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Finding properly efficient solutions of nonconvex multiobjective optimization problems with a minimum bound for trade-offs

Narges Hoseinpoor, Mehrdad Ghaznavi*

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

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Abstract

In the presented paper, we investigate efficient solutions to optimization problems with multiple criteria and bounded trade-offs. A nonlinear optimization problem to find the relationships between the upper bound for trade-offs and objective functions is presented. Due to this problem, we determine some properly efficient points that are closer to the ideal point. To this end, we apply the extended form of the generalized Tchebycheff norm. Note that all the presented results work for general problems and no convexity assumption is needed.

Keywords: Multiobjective optimization, Tchebycheff norm, Trade-off, Properly efficient solution
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1 Introduction

Multiobjective programming is a branch of mathematical programming concerned with decision problems distinguished by several competing objective functions that are to be optimized over feasible points. In recent years, the developed optimization problems are mostly multicriteria or multiobjective [13, 14, 18, 19, 20, 22, 23, 24, 25].

The Pareto concept of solutions (efficient solutions) is used in multiobjective optimization instead of optimality. An optimal solution of a multiobjective optimization problem does not allow a better objective value while preserving the same values on the others. Development of some objective function can only be gained at the expense of the deterioration of at least one another objective function. These trade-offs between objective functions can be measured.

In some multiobjective optimization problems, optimal solutions will have unbounded trade-offs. Efficient solutions with bounded trade-offs are called properly efficient solutions. Identification of such solutions is very important in the theoretical development and the practical applications of multiobjective optimization problems. These solutions are utilized in sketching interactive algorithms [17], approximating utility (value) functions [21], and creating effective stock portfolios [12], for instance. The notion of proper efficiency was presented firstly by Kuhn and Tucker [16]. This solution was more precisely introduced by Geoffrion [8] for multiobjective optimization problems. Thereafter, some expands of proper efficiency were presented by Borwein [3], Benson [2], and Henig [9].

Most methods introduced for solving multiobjective optimization problems find efficient solutions, and concern the elicitation of the decision-makers precedence [1, 10]. To support the decision-maker that the selected decisions are the right ones, it is significant to realize the relationships between the trade-offs and different efficient solutions

*Corresponding author
Email addresses: ha.majd32@yahoo.com (Narges Hoseinpoor), ghaznavi@shahroodut.ac.ir (Mehrdad Ghaznavi)

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of multiobjective problems. Indeed, a trade-off is concerned with the construction of the problem and estimates the varying in one objective function concerning the varying in another one, when changing from a feasible point to another one. Eskelinen and Miettinen [7] presented a trade-off analysis method. This approach can be integrated into any implementation of an interactive or a posteriori multiobjective optimization technique using an achievement scalarizing function an equivalent scalarization formulation for creating efficient solutions. Khaledian and Soleimani-Damaneh [15] investigated efficient solutions of multiobjective optimization problems with trade-offs. Hozzar et al. [11] proposed some optimization problems to investigate trade-offs between criteria. By applying these problems, they introduced several properly efficient solutions which are closer to the ideal solution and proposed these matter under convexity by weighted sum scalarization.

In this paper, efficient solutions in which the trade-offs among criteria are not unbounded by the extended form of the generalized Tchebycheff norm [4, 26], are studied without any convexity assumption. We introduce a new method for getting properly efficient solutions. We would like to investigate the decision-maker preferences. For this purpose, we utilize the developed form of the generalized Tchebycheff norm and Geoffrion's definition of proper efficiency. Some solutions which are close to the ideal point are important for practical applications because they imply the majority of decision-maker preferences. Thus, we consider proper efficiency so that one can determine an upper bound for bounded trade-offs. Also, one can obtain some solutions that the upper bound of those are the least value than others. Moreover, these solutions are the closest solution to the ideal point.

The outline of this article is categorized as follows: We briefly review some necessary preliminaries that are used throughout the paper, in Section 2. Then some notes about proper efficiency and trade-off are introduced in Section 3, and we propose an optimization problem for determining an efficient properly efficient solution with the upper bound. Finally, conclusions are given in Section 4.

2 Fundamental concepts and terminologies

First, we state some fundamental definitions to facilitate working with multiobjective optimization problems (MOP) which are applied in the outline of the article. Consider the n-dimensional Euclidean space $\mathbb{R}^n$. Let $x_i, y_i \in \mathbb{R}$. $x_i \geqslant y_i$ if and only if $x_i - y_i \geqslant 0$. If $x, y \in \mathbb{R}^n$, then we gain the following rules for all $i = 1, 2, \dots, n$

$$x \geqq y \text{ if and only if } x_i \geqslant y_i \text{ for all } i.$$

Essentially this means that the preorder $\geqq$ in $\mathbb{R}^n$ denotes the canonical order.

$$x \geq y \text{ if and only if } x \geqq y \text{ and } x \neq y,$$

in fact, $\geq$ demonstrates that any component of x is larger than or equal to any component of y and $x \neq y$.

$$x > y \text{ if and only if } x_i > y_i \text{ for all } i,$$

therefore $>$ represents the standard strict inequality, component by component. Henceforth, it will be assumed that $X \subseteq \mathbb{R}^m$ is a feasible set. Investigate a multiobjective optimization problem (MOP) in general, as follows:

$$MOP \quad \min_{x\in X} \quad f(x) = (f_1(x), f_2(x), \cdots, f_n(x)), \tag{2.1}$$

with non-empty feasible set $X \subseteq \mathbb{R}^m$ and objective functions $f_i : X \to \mathbb{R}$, for each $i = 1, 2, \dots, n$. Assume that all the criteria f_i for every $i = 1, 2, \dots, n$ are bounded over the set X. A review of the definitions of efficiency for MOP (2.1) is provided as follows.

Definition 2.1. [6] Suppose that $x^* \in X$ is a feasible point. Then, x^* is said an efficient (a Pareto optimal) solution of MOP (2.1) if there is no other $x \in X$ such that $f(x) \leq f(x^*)$, i.e. there is no $x \in X$ such that $f_i(x) \leqslant f_i(x^*)$ for each $i = 1, 2, \dots, n$ and $f_j(x) < f_j(x^*)$ for at least one index $j \in \{1, 2, \dots, n\}$.

Definition 2.2. [6, 8] A feasible point x^* is said a properly efficient (properly Pareto optimal) solution of MOP (2.1) if it is efficient and there is a real positive scalar M such that, for every $i \in \{1, 2, \dots, n\}$ and every $x \in X$ satisfying $f_i(x) < f_i(x^*)$, there is an index $j \in \{1, 2, \dots, n\}$ such that $f_j(x^*) < f_j(x)$ and $(f_i(x^*) - f_i(x))/(f_j(x) - f_j(x^*)) \leqslant M$.

Definition 2.3. The point $y^I = (y_1^I, y_2^I, \dots, y_n^I)$ indicated by

$$y_i^I = \min_{x \in X} f_i(x), \qquad i = 1, 2, \dots, n,$$

is called the ideal point of MOP (2.1).

In the presented problems it is necessary to have a utopia vector $u = (u_1, u_2, \dots, u_n)$, that its components are defined as

$$u_i = \min_{x \in X} f_i(x) - \gamma_i, \qquad i = 1, 2, \dots, n,$$

for a small scalar $\gamma_i > 0$. If $\min_{x \in X} f_i(x)$ for all $i = 1, 2, \dots, n$ exist and are finite, then the utopia vector u is well-defined.

Tchebycheff norms are utilized to optimize the distance between the current point and the ideal point. Rather than being an L_1 metric as applied in traditional linear programming, or an L_2 metric applied in least-squares recurrence, Tchebycheff norms minimize an L_∞ metric.

Definition 2.4. [4] The generalized Tchebycheff norm $\|.\|_\beta^\alpha$ is a real-valued function on $\mathbb{R}^n$ described by $\|y\|_\beta^\alpha = \max_{1 \leqslant i \leqslant n} \beta_i |(I_\alpha^{-1} y)_i|$, with $\alpha \in \mathbb{R}$, positive vector $\beta \in \mathbb{R}^n$ and $n \times n$ matrix I_α such that

$$(I_\alpha)_{ij} = \begin{cases} 1, & i = j, \\ \alpha, & i \neq j. \end{cases}$$

It is obvious that if we set $\alpha = 0$ and $\beta_i = 1$ for all $i \in \{1, 2, \dots, n\}$, then $\|y\|_\beta^\alpha$ reduces to $\|y\|_\infty$.

Lemma 2.5. [4] Let $\frac{-1}{2n} < \alpha \leqslant 0$. Then the matrix I_α is nonsingular and every element of I_α^{-1} is nonnegative. Specifically, if $\frac{-1}{2n} < \alpha < 0$ then every element of I_α^{-1} is positive.

Now, we recall an approach for solving multiobjective optimization problems which can generate properly Pareto points. This technique was introduced by Choo and Atkins in [4] and is as follows:

$$\begin{array}{ll} \min & \|f(x) - u\|_\beta^\alpha \\ s.t. & x \in X. \end{array} \tag{2.2}$$

Definition 2.6. [5] Consider the following problem:

$$\begin{array}{ll} \min & g(x, y) \\ \min & f(x) \\ s.t. & x \in X_1 = \{x \in \mathbb{R}^m : g_i(x, y) \leqslant 0,\ i = 1, 2, \dots, t\} \\ s.t. & (x, y) \in X_2 = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : h_j(x, y) \leqslant 0,\ j = 1, 2, \dots, s\}. \end{array} \tag{2.3}$$

The problem (2.3) is said to be a bi-level optimization problem if it is shown as follows:

$$\begin{array}{ll} \min & g(x, y) \\ s.t. & (x, y) \in X_2 \bigcap {X_f}^* \end{array}$$

where ${X_f}^*$ is the optimal solution set of following problem:

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in X_1 = \{x \in \mathbb{R}^m : g_i(x, y) \leqslant 0,\ i = 1, 2, \dots, t\}. \end{array}$$

Definition 2.7. [11] Suppose that x^* and $\bar{x}$ are two different efficient solutions of MOP (2.1) such that they satisfy $f_i(\bar{x}) < f_i(x^*)$ for index $i \in \{1, 2, \dots, n\}$ and $f_j(x^*) < f_j(\bar{x})$ for index $j \in \{1, 2, \dots, n\} \backslash \{i\}$. The following relation is called a trade-off among criteria f_i and f_j at x^* and $\bar{x}$

$$\frac{f_i(x^*) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x^*)}.$$

3 Establishing upper bound for trade-offs

Definition 2.2 is proposed based on finite trade-offs. Therefore, according to this definition, properly efficient solutions are those efficient solutions that have bounded trade-offs among the objective functions. In the following, the relationships between the scalarized problem (2.2) and properly efficient solutions of MOP (2.1) are investigated.

Choo and Atkins in [4] showed that an optimal solution of the problem (2.2) with $\beta > 0$ and $\frac{-1}{2n} < \alpha < 0$ is a properly efficient solution. At first, the relation between the concept of trade-off and the scalarized problem (2.2) is considered.

In the proof of Theorem 3.1 [4], Choo and Atkins showed that the parameter α is related to the uniform bound of the marginal rates of substitution between criteria ($M = -1/\alpha$). Suppose that $\tilde{x}$ is a properly efficient solution of MOP (2.1) and assume $\frac{-1}{2n} < \alpha < 0$, they showed that if for every $i \in \{1, 2, \ldots, n\}$ and every $x \in X$ satisfying $f_i(x) < f_i(\tilde{x})$, there is an index $j \in \{1, 2, \ldots, n\}$ such that $f_j(\tilde{x}) < f_j(x)$ then

$$\frac{f_i(\tilde{x}) - f_i(x)}{f_j(x) - f_j(\tilde{x})} \leqslant \frac{1}{-\alpha}.$$

The value $\frac{1}{-\alpha}$ can be a good value for M in the original definition of Geoffrion proper efficiency. Therefore, this value is an appropriate upper bound for trade-offs among objective functions. Now, assume that $\frac{-1}{2n} < \alpha \leqslant 0$ and A is an $n \times n$ matrix as follows

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We have $A^2 = nA$. According to Definition 2.4, we can write $I_\alpha = \alpha A + (1-\alpha)I$. Now, we obtain the inverse of I_α. Suppose that $I_\alpha^{-1} = dA + \frac{1}{1-\alpha}I$. We will find d. Note that we have

$$I_\alpha I_\alpha^{-1} = \Big(\alpha A + (1-\alpha)I\Big)\Big(dA + \frac{1}{1-\alpha}I\Big) = I.$$

This implies $d = \frac{-\alpha}{(1-\alpha)(n\alpha+1-\alpha)}$. Hence, we have

$$(I_\alpha)^{-1}_{ij} = \begin{cases} \dfrac{n\alpha + 1 - 2\alpha}{(1-\alpha)(n\alpha+1-\alpha)}, & i = j, \\[2ex] \dfrac{-\alpha}{(1-\alpha)(n\alpha+1-\alpha)}, & i \neq j. \end{cases} \tag{3.1}$$

Due to Lemma 2.5, $(I_\alpha)^{-1}_{ij} \geqslant 0$. Let

$$\begin{aligned} \|f(x) - u\|^\alpha_\beta &= \max_{1 \leqslant i \leqslant n} \beta_i (I_\alpha^{-1}(f(x) - u))_i = \beta_k (I_\alpha^{-1}(f(x) - u))_k = \beta_k \sum_{j=1}^{n} (I_\alpha^{-1})_{kj}(f_j(x) - u_j) \\ &= \beta_k \Big(\frac{n\alpha + 1 - 2\alpha}{(1-\alpha)(n\alpha+1-\alpha)}(f_k(x) - u_k) + \frac{-\alpha(n-1)}{(1-\alpha)(n\alpha+1-\alpha)} \sum_{j \neq k}(f_j(x) - u_j) \Big). \end{aligned}$$

We have

$$\frac{n\alpha + 1 - 2\alpha}{(1-\alpha)(n\alpha+1-\alpha)} = \frac{1 - 1/n}{1-\alpha} + \frac{1/n}{n\alpha+1-\alpha} \leqslant \frac{1}{1-\alpha} + \frac{1}{n\alpha+1-\alpha} \leqslant \frac{1}{1-\alpha} + \frac{1}{-\alpha} \leqslant \frac{2}{-\alpha}, \tag{3.2}$$

and

$$\frac{-\alpha n + \alpha}{(1-\alpha)(n\alpha+1-\alpha)} \leqslant \frac{-1 + 1/n}{1-\alpha} + \frac{1 - 1/n}{n\alpha+1-\alpha} \leqslant \frac{1 - 1/n}{n\alpha+1-\alpha} \leqslant \frac{1}{n\alpha+1-\alpha} \leqslant \frac{1}{-\alpha}. \tag{3.3}$$

Note that due to the above descriptions

$$\|f(x) - u\|^\alpha_\beta \leqslant \beta_k \Big(\frac{2}{-\alpha}(f_k(x) - u_k) + \frac{1}{-\alpha} \sum_{j \neq k}(f_j(x) - u_j) \Big),$$

it is clear that if β is constant and $-\alpha$ is increased, then $f(x)$ tends to u. Motivated by this discussion, it is clear that if we are going to obtain some properly efficient solutions that are the closest properly efficient solutions to the ideal point with the upper bound $1/(-\alpha)$, the value $-\alpha$ must be increased. Hence, we present the following bi-level optimization problem to generate the closest properly efficient solution to the ideal point:

$$\begin{array}{ll} \min & \gamma \\ \min & \|f(x)-u\|_\beta^\alpha \\ s.t. & x \in X, \\ & -\alpha \leqslant \gamma, \\ s.t. & \gamma \geqslant 0. \end{array} \tag{3.4}$$

Theorem 3.1. Assume that $(\tilde{x}, \tilde{\alpha}, \tilde{\gamma})$ is an optimal solution of the bi-level optimization problem (3.4). The feasible point $\tilde{x} \in X$ is a properly efficient solution of MOP (2.1) if and only if $-\tilde{\alpha} > 0$.

Proof . Suppose that $\tilde{x}$ is a properly efficient solution of MOP (2.1). By applying Theorem 3.1 in [4] and optimality of $\tilde{x}$ for the problem (3.4), we have

$$\begin{array}{ll} \tilde{x} = & \arg\min \|f(x)-u\|_\beta^{\tilde{\alpha}} \\ s.t. & x \in X, \end{array}$$

thus $0 < -\tilde{\alpha} < \gamma$. If $0 < -\tilde{\alpha}$, then due to Theorem 3.1 in [4], $\tilde{x} \in X$ is a properly efficient solution of MOP (2.1). The proof is complete. □

We consider $X_l = \{x \in X : -\alpha \leqslant \gamma\}$.

Theorem 3.2. If $(\tilde{x}, \tilde{\alpha}, \tilde{\gamma})$ is an optimal solution of the bi-level optimization problem (3.4) with $\tilde{\gamma} > 0$, then $\tilde{\gamma}$ is an upper bound for any trade-off of properly efficient solutions.

Proof . By contradiction, assume that there is an $\hat{x} \in X_l$ and an index $q \in \{1, 2, \dots, n\}$ such that $f_q(\tilde{x}) > f_q(\hat{x})$ and for all $j \neq q$ with $f_j(\tilde{x}) < f_j(\hat{x})$ we have

$$\frac{f_q(\tilde{x}) - f_q(\hat{x})}{f_j(\hat{x}) - f_j(\tilde{x})} > \tilde{\gamma}.$$

Let

$$\max_{1 \leqslant i \leqslant n} \beta_i (I_{\tilde{\alpha}}^{-1}(f(\hat{x}) - u))_i = \beta_l (I_{\tilde{\alpha}}^{-1}(f(\hat{x}) - u))_l = \beta_l \sum_{j=1}^{n} (I_{\tilde{\alpha}}^{-1})_{lj} (f_j(\hat{x}) - u_j) \tag{3.5}$$

and

$$\max_{1 \leqslant i \leqslant n} \beta_i (I_{\tilde{\alpha}}^{-1}(f(\tilde{x}) - u))_i = \beta_k (I_{\tilde{\alpha}}^{-1}(f(\tilde{x}) - u))_k = \beta_k \sum_{j=1}^{n} (I_{\tilde{\alpha}}^{-1})_{kj} (f_j(\tilde{x}) - u_j). \tag{3.6}$$

According to relations (3.2) and (3.3), we have

$$\frac{\sum_{j \neq q} (I_{\tilde{\alpha}}^{-1})_{lj}}{(I_{\tilde{\alpha}}^{-1})_{lq}} \leqslant \frac{1/(-\tilde{\alpha})}{2/(-\tilde{\alpha})} \leqslant \frac{1}{2} \leqslant \frac{1}{-\tilde{\alpha}}.$$

Due to Lemma 2.5, we have

$$\frac{f_q(\tilde{x}) - f_q(\hat{x})}{f_j(\hat{x}) - f_j(\tilde{x})} > \tilde{\gamma} = \frac{1}{-\tilde{\alpha}} \geqslant \frac{\sum_{j \neq q} (I_{\tilde{\alpha}}^{-1})_{lj}}{(I_{\tilde{\alpha}}^{-1})_{lq}} > 0. \tag{3.7}$$

Since $f_q(\tilde{x}) - f_q(\hat{x})$ is positive, the following relation is trivially true if $f_j(\hat{x}) < f_j(\tilde{x})$,

$$f_q(\tilde{x}) - f_q(\hat{x}) > \frac{1}{-\tilde{\alpha}} (f_j(\hat{x}) - f_j(\tilde{x})).$$

Hence, by applying these inequalities and multiplying any of them by its corresponding $(I_{\tilde{\alpha}}^{-1})_{lj}$ for all $j \neq q$ and summing them over $j \neq q$, we have

$$\sum_{j \neq q} (I_{\tilde{\alpha}}^{-1})_{lj} (f_q(\tilde{x}) - f_q(\hat{x})) > \frac{1}{-\tilde{\alpha}} \sum_{j \neq q} (I_{\tilde{\alpha}}^{-1})_{lj} (f_j(\hat{x}) - f_j(\tilde{x})).$$

From relation (3.7), it follows that $(I_{\tilde{\alpha}}^{-1})_{lq} \geqslant -\tilde{\alpha}\sum_{j\neq q}(I_{\tilde{\alpha}}^{-1})_{lj}$. Therefore,

$$(I_{\tilde{\alpha}}^{-1})_{lq}(f_q(\tilde{x}) - f_q(\hat{x})) \geqslant -\tilde{\alpha}\sum_{j\neq q}(I_{\tilde{\alpha}}^{-1})_{lj}(f_q(\tilde{x}) - f_q(\hat{x})) > \sum_{j\neq q}(I_{\tilde{\alpha}}^{-1})_{lj}(f_j(\hat{x}) - f_j(\tilde{x})).$$

Hence

$$\sum_{j=1}^{n}(I_{\tilde{\alpha}}^{-1})_{lj}(f_j(\hat{x}) - f_j(\tilde{x})) < 0. \tag{3.8}$$

Based on relations (3.5), (3.6) and (3.8), it follows that

$$\beta_l\sum_{j=1}^{n}(I_{\tilde{\alpha}}^{-1})_{lj}(f_j(\hat{x}) - u_j) < \beta_l\sum_{j=1}^{n}(I_{\tilde{\alpha}}^{-1})_{lj}(f_j(\tilde{x}) - u_j) \leqslant \beta_k\sum_{j=1}^{n}(I_{\tilde{\alpha}}^{-1})_{kj}(f_j(\tilde{x}) - u_j).$$

Therefore, $\|f(\hat{x}) - u\|_{\beta}^{\tilde{\alpha}} < \|f(\tilde{x}) - u\|_{\beta}^{\tilde{\alpha}}$, which is a contradiction to $\tilde{x}$ being an optimal solution of the bi-level optimization problem (3.4). □

Example 3.3. Establish the following biobjective optimization problem.

$$\begin{array}{ll} \min & (-3x_1 - 2x_2 + 3, -x_1 - 3x_2 + 1) \\ \text{s.t.} & (x_1 - 1)^3 + x_2 \leqslant 0, \\ & -x_1 \leqslant 0, \quad -x_2 \leqslant 0, \quad -x_2 \leqslant -4. \end{array}$$

The optimal solution of the bi-level optimization problem (3.4) with $\beta = 0.78$ is $\tilde{x}_1 = 0$, $\tilde{x}_2 = 1$, and $\tilde{\alpha} = -0.001$. Due to Theorem 3.1, $(\tilde{x}_1, \tilde{x}_2) = (0, 1)$ is a properly efficient solution. The point $(x_1, x_2) = (1, 0)$ is also another properly efficient solution, because the following inequality is satisfied:

$$\frac{f_2(x^*) - f_2(x)}{f_1(x) - f_1(x^*)} \leqslant \frac{x_1 + 3x_2 - 1}{-3x_1 - 2x_2 + 3} \leqslant \frac{x_1 + 3(1 - x_1)^3 - 1}{-3x_1 - 2(1 - x_1)^3 + 3} = \frac{3(1 - x_1)^2 - 1}{-2(1 - x_1)^2 + 3} \leqslant 2.$$

This follows from $(1 - x_1)^3 \geqslant x_2 \geqslant 0$. Therefore, the problem (3.4) with $\tilde{\alpha} = -0.001$ implies the closest properly efficient solution to the ideal point $(u_1, u_2) = (0.1193, -2)$.

4 Conclusions

In the present research, we established an approach for finding properly efficient solutions with bounded trade-offs, so that they imply the decision-maker preferences. Note that in multiobjective optimization, decision-makers commonly would like to apply some solutions which are close to the ideal point. We investigate this approach by the extended form of the generalized Tchebycheff norm. Unlike most scalarization techniques, this technique works for general problems, without additional conditions, like the convexity assumption.

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