{*}\right|,x-x^{*}\right\rangle\geq 0.$$

In view of Lemma 3.2, we have

$$\begin{aligned}
\left\langle\mathcal{T}_{\epsilon}x^{*}+\mathcal{B}_{\epsilon}\left|x^{*}\right|,x-x^{*}\right\rangle &\geq -\frac{1}{\rho}\left\|\mathcal{T}_{\epsilon}x^{*}+\mathcal{B}_{\epsilon}\left|x^{*}\right|\right\|^{2}-\frac{1}{4\rho}\left\|x-x^{*}\right\|^{2}\\
\text{or}\quad \left\langle(\mathcal{T}+\epsilon I)x^{*}+(\mathcal{B}+\epsilon I)\left|x^{*}\right|,x-x^{*}\right\rangle &\geq -\frac{1}{\rho}\left\|(\mathcal{T}+\epsilon I)x^{*}+(\mathcal{B}+\epsilon I)\left|x^{*}\right|\right\|^{2}-\frac{1}{4\rho}\left\|x-x^{*}\right\|^{2}\\
\text{or}\quad \left\langle\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*},x-x^{*}\right\rangle &\geq -\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}-\frac{1}{4\rho}\left\|x-x^{*}\right\|^{2}. \qquad (3.17)
\end{aligned}$$

From (3.12), using the strong monotonicity of the operators $\mathcal{T}_{\epsilon}$ and $\mathcal{B}_{\epsilon}$ with constants $(\alpha+\epsilon)_{\mathcal{T}}>0$ and $(\alpha+\epsilon)_{\mathcal{B}}>0$, respectively and (3.17), we have

$$\begin{aligned}
D_{\rho,\delta}(x)=&\left\langle\mathcal{T}_{\epsilon}x+\mathcal{B}_{\epsilon}\left|x\right|,\mathcal{R}_{\rho}(x)-\mathcal{R}_{\delta}(x)\right\rangle-\frac{1}{2\rho}\left\|\mathcal{R}_{\rho}(x)\right\|^{2}+\frac{1}{2\delta}\left\|\mathcal{R}_{\delta}(x)\right\|^{2}\\
=&\left\langle\mathcal{T}x+\mathcal{B}\left|x\right|+2\epsilon x,\mathcal{R}_{\rho}(x)-\mathcal{R}_{\delta}(x)\right\rangle-\frac{1}{2\rho}\left\|\mathcal{R}_{\rho}(x)\right\|^{2}+\frac{1}{2\delta}\left\|\mathcal{R}_{\delta}(x)\right\|^{2}\\
=&\left\langle\mathcal{T}x+\mathcal{B}\left|x\right|+2\epsilon x,x-x^{*}\right\rangle-\frac{1}{2\rho}\left\|x-x^{*}\right\|^{2}+\frac{1}{2\delta}\left\|x-x^{*}\right\|^{2}\\
=&\left\langle\mathcal{T}x+\mathcal{B}\left|x\right|-\mathcal{T}x^{*}+\mathcal{T}x^{*}-\mathcal{B}\left|x^{*}\right|+\mathcal{B}\left|x^{*}\right|+2\epsilon x-2\epsilon x^{*}+2\epsilon x^{*},x-x^{*}\right\rangle-\frac{1}{2\rho}\left\|x-x^{*}\right\|^{2}+\frac{1}{2\delta}\left\|x-x^{*}\right\|^{2}\\
=&\left\langle\mathcal{T}x-\mathcal{T}x^{*},x-x^{*}\right\rangle+\left\langle\mathcal{B}\left|x\right|-\mathcal{B}\left|x^{*}\right|,x-x^{*}\right\rangle+\left\langle\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*},x-x^{*}\right\rangle\\
&+\left\langle 2\epsilon x-2\epsilon x^{*},x-x^{*}\right\rangle-\frac{1}{2\rho}\left\|x-x^{*}\right\|^{2}+\frac{1}{2\delta}\left\|x-x^{*}\right\|^{2}\\
\geq&(\alpha+\epsilon)_{\mathcal{T}}\left\|x-x^{*}\right\|^{2}+(\alpha+\epsilon)_{\mathcal{B}}\left\|x-x^{*}\right\|^{2}-\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}\\
&-\frac{1}{4\rho}\left\|x-x^{*}\right\|^{2}+2\epsilon\left\|x-x^{*}\right\|^{2}-\frac{1}{2\rho}\left\|x-x^{*}\right\|^{2}+\frac{1}{2\delta}\left\|x-x^{*}\right\|^{2}\\
=&\left((\alpha+\epsilon)_{\mathcal{T}}+(\alpha+\epsilon)_{\mathcal{B}}-\frac{1}{4\rho}-\frac{1}{2\rho}+\frac{1}{2\delta}+2\epsilon\right)\left\|x-x^{*}\right\|^{2}-\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}\\
=&\left((\alpha+\epsilon)_{\mathcal{T}}+(\alpha+\epsilon)_{\mathcal{B}}-\left(\frac{3\delta-2\rho-8\rho\delta\epsilon}{4\rho\delta}\right)\right)\left\|x-x^{*}\right\|^{2}-\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}\\
=&\frac{4\rho\delta\left((\alpha+\epsilon)_{\mathcal{T}}+(\alpha+\epsilon)_{\mathcal{B}}\right)-3\delta+2\rho+8\rho\delta\epsilon}{4\rho\delta}\left\|x-x^{*}\right\|^{2}-\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}.
\end{aligned}$$

Therefore

$$\|x-x^{*}\|^{2}\leq\frac{4\rho\delta}{4\rho\delta\left((\alpha+\epsilon)_{\mathcal{T}}+(\alpha+\epsilon)_{\mathcal{B}}\right)-3\delta+2\rho+8\rho\delta\epsilon}[\mathcal{D}_{\rho,\delta(x)}+\frac{1}{\rho}\left\|\mathcal{T}x^{*}+\mathcal{B}\left|x^{*}\right|+2\epsilon x^{*}\right\|^{2}].$$

□

Conclusions

We introduced and studied numerous merit functions for a new type of variational inequalities, namely perturbed absolute value variational inequalities, in this study. These merit functions are used to calculate error bounds for the estimated solution of absolute value variational inequalities and the related optimization problems. The results presented in this paper may be regarded as a primary contribution in this fascinating domain. Interested researchers are encouraged to investigate the applications of perturbed absolute value variational inequalities in a wide range of pure and applied areas. The suggestions in this paper might be applied in future research.

References

[1] D. Aussel, R. Correa, and M. Marechal, *Gap functions for quasivariational inequalities and generalized Nash equilibrium problems,* J. Optim. Theory Appl. **151** (2011), 474–488.

[2] D. Aussel, R. Gupta, and A. Mehra, *Gap functions and error bounds for inverse quasi-variational inequality problems,* J. Math. Anal. Appl. **407** (2013), 270–280.

[3] S. Batool, M.A. Noor, and K.I. Noor, *Absolute value variational inequalities and dynamical systems,* Int. J. Anal. Appl. **18** (2020), 337–355.

[4] S. Batool, M.A. Noor, and K.I. Noor, *Merit functions for absolute value variational inequalities,* AIMS Math. **6** (2021), 12133–12147.

[5] M.I. Bloach and M.A. Noor, *Perturbed mixed variational-like inequalities,* AIMS Mathematics 5 (2020) 2153-2162.

[6] M. Bohner, A. Kashuri, P. Mohammed, and J.E.N. Valdes, *Hermite-Hadamard-type inequalities for conformable integrals,* Hacet. J. Math. Stat. **51** (2022), no. 3, 775–786.

[7] S.S. Chang, S.Salahuddin, M. Liu, X.R. Wang, and J.F. Tang, *Error bounds for generalized vector inverse quasi-variational inequality problems with point to set mappings,* AIMS Math. **6** (2020), 1800–1815.

[8] J. Dutta, *Gap functions and error bounds for variational and generalized variational inequalities,* Vietnam J. Math. **40** (2012), 231–253.

[9] R. Gupta and A. Mehra, *Gap functions and error bounds for quasi variational inequalities,* J. Glob. Optim. **53** (2012), 737–748.

[10] S.L. Hu and Z.H. Huang, *A note on absolute value equations,* Optim. Lett. **4** (2010), 417–424.

[11] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* SIAM Publishing Co., Philadelphia, USA, 2000.

[12] T. Kouichi, *On gap functions for quasi-variational inequalities,* Abstr. Appl. Anal. **2008** (2008), 531361.

[13] G.Y. Li and K.F. Ng, *Error bounds of generalized D-gap functions for nonsmooth and nonmonotone variational inequality problems,* SIAM J. Optim. 20 (2009) 667-690.

[14] J.L. Lions and G. Stampacchia, *Variational inequalities,* Commun. Pur. Appl. Math. **20** (1967), 493–519.

[15] O.L. Mangasarian, *The linear complementarity problem as a separable bilinear program,* J. Glob. Optim. **6** (1995), 153–161.

[16] O.L. Mangasarian and R.R. Meyer, *Absolute value equations,* Linear Algebra Appl. **419** (2006), 359-,367.

[17] M.A. Noor, *On variational inequalities,* Ph.D Thesis, Brunel University, London, UK, 1975.

[18] M.A. Noor, *Merit functions for general variational inequalities,* J. Math. Anal. Appl. **316** (2006), 736–752.

[19] M.A. Noor, *On merit functions for quasi variational inequalities,* J. Math. Inequal. **1** (2007), 259–268.

[20] M.A. Noor and K.I. Noor, *From representation theorems to variational inequalities,* Computational Mathematics and Variational Analysis, Springer 2020, pp. 261–277.

[21] M.A. Noor, K.I. Noor, and S. Batool, *On generalized absolute value equations,* UPB Sci. Bull. Series A **80** (2018), 63–70.

[22] B. Qu, C.Y. Wang, and Z.J. Hang, *Convergence and error bound of a method for solving variational inequality problems via the generalized D-gap function,* J. Optim. Theory App. **119** (2003), 535–552.

[23] O. Prokopyev, *On equivalent reformulation for absolute value equations,* Comput. Optim. Appl. **44** (2009), 363.

[24] T. Ram and R. Kour, *Mixed variational like inequalities involving perturbed operator,* J. Math. Comput. Sci. **12** (2022), 1–10.

[25] J. Rohn, *A theorem of the alternatives for the equation Ax+B—x—=b,* Linear Multilinear. A. **52** (2004), 421–426.

[26] S.K. Sahoo, Y.S. Hamed, P.O. Mohammed, B. Kodamasingh, and K. Nonlaopon, *New midpoint type Hermite-Hadamard-Mercer inequalities pertaining to Caputo-Fabrizio fractional operators,* Alexandria Engin. J. **65** (2023), 689–698.

[27] M.V. Solodov, *Merit functions and error bounds for generalized variational inequalities,* J. Math. Anal. Appl. **287** (2003), 405–414.

[28] M.V. Solodov and P. Tseng, *Some methods based on the D-gap function for solving monotone variational inequalities,* Comput. Optim. Appl. **17** (2000), 255–277.

[29] H.M. Srivastava, M. Tariq, P.O. Mohammed, H. Alrweili, E. Al-Sarairah, and M. De La Sen, *On modified integral inequalities for a generalized class of convexity and applications,* Axioms **12** (2023), no. 2, 162.

[30] G. Stampacchia, *Formes bilineaires coercivities sur less ensembles convexes*, C.R. Acad. Sci. Paris **258** (1964), 4413–4416.