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On stability results for a nonlinear generalized fractional hybrid pantograph equation involving deformable derivative

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Abstract

The pantograph equation is a special type of delay differential equation with applications in quantum mechanics and electrodynamics. A generalized hybrid pantograph equation of fractional order involving deformable derivative is considered in this work to carry out the stability analysis. The existence of solutions is established by employing the measure of noncompactness and Darbo's fixed point theorem while the contraction mapping principle is used for proving the uniqueness of the solution. The link between the right-hand term of the given equation and the order of the deformable derivative is established. The paper presents the results on Ulam-Hyers stability and the generalized Ulam-Hyers stability of the proposed equation. Numerical simulations are provided to demonstrate the performed theoretical analysis.

Keywords: Deformable derivative, Pantograph equations, Darbo fixed point, Initial value problem
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1 Introduction

The concept of fractional order differentiation and integration has been known since 1695. However, various methods and related theory were developed only during the 20^{th} century and thus an instantaneous growth of the subject is evident via the interest of the researchers towards the field. Recent advancements in fractional calculus with evolution of different types of derivative (conformable [29], deformable [2], M-conformable fractional derivative [31], so on) has received the attention of many scientists who are working in applied sciences and engineering and this because fractional differential equations provide better description for several real world applications. Indeed, fractional calculus has strengthened the modeling capability of researchers in fields like quantum mechanics, solid state physics, optical physics, chemical engineering, population dynamics, control systems, fractional multi-pantograph systems, diffusion models and astronomy [25, 30, 28].

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Pantograph, a mechanical device used to collect the current through a suspended wire in an electric train or tram, was mathematically modeled in [32, 24]. Currently, the half pantograph is more common for compactness and it provides a responsible design with single arm for trains moving at high speed. Several authors have studied the pantograph equations considering various aspects and different derivative operators. For completeness, we report some of them. In [19], the authors developed the Runge-Kutta methods for the following multi-pantograph equation

$$u'(t) = \xi u(t) + \sum_{i=1}^{m} \sigma_i(t) u(\lambda_i(t)) + f(t), t \geq 0. \tag{1.1}$$

However in [20], Liu and Li considered the nonlinear generalized multi-pantograph equation given by

$$\begin{aligned} u'(t) =& \Psi(t, u(t), u(\lambda t), \dots, u(\lambda_m(t))),\ 0 \leq t \leq T, \\ u(0) =& u_0. \end{aligned} \tag{1.2}$$

and discussed the properties of the solution. Recently in [6], Pantograph equation with fractional order of the form

$$\begin{aligned} D^{\alpha} u(t) =& \Psi(t, u(t), u(\lambda t)),\ 0 \leq t \leq T, \\ u(0) =& u_0. \end{aligned} \tag{1.3}$$

where D is the Riemann–Liouville fractional operator. The existence of solutions of (1.3) is obtained using fractional calculus and fixed point theorems. In [10], the existence results for generalized hybrid type pantograph equation of fractional order given by

$$\begin{aligned} D^{\alpha} \frac{u(t)}{\Theta(t, u(t), y(\mu t))} =& \Psi(t, u(t), u(\sigma t)),\ 0 < t < 1, \\ u(0) =& 0, \end{aligned} \tag{1.4}$$

where $\alpha, \mu, \sigma \in (0, 1)$. Further results regarding equation (1.4) were established by Karimov et al in [15]. The existence and uniqueness results for nonlinear neutral pantograph equations with generalized fractional derivative was the topic of the paper [32]. In [29], the authors explored the existence and uniqueness for a coupled Caputo conformable system of pantograph equation. The analysis of impulsive boundary value pantograph problems via Caputo proportional fractional derivative under Mittag-Leffler functions was discussed in [17]. The asymptotic stability results of discrete fractional pantograph equations with nonlocal initial conditions was carried out in [3].

Deformable derivative was developed in [2] to overcome the shortcoming of conformable derivative defined by R. Khalil in [29] which lacks to include zero and negative numbers. The definition of the deformable derivative uses limit approach as of classical differential equations while the range of the parameters varying over unit interval. The term "deformable" refers to the intrinsic property of continuously deforming function to derivative. Thus, the deformable derivative is linearly related to the usual derivative while it can be viewed as a derivative of fractional order. The properties of deformable derivatives were provided in [34] whereas the existence and uniqueness results of the deformable fractional equation were illustrated in [12, 23]. To the best of authors expectations, there is no papers in the literature concerning the pantograph equation within the deformable fractional derivative.

Inspired by the above mentioned works and motivated by the advantage of deformable derivative over other types of fractional derivatives, we carry out the stability analysis for the generalized hybrid type fractional pantograph equation of the form:

$$\begin{cases} \mathcal{D}^{\theta}_{0^+}\Big(\dfrac{u(t)}{\varphi\left(t, u(t), u(\psi(t))\right)}\Big) = h\left(t, u(t), u(\phi(t))\right),\quad 0 \leq t \leq 1 \\ u(0) = 0 \end{cases} \tag{1.5}$$

where $\mathcal{D}^{\theta}$ is the deformable fractional derivative, $\theta \in (0,\, 1)$ satisfying $\delta + \theta = 1$ for some $\delta > 0$. Let $L > 0$ be such that $L = \sup_{t \in [0,\, 1]} |\varphi\left(t, 0, 0\right)|$ and assume the following assumptions:

(A_1) $\varphi \in C\Big([0,\, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} - \{0\}\Big), h \in C\Big([0,\, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big)$ and $\phi, \psi \in C\Big([0,\, 1],\, [0,\, 1]\Big)$,

(A_2) $|\varphi\left(t, x_1, x_2\right) - \varphi\left(t, y_1, y_2\right)| \leq \max\Big(|x_1 - y_1|, |x_2 - y_2|\Big)$,

$|h\left(t, x_1, x_2\right) - h\left(t, y_1, y_2\right)| \leq \max\Big(|x_1 - y_1|, |x_2 - y_2|\Big),\quad t \in [0,\, 1], x_1, x_2, y_1, y_2 \in \mathbb{R}$,

(A_3) $\exists\, M > 0,$ such that $|h(t, x_1, x_2)| \leq M,\ \forall\, x_1, x_2 \in \mathbb{R}.$

The paper is structured in the following way. Section 2 provides basic mathematical requirements for the theoretical analysis. The results of existence and uniqueness of solutions are presented in section 3. Stability analysis are performed in the sense of Ulam and Hyers in section 4 followed with examples and simulations in section 5.

2 Prerequisites

In this section we will recall some important and basic definitions, preliminary facts and properties of deformable derivative which will be used in this paper. By $C([0,\ 1], \mathbb{R})$ we denote the Banach space of continuous functions from $[0,\ 1]$ into $\mathbb{R}$ endowed with the norm

$$\|u\| = \sup_{t\in[0,\ 1]} |u(t)|$$

Definition 2.1. [34, 12] Let $h : [a,\ b] \longrightarrow \mathbb{R},\ \theta, \delta$ positive numbers with $0 \leq \theta \leq 1$ and $\theta + \delta = 1$. The deformable derivative of h of order θ at $t \in (a,\ b)$ is defined by

$$\left(\mathcal{D}^{\theta} h\right)(t) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\delta)\, h\,(t+\varepsilon\theta) - h(t)}{\varepsilon} \tag{2.1}$$

If the limit exists, h is $\theta-$differentiable at t

Remark 2.2. If $\theta = 1$, then $\delta = 0$, we recover the usual derivative. This shows that the deformable derivative is more general than the usual derivative.

Definition 2.3. [34, 12] For $\theta \in (0,\ 1]$, the $\theta-$integral of the function $h \in L^1([a,\ b], \mathbb{R}_+)$ is defined by

$$\left(I_a^{\theta} h\right)(t) = \frac{1}{\theta} e^{-\frac{\delta}{\theta}t} \int_a^t e^{\frac{\delta}{\theta}s} h(s)\, ds, \quad t \in [a,\ b], \tag{2.2}$$

where $\theta + \delta = 1$. when $a = 0$ we write $\left(I^{\theta} h\right)$ instead of writing $\left(I_0^{\theta} h\right)$.

The following theorem is an important tool in our work, it gathers the most important properties of the operators $\mathcal{D}^{\theta}, I_a^{\theta}$ which will be used in the sequel.

Theorem 2.4. [34, 12, 22, 21] Let $\theta, \theta_1, \theta_2 \in (0,\ 1]$ be such that $\theta + \delta = 1$ and $\theta_i + \delta_i = 1$ for $i = 1, 2$. Then,

1. The operators $\mathcal{D}^{\theta}$ and I_a^{θ} are linear.
2. The operators $\mathcal{D}^{\theta}$ and I_a^{θ} are commutative.
3. $\mathcal{D}^{\theta}(\tau) = \delta\tau,$ for all constant $\tau \in \mathbb{R}$
4. $\mathcal{D}^{\theta}(hg) = \left(\mathcal{D}^{\theta} h\right) g + \theta h \mathcal{D} g.$
5. Let h be continuous on $[a,\ b]$. Then, $I_a^{\theta} h$ is $\theta-$differentiable in (a, b) and we have

$$\mathcal{D}^{\theta}\left(I_a^{\theta} h\right)(t) = h(t) \tag{2.3}$$

$$I_a^{\theta}\left(\mathcal{D}^{\theta} h\right)(t) = h(t) - e^{\frac{\delta}{\theta}(a-t)} h(a). \tag{2.4}$$

Lemma 2.5. [12] Let $\theta \in (0,\ 1]$. The differential equation

$$\left(\mathcal{D}^{\theta} h\right)(t) = 0$$

has solutions

$$h(t) = \tau e^{-\frac{\delta}{\theta}t},$$

where $\tau \in \mathbb{R}$ is a constant.

Theorem 2.6. [18] (Banach Contraction Mapping Principle) A Contraction Mapping on a complete metric space has exactly one fixed point.

Now, we recall Darbo fixed point theorem which will be very useful for our work.

Theorem 2.7. [7, 1, 15] Let Λ be a nonempty, closed and convex subset of a Banach space X and let $T:\ \Lambda \longrightarrow \Lambda$ be a continuous mapping. Assume that there exists a constant $\tau \in [0,\ 1)$ such that

$$\mu\left(TY\right) \leq \tau\mu\left(Y\right)$$

for any nonempty subset Y of Λ, where μ is a measure of noncompactness defined in X. Then T has a fixed point in Λ.

For more details on non-compactness measures and Darbo's fixed points theorems, the reader is invited to consult [7, 1, 15, 8, 5]. We denote by $\mathcal{M}_X$ the family of all nonempty bounded subset of X. The following definition constitutes an important tool for our purposes.

Definition 2.8. [15, 9] Let X be a Banach algebra. A measure of noncompactness μ is said to satisfy condition (m) if it satisfies the following condition:

$$\mu\left(YZ\right) \leq \|Y\|\mu(Z) + \|Z\|\mu(Y)$$

for any $Y, Z \in \mathcal{M}_X$, where $YZ = \left\{yz\ /\ y \in Y, z \in Z\right\}$.

Let us mention that the Banach space $\left(C\left[a,\ b\right], \|.\|\right)$ is a Banach algebra, where the multiplication is defined as the usual product of real functions and $\|u\| = \sup\limits_{t\in[a,\ b]} u(t),\ \ u \in C\left[a,\ b\right]$. The concept of measure of noncompactness in $C\left[a,\ b\right]$ is needed in the sequel. Let $\varepsilon > 0$, for a fixed set $Y \in \mathcal{M}_X$ and $y \in Y$, the modulus of continuity of y is defined as follow:

$$\omega\left(y, \varepsilon\right) = \sup\left\{\left|y(t) - y(s)\right|:\ t, s \in \left[a,\ b\right],\ \left|t - s\right| \leq \varepsilon\right\}.$$

As cited in [15], it was proven in [8] that

$$\omega_0(Y) = \lim_{\varepsilon\to 0}\omega\left(Y, \varepsilon\right)$$

is a measure of noncompactness in $C\left[a,\ b\right]$, with

$$\omega\left(Y, \varepsilon\right) = \sup\left\{\omega\left(y, \varepsilon\right):\ y \in Y\right\}.$$

3 Main result

Let $X = \left(C\left[0,\ 1\right], \mathbb{R}\right)$ be the Banach space of real functions defined and continuous on $[0,\ 1]$ equipped with the usual norm given by $\|u\| = \sup\limits_{t\in[0,1]} u(t),\ \ u \in X$.

Lemma 3.1. Let $h \in C\left(\left[0,\ 1\right], \mathbb{R}\right)$ and $g \in C\left(\left[0,\ 1\right], \mathbb{R}^*\right)$. The function $u \in C\left(\left[0, 1\right], \mathbb{R}\right)$ such that

$$u(t) = \frac{1}{\theta}g(t)e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h(s)\,ds$$

is a solution for the fractional initial value problem

$$\begin{cases}\mathcal{D}^\theta\left(\dfrac{u(t)}{g(t)}\right) = h(t), & t \in [0,\ 1]\\ u(0) = 0\end{cases}$$

where $\mathcal{D}^\theta$ is the deformable fractional derivative of order θ with $\theta + \delta = 1, 0 \leq \theta \leq 1,$ and $\delta \neq 0$.

Proof . Since $\dfrac{u}{g}$ is continuous on $[0, 1]$ and h is a continuous anti-θ-derivative of $\dfrac{u}{g}$ over $[0, 1]$, we have

$$I_0^\theta\left(\frac{u}{g}\right)(t) = I_0^\theta\left(h\right)(t)$$

by (2.4), we obtain

$$\frac{u(t)}{g(t)} - u(0)e^{-\frac{\delta}{\theta}} = \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h(s)ds,$$

using the initial condition $u(0)=0$, it follows that $u(t) = \frac{1}{\theta}g(t)e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h(s)\,ds$. $\square$

Theorem 3.2. Assume that $(A_1)-(A_3)$ hold. If

$$M < \delta \tag{3.1}$$

then the Problem (1.5) has at least one solution.

Proof . We will prove this theorem in frame of Theorem(2.7). In view of Lemma 3.1, we define on $C([0,\,1])$ the operator $\mathbb{F}$ by

$$\mathbb{F}u(t) = \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\varphi\left(t,u(t),u(\psi(t))\right)\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds. \tag{3.2}$$

For any $u\in C([0,\,1])$ and $t\in[0,\,1]$, it is clear that $\mathbb{F}u=(\Psi u)(\Upsilon u)$, where Ψ,Υ are the operators defined by

$$\Psi u(t) = \varphi\left(t,u(t),u(\psi(t))\right)$$

and

$$\Upsilon u(t) = \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds.$$

The proof will be done in four steps:

Step 1: $\mathbb{F}$ is well defined,i.e, $\mathbb{F}u\in C[0,\,1]$ for any $u\in C[0,\,1]$. Let us prove that $\Psi u,\Upsilon u$ are in $C[0,\,1]$. let (t_n) a sequence in $[0,\,1]$ which converge to t_0 in $[0,\,1]$ as $n\to+\infty$. It can be easily seen that that for any $u\in C[0,\,1]$, we have Ψu is continuous on $[0,\,1]$. Indeed

$$\Big|\Psi u(t_n)-\Psi u(t_0)\Big| \quad = \quad \Big|\varphi\left(t_n,u(t_n),u(\psi(t_n))\right)-\varphi\left(t_0,u(t_0),u(\psi(t_0))\right)\Big| \underset{n\to+\infty}{\longrightarrow} 0,$$

on the other hand, taking into account that $t_0\le t_n$ we obtain

$$\begin{aligned}
\Big|\Upsilon u(t_n)-\Upsilon u(t_0)\Big| =&\Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_n}\int_0^{t_n}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds-\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_0}\int_0^{t_0}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\Big|\\
\le&\Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_n}\int_0^{t_n}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds-\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_0}\int_0^{t_n}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\Big|\\
&+\Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_0}\int_0^{t_n}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds-\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_0}\int_0^{t_0}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\Big|\\
\le&\frac{1}{\theta}\left(e^{-\frac{\delta}{\theta}t_n}-e^{-\frac{\delta}{\theta}t_0}\right)\int_0^{t_n}e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)\Big|ds+\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_0}\int_{t_0}^{t_n}e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)\Big|ds\\
\le&\frac{1}{\theta}M\Big[\left(e^{-\frac{\delta}{\theta}t_n}-e^{-\frac{\delta}{\theta}t_0}\right)\int_0^{t_n}e^{\frac{\delta}{\theta}s}ds+e^{-\frac{\delta}{\theta}t_0}\int_{t_0}^{t_n}e^{\frac{\delta}{\theta}s}ds\Big]\\
\le&\frac{1}{\delta}M\Big[\left(e^{-\frac{\delta}{\theta}t_n}-e^{-\frac{\delta}{\theta}t_0}\right)\left(e^{\frac{\delta}{\theta}t_n}-1\right)+e^{-\frac{\delta}{\theta}t_0}\left(e^{\frac{\delta}{\theta}t_n}-e^{\frac{\delta}{\theta}t_0}\right)\Big]\\
\le&\frac{1}{\delta}M\Big[e^{-\frac{\delta}{\theta}t_0}\left(e^{-\frac{\delta}{\theta}(t_n-t_0)}-1\right)\left(e^{\frac{\delta}{\theta}t_n}-1\right)+\left(e^{\frac{\delta}{\theta}(t_n-t_0)}-1\right)\Big]\underset{n\to+\infty}{\longrightarrow}0.
\end{aligned}$$

Step 2: Selecting $r\ge\frac{ML}{\delta-M}$. We claim that $\mathbb{F}(B_r)\subset B_r,\ B_r=\{u\in C[0,\,1]\ /\ \|u\|\le r\}$ Indeed, take $u\in B_r$ then for any $t\in[0,\,1]$, we have

$$\Big|\mathbb{F}u(t)\Big|\le\Big|\varphi\left(t,u(t),u(\psi(t))\right)\Big|\Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\Big|$$

$$\begin{aligned}
\Big|\varphi\left(t,u(t),u(\psi(t))\right)\Big| \quad &\le \quad \Big|\varphi\left(t,u(t),u(\psi(t))\right)-\varphi\left(t,0,0\right)\Big|+\Big|\varphi\left(t,0,0\right)\Big|\\
&\le \quad \max\Big(|u(t)|,|u(\psi(t)|\Big)+L\\
&\le \quad \|u\|+L\\
&\le \quad r+L
\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\right| &\leq \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)\Big|\,ds\\ &\leq \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}M\int_0^t e^{\frac{\delta}{\theta}s}\,ds\\ &\leq \frac{1}{\delta}e^{-\frac{\delta}{\theta}t}M\left(e^{\frac{\delta}{\theta}t}-1\right)\\ &\leq \frac{1}{\delta}M\left(1-e^{-\frac{\delta}{\theta}t}\right)\end{aligned}$$

we deduce that

$$\begin{aligned}\Big|\mathbb{F}u(t)\Big| &\leq \frac{1}{\delta}\left(r+L\right)M\left(1-e^{-\frac{\delta}{\theta}t}\right)\\ \Big\|\mathbb{F}u\Big\| &\leq \frac{1}{\delta}\left(r+L\right)M\left(1-e^{-\frac{\delta}{\theta}}\right)\\ &\leq \frac{M}{\delta}\left(r+L\right)\\ &\leq r.\end{aligned}$$

This last result yields from the selection of r. That is $\mathbb{F}(B_r)\subset B_r$. Moreover, for any u in B_r we have the following estimations

$$\|\Psi u\|\leq r+L$$

and

$$\begin{aligned}\|\Upsilon u\| &\leq \frac{M}{\delta}\left(1-e^{-\frac{\delta}{\theta}}\right)\\ &\leq \frac{M}{\delta}\end{aligned}$$

Step 3: We prove that $\mathbb{F}$ is continuous on the Ball B_r.

Claim 1: Ψ is continuous. For this purpose let us consider $u,v\in B_r$ with $\|u-v\|\underset{u\to v}{\longrightarrow}0$.

$$\begin{aligned}\Big|\Psi u(t)-\Psi v(t)\Big| &= \Big|\varphi\left(t,u(t),u(\psi(t))\right)-\varphi\left(t,v(t),v(\psi(t))\right)\Big|\\ &\leq \max\Big(|u(t)-v(t)|,|u(\psi(t)-v(\psi(t)|\Big)\end{aligned}$$

hence, $\Big\|\Psi u-\Psi v\Big\|\leq\|u-v\|\underset{u\to v}{\longrightarrow}0$.

Claim 2: Υ is continuous. Indeed, let ρ be a real positive number and $u,v\in B_r$ such that $\|u-v\|\leq\rho$. For any $t\in[0,\,1]$ we have

$$\begin{aligned}\Big|\Upsilon u(t)-\Upsilon v(t)\Big| &= \Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\Big[\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds-\int_0^t e^{\frac{\delta}{\theta}s}h\left(s,v(s),v(\phi(s))\right)\,ds\Big]\Big|\\ &\leq \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)-h\left(s,v(s),v(\phi(s))\right)\Big|\,ds\\ &\leq \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\int_0^t e^{\frac{\delta}{\theta}s}\max\Big(|u(s)-v(s|,|u(\phi(s))-v(\phi(s))|\Big)\,ds\\ &\leq \frac{1}{\delta}e^{-\frac{\delta}{\theta}t}\|u-v\|\left(e^{\frac{\delta}{\theta}t}-1\right)\\ &\leq \frac{1}{\delta}\|u-v\|\left(1-e^{-\frac{\delta}{\theta}t}\right)\\ \Big\|\Upsilon u-\Upsilon v\Big\| &\leq \frac{1}{\delta}\|u-v\|\left(1-e^{-\frac{\delta}{\theta}}\right)\\ &\leq \frac{1}{\delta}\rho\left(1-e^{-\frac{\delta}{\theta}}\right)\underset{\rho\to 0}{\longrightarrow}0\end{aligned}$$

Since $\mathbb{F}=\Psi\Upsilon$, we deduce the continuity of $\mathbb{F}$.

Step 4: We show that F is a contraction with respect to some measure of noncompactness in $(C[0,\,1])$. Let us consider the operator $\mathbb{F}:\ B_r \longrightarrow B_r$ defined by $\mathbb{F}u=(\Psi u)(\Upsilon u)$. Let Y a nonempty set of B_r and $u\in Y$, we need to estimate $\omega_0(\Psi u)$ and $\omega_0(\Upsilon u)$. As ψ is uniformly continuous we have

$$\forall\,\varepsilon>0,\ \exists\,\eta>0 \text{ such that } |t_1-t_2|\le\eta \Longrightarrow |\psi(t_1)-\psi(t_2)|<\varepsilon.$$

According to this, let take $t_1,t_2\in[0,\ 1]$ with $|t_1-t_2|\le\eta<\varepsilon$.

$$\begin{aligned}
\Big|\Psi u(t_1)-\Psi u(t_2)\Big| &= \Big|\varphi\left(t_1,u(t_1),u(\psi(t_1))\right)-\varphi\left(t_2,u(t_2),u(\psi(t_2))\right)\Big|\\
&= \Big|\varphi\left(t_1,u(t_1),u(\psi(t_1))\right)+\varphi\left(t_1,u(t_2),u(\psi(t_2))\right)-\varphi\left(t_1,u(t_2),u(\psi(t_2))\right)-\varphi\left(t_2,u(t_2),u(\psi(t_2))\right)\Big|\\
&\le \Big|\varphi\left(t_1,u(t_1),u(\psi(t_1))\right)-\varphi\left(t_1,u(t_2),u(\psi(t_2))\right)\Big|+\Big|\varphi\left(t_1,u(t_2),u(\psi(t_2))\right)-\varphi\left(t_2,u(t_2),u(\psi(t_2))\right)\Big|\\
&\le \max\left(\Big|u(t_1)-u(t_2)\Big|,\Big|u(\psi(t_1))-u(\psi(t_2))\Big|\right)+\omega\left(\varphi,\varepsilon\right)\\
&\le \omega\left(Y,\varepsilon\right)+\omega\left(\varphi,\varepsilon\right)
\end{aligned}$$

where

$$\omega\left(\varphi,\varepsilon\right)=\sup\left\{\Big|\varphi\left(t_1,u(t_2),u(\psi(t_2))\right)-\varphi\left(t_2,u(t_2),u(\psi(t_2))\right)\Big|,\ |t_1-t_2|\le\varepsilon,\ t_1,t_2\in[0,\ 1]\right\},\ \omega\left(Y,\varepsilon\right)=\sup\left\{\omega\left(u,\varepsilon\right),\ \ u\in Y\right\}$$

and

$$\omega\left(u,\varepsilon\right)=\sup\left\{\Big|u(t_1)-u(t_2)\Big|,\ |t_1-t_2|\le\varepsilon,\ t_1,t_2\in[0,\ 1]\right\}.$$

In view of the uniform continuity of $\varphi\left(t,z_1,z_2\right)$ we have $\lim\limits_{\varepsilon\to 0}\omega\left(\varphi,\varepsilon\right)=0$. Therefore,

$$\omega_0\left(\Psi Y,\varepsilon\right)\le\omega_0\left(Y\right).$$

Now, we move on to estimating $\omega_0(\Upsilon u)$. For this purpose, let $0<t_1<t_2<1$ and $|t_1-t_2|\le\upsilon<\varepsilon$, for a given $\varepsilon>0$. For any $u\in Y$ we have

$$\begin{aligned}
\Big|\Upsilon u(t_1)-\Upsilon u(t_2)\Big| &= \Big|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_1}\int_0^{t_1}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds-\frac{1}{\theta}e^{-\frac{\delta}{\theta}t_2}\int_0^{t_2}e^{\frac{\delta}{\theta}s}h\left(s,u(s),u(\phi(s))\right)\,ds\Big|\\
&\le \frac{1}{\theta}e^{-\frac{\delta}{\theta}t_1}\int_{t_1}^{t_2}e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)\Big|\,ds+\frac{1}{\theta}\Big|e^{-\frac{\delta}{\theta}t_1}-e^{-\frac{\delta}{\theta}t_2}\Big|\Big|\int_0^{t_2}e^{\frac{\delta}{\theta}s}\Big|h\left(s,u(s),u(\phi(s))\right)\Big|\,ds\\
&\le \frac{M}{\delta}e^{-\frac{\delta}{\theta}t_1}\int_{t_1}^{t_2}e^{\frac{\delta}{\theta}s}\,ds+\frac{M}{\delta}\Big|e^{-\frac{\delta}{\theta}t_1}-e^{-\frac{\delta}{\theta}t_2}\Big|\Big|\int_0^{t_2}e^{\frac{\delta}{\theta}s}\,ds\\
&\le \frac{M}{\delta}\left(e^{-\frac{\delta}{\theta}(t_1-t_2)}-1\right)+\frac{M}{\delta}\left(e^{-\frac{\delta}{\theta}(t_1-t_2)}-1\right)-\frac{M}{\delta}\left(e^{-\frac{\delta}{\theta}t_1}-e^{-\frac{\delta}{\theta}t_2}\right)\\
&\le \frac{2M}{\delta}\left(e^{-\frac{\delta}{\theta}(t_1-t_2)}-1\right)\\
&\le \frac{2M}{\delta}\left(e^{\frac{\delta}{\theta}\varepsilon}-1\right)
\end{aligned}$$

Then

$$\omega\left(\Upsilon u,\varepsilon\right)\le\frac{2M}{\delta}\left(e^{\frac{\delta}{\theta}\varepsilon}-1\right)$$

and

$$\omega_0\left(\Upsilon u\right)=0.$$

Now, we can deduce the estimate of $\omega_0\left(\mathbb{F}Y\right)$. Using the fact that $\omega_0\left(YZ\right)\le\|Y\|\omega_0(Z)+\|Z\|\omega_0(Y)$ and the estimations of $\|\Psi Y\|,\|\Upsilon Y\|$ we have

$$\begin{aligned}
\omega_0\left(\mathbb{F}Y\right) &= \omega_0\left(\left(\Psi Y\right)\left(\Upsilon Y\right)\right)\\
&\le \|\Psi Y\|\omega_0\left(\Upsilon Y\right)+\|\Upsilon Y\|\omega_0\left(\Psi Y\right)\\
&\le \frac{M}{\delta}\omega_0\left(Y\right)\\
&\le \sigma\omega_0\left(Y\right)
\end{aligned}$$

where $0 < \sigma = \dfrac{M}{\delta} < 1$. That is F is a contraction with respect to the measure of noncompactness ω_0. From steps $(1) - (4)$ the operator $F:\ B_r \longrightarrow B_r$ is a continuous contraction with respect to the measure of noncompactness ω_0, then by Theorem 2.7 the operator F has at least a fixed point in B_r. Therefore, the Problem (1.5) has at least a solution. □

Theorem 3.3. Assume that the conditions $(A_1) - (A_3)$ hold and let r be such that $r < \delta - M - L$ If

$$\delta - M > max\left(L, \sqrt{L\delta}\right) \tag{3.3}$$

then problem (1.5) has a unique solution on B_r.

Proof . We prove that the operator $\mathbb{F}$ is a contraction. Let $u, u^* \in C([0,1],\mathbb{R})$ and $t \in [0,1]$, we have

$$\begin{aligned}
|\mathbb{F}u(t) - \mathbb{F}u^*(t)| =& \Bigg|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\varphi\left(t, u(t), u(\psi(t))\right)\int_0^t e^{\frac{\delta}{\theta}s}h\left(s, u(s), u(\phi(s))\right)\, ds \\
&- \frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\varphi\left(t, u^*(t), u^*(\phi(t))\right)\int_0^t e^{\frac{\delta}{\theta}s}h\left(s, u^*(s), u^*(\phi(s))\right)\, ds\Bigg| \\
\leq& \left|\frac{1}{\theta}e^{-\frac{\delta}{\theta}t}\varphi\left(t, u^*(t), u^*(\psi(t))\right)\right|\Bigg[\int_0^t e^{\frac{\delta}{\theta}s}\big|h\left(s, u(s), u(\phi(s))\right) \\
&- h\left(s, u^*(s), u^*(\phi(s))\right)\big|ds\Bigg] + \left|\frac{1}{\theta}e^{-\frac{\delta}{\theta}}\right|\big|\varphi\left(t, u(t), u(\psi(t))\right) \\
&- \varphi\left(t, u^*(t), u^*(\psi(t))\right)\big|\left|\int_0^t e^{\frac{\delta}{\theta}s}h\left(s, u^*(s), u^*(\phi(s))\right)\, ds\right| \\
\leq& \frac{1}{\delta}e^{-\frac{\delta}{\theta}t}\left[(r+L)\left(e^{\frac{\delta}{\theta}t} - 1\right) + M\left(e^{\frac{\delta}{\theta}t} - 1\right)\right]\|u - u^*\| \\
\leq& \frac{1}{\delta}\left(1 - e^{\frac{\delta}{\theta}t}\right)(r + L + M)\|u - u^*\| \\
\|\mathbb{F}u(t) - \mathbb{F}u^*(t)\| \leq& \frac{1}{\delta}\left(1 - e^{\frac{\delta}{\theta}}\right)(r + L + M)\|u - u^*\| \\
\leq& \frac{1}{\delta}(r + L + M)\|u - u^*\|.
\end{aligned}$$

Since $\dfrac{ML}{\delta - M} \leq r < \delta - M - L$ the mapping $\mathbb{F}$ is a contraction such that $\mathbb{F}(B_r) \subset B_r$. Hence, by Banach contraction principle the operator $\mathbb{F}$ has a unique fixed point. □

4 Ulam-Hyers Stability

Stability analysis is a popular topic among the mathematicians and is crucial in understanding the dynamics of the mathematical models constructed representing real world phenomena. Differential equations with quadratic perturbations of fractional order has been of recent interest and stability of the equation are carried out in [11, 3, 26, 13, 4]. This section is devoted for establishing the Ulam- Hyers stability results for the considered problem (1.5).

$$(Q_1) \quad \varphi \in C\Big([0,\,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+ - \{0\}\Big), h \in C\Big([0,\,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big) \text{ and } \phi, \psi \in C\Big([0,\,1],\,[0,\,1]\Big).$$

Definition 4.1. [27] [Ulam-Hyers Stability]If there exist a real number $\mathbb{P} > 0$ such that for $\varepsilon > 0$ and for every solution $u_1 \in C([0,1],\mathbb{R})$ of the inequality

$$\left|\mathcal{D}^{\theta}_{0^+}\Big(\frac{u_1(t)}{\varphi\left(t, u_1(t), u_1(\psi(t))\right)}\Big) - h\left(t, u_1(t), u_1(\phi(t))\right)\right| < \varepsilon,\ t \in [0,1], \tag{4.1}$$

there exists $u \in C([0,1],\mathbb{R})$ that solves (1.5) with

$$|u_1 - u| < \mathbb{P}\varepsilon. \tag{4.2}$$

Then the hybrid fractional pantograph equation via deformable derivative in (1.5) is Ulam-Hyers stable.

Definition 4.2. [27] [Generalized Ulam-Hyers Stability] If $\zeta \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\zeta(0) = 0$ exists, such that for every $u_1 \in C([0,1], \mathbb{R})$ that solves (4.1), there exists $u \in C([0,1], \mathbb{R})$ of (1.5) with

$$|u_1 - u| < \zeta(\varepsilon). \tag{4.3}$$

Then the initial value fractional hybrid pantograph equation involving deformable derivative (1.5) is generalized Ulam-Hyers stable.

Theorem 4.3. Assume that conditions $(Q_1), (A_2), (A_3)$ are satisfied. If

$$\Delta = \frac{1}{\delta}(r + L + M) < 1 \tag{4.4}$$

and (3.3) hold, then there exists $\mathbb{P} > 0$ and a unique $u \in C([0,1], \mathbb{R})$ such that

$$\|u_1 - u\| \leq \mathbb{P}\varepsilon. \tag{4.5}$$

That is (1.5) is Ulam-Hyers stable.

Proof . Let $u_1 \in\in C([0,1], \mathbb{R})$ such that

$$\left| \mathcal{D}^{\theta}_{0+}\left(\frac{u_1(t)}{\varphi\left(t, u_1(t), u_1(\psi(t))\right)}\right) - h\left(t, u_1(t), u_1(\phi(t))\right) \right| < \varepsilon, \quad \text{with} \quad u_1(0) = 0 \quad t \in [0,1]. \tag{4.6}$$

Integrating 4.6 and using $u_1(0) = 0$, we have

$$\left(\frac{u_1(t)}{\varphi\left(t, u_1(t), u_1(\psi(t))\right)}\right) - I^{\theta}_0\left(h\left(t, u_1(t), u_1(\phi(t))\right)\right) \leq I^{\theta}_0(\varepsilon).$$

Hence,

$$\left| u_1(t) - \varphi\left(t, u_1(t), u_1(\psi(t))\right) I^{\theta}_0\left(h\left(t, u_1(t), u_1(\phi(t))\right)\right) \right| \leq \left| I^{\theta}_0(\varepsilon) \right| \left| \varphi\left(t, u_1(t), u_1(\psi(t))\right) \right|.$$

That is

$$\begin{aligned}
\left| u_1(t) - \mathbb{F}(u_1)(t) \right| &\leq \frac{1}{\theta} e^{-\frac{\delta}{\theta}t} \int_0^t e^{\frac{\delta}{\theta}s} \varepsilon \left| \varphi\left(t, u_1(t), u_1(\psi(t))\right) \right| \\
&\leq \frac{1}{\delta}\left(1 - e^{-\frac{\delta}{\theta}t}\right)(r + L)\varepsilon \\
\left\| u_1 - \mathbb{F}(u_1) \right\| &\leq \frac{1}{\delta}\left(1 - e^{\frac{\delta}{\theta}}\right)(r + L)\varepsilon \\
&\leq \frac{1}{\delta}(r + L)\varepsilon.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left\| \mathbb{F}(u_1) - u \right\| &= \left\| \mathbb{F}(u_1) - \mathbb{F}(u) \right\| \\
&\leq \frac{1}{\delta}(r + L + M)\left\| u_1 - u \right\|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left\| u_1 - u \right\| &\leq \left\| u_1 - \mathbb{F}(u_1) \right\| + \left\| \mathbb{F}(u_1) - u \right\| \\
&\leq \left\| u_1 - \mathbb{F}(u_1) \right\| + \left\| \mathbb{F}(u_1) - \mathbb{F}(u) \right\| \\
&\leq \frac{1}{\delta}(r + L)\varepsilon + \frac{1}{\delta}(r + L + M)\left\| u_1 - u \right\|,
\end{aligned}$$

then,
$$\left(1-\frac{1}{\delta}\left(r+L+M\right)\right)\left\|u_1-u\right\| \leq \frac{1}{\delta}\left(r+L\right)\varepsilon.$$

Since $r+L+M<\delta$ we have
$$\left\|u_1-u\right\| \leq \frac{(r+L)}{\delta-(r+L+M)}\varepsilon,$$
setting $\mathbb{P}=\dfrac{(r+L)}{\delta-(r+L+M)}$ we have
$$\left\|u_1-u\right\| \leq \mathbb{P}\varepsilon.$$

Therefore, fractional hybrid pantograph equation via deformable derivative in (1.5) is Ulam-Hyers stable and the generalized Ulam-Hyers stability is obtained by replacing $\mathbb{P}\varepsilon=\zeta(\varepsilon)$ with $\zeta(0)=0$. □

5 Examples

Example 5.1. Let us consider the following fractional hybrid problem
$$\begin{cases} \mathcal{D}^{\frac{3}{4}}_{0^+}\left(\dfrac{B\left(1+t\right)u(t)}{A\Big[u(t)+u(e^{-t})+1\Big]}\right)=\dfrac{\alpha\cos\left(2\pi t\right)}{\beta\left(1+t\right)^2}\left[\dfrac{u^2(t)}{u^2(t)+1}+\dfrac{u^2(t^2)}{u^2(t^2)+1}\right],\ \ 0<t<1 \\ u(0)=0 \end{cases} \tag{5.1}$$
where $\alpha,\beta,A,B>0$ with $24\alpha-5\beta<0,\ 2A<B$. Let us point out that the Problem 5.1 is similar to the Problem 1.5 where
$$\varphi\left(t,u,v\right)=\frac{A}{B\left(1+t\right)}\left(u+v+1\right),\ \ \psi(t)=e^{-t}$$
$$h\left(t,u,v\right)=\frac{\alpha\cos\left(2\pi t\right)}{\beta\left(1+t\right)^2}\left(\frac{u^2}{u^2+1}+\frac{v^2}{v^2+1}+1\right),\ \phi(t)=t^2,\theta=\frac{3}{4},\ \delta=\frac{1}{4},\ M=\frac{3\alpha}{\beta},\ L=\frac{A}{B},\ r\geq\frac{24A\alpha}{B\left(5\beta-24\alpha\right)}.$$

We easily check that assumptions (A_2) and $(A3)$ hold. On the other hand $M<\delta$ since $24\alpha-5\beta<0$. Furthermore, assumption A_1 hold for φ and h. Indeed, for any $t\in[0,\,1]$ and $x,y,u,w\in\mathbb{R}$, we have
$$\begin{aligned}\Big|\varphi\left(t,x,y\right)-\varphi\left(t,u,v\right)\Big| &\leq \frac{A}{B(1+t)}\left[\Big|x-u\Big|+\Big|y-v\Big|\right]\\ &\leq \frac{2A}{B}\max\left(\Big|x-u\Big|,\Big|y-v\Big|\right)\\ &\leq \max\left(\Big|x-u\Big|,\Big|y-v\Big|\right).\end{aligned}$$

In the same way
$$\begin{aligned}\Big|h\left(t,x,y\right)-h\left(t,u,v\right)\Big| \leq &\frac{\alpha\cos\left(2\pi t\right)}{\beta\left(1+t\right)^2}\Bigg[\left(\frac{|x|}{\left(x^2+1\right)\left(u^2+1\right)}+\frac{|u|}{\left(x^2+1\right)\left(u^2+1\right)}\right)\Big|x-u\Big|\\ &+\left(\frac{|y|}{\left(y^2+1\right)\left(v^2+1\right)}+\frac{|v|}{\left(y^2+1\right)\left(y^2+1\right)}\right)\Big|y-v\Big|\Bigg]\end{aligned}$$

$$\begin{aligned}\Big|h\left(t,x,y\right)-h\left(t,u,v\right)\Big| &\leq \frac{4\alpha}{\beta}\max\left(\Big|x-u\Big|,\Big|y-v\Big|\right)\\ &\leq \max\left(\Big|x-u\Big|,\Big|y-v\Big|\right).\end{aligned}$$

All conditions of Theorem 3.2 are satisfied, then the fractional order problem 5.1 has at least one solution. Let the parameters A, B, α and β take the values $0.3, 5, 0.1$ and 7.5 respectively. From the condition (3.3), we get $\delta - M = 0.21$ which is clearly greater than $\max\{0.06, 0.1224\}$. Thus, the considered problem (5.1) has an unique solution. Using these values we now check the Ulam-Hyers stability of (5.1) from Theorem 4.3. Substituting the parameters value in Δ defined in (4.4), we yield $\Delta = 0.41641$ which is clearly less than 1. Further, we shall illustrate the impact of the parameters α and β on the stability criterion by varying $\alpha \in [0.05, 0.2]$and $\beta \in [4, 8]$ such that the condition $24\alpha - 5\beta < 0$ is satisfied. Table 1 presents the value of Δ for different choices of α and β and corresponding the 3-dimensional plot is displayed in Figure 1. From the tabulation and numerical simulation, we ensure the Ulam-Hyers stability of the considered problem (5.1) for the assumed parameter values.

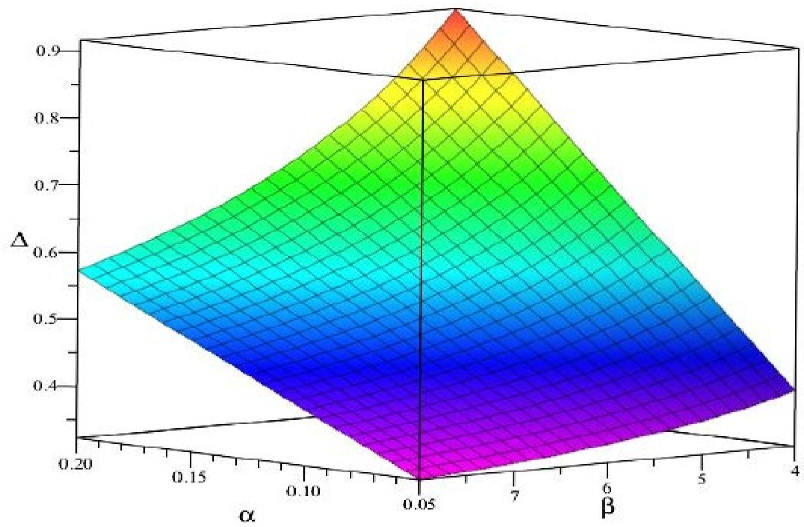

Figure 1: Impact of the change of values for the parameters α and β on Δ.

Table 1: Impact of the change of values for the parameters α and β on Δ.

β	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$	$\alpha = 0.2$
	Δ			
4.0	0.40531	0.57272	0.74268	0.91578
4.5	0.38685	0.53532	0.68571	0.83841
5.0	0.37210	0.50548	0.64037	0.77702
5.5	0.36004	0.48113	0.60342	0.72711
6.0	0.35000	0.46086	0.57272	0.68571
6.5	0.34150	0.44375	0.54681	0.65081
7.0	0.33423	0.42909	0.52465	0.62100
7.5	0.32793	0.41641	0.50548	0.59522
8.0	0.32242	0.40531	0.48873	0.57272

Example 5.2. This example establishes the Ulam-Hyers stability result for hybrid fractional problem of the form

$$\begin{cases} \mathcal{D}^{0.6}_{0^+}\left(\dfrac{u(t)}{\cos\left(u\left(\frac{t}{1+t}\right)\right) - K}\right) = A\sin(u^2(t)) + B\left[\dfrac{u(t)}{u(t)+1}\right], \quad 0 < t < 1 \\ u(0) = 0 \end{cases} \tag{5.2}$$

Comparison of Equation (5.2) with (1.5) yields $\varphi(t, u, v) = \cos\left(u\left(\frac{t}{1+t}\right)\right) - K$, $\psi(t) = \frac{t}{1+t}$. $h(t, u, v) =$

$A\sin(u^2(t)) + B\left[\dfrac{u(t)}{u(t)+1}\right]$. We know that $L = \sup_{t\in[0,\,1]} |\varphi(t,0,0)| = 1-K$. and from the assumption (A_3) we get $M = A + B$. Let us assume $A = \dfrac{1}{10}, B = \dfrac{1}{10}$ and $K = 0.92$. Calculations yield the following values $\theta = 0.6$, $\delta = 0.4$, $M = 0.2$, $L = 0.08$. Direct observations ensure the satisfaction of condition (3.1) presented in Theorem 3.2. Firstly, the condition (3.3) yields $\delta - M = 0.2 > \max\{0.08, 0.1788\}$ ensuring that the problem (5.2) has a unique solution. We shall now numerically analyse the stability of the considered problem (5.2) in the sense of Ulam and Hyers. The value of r defined in Theorem 3.2 is 0.08 and the condition Δ in (4.4) is evaluated as $0.9 < 1$. Thus, for the assumed parameter values the fractional order hybrid equation is Ulam-Hyers stable. We now proceed the demonstrate the impact of the M and L by suitable choice of K, A, B. Table 2 and Figure 2 demonstrates the impact of value of $M \in [0.01, 0.2]$ and $L \in [0.02, 0.08]$.

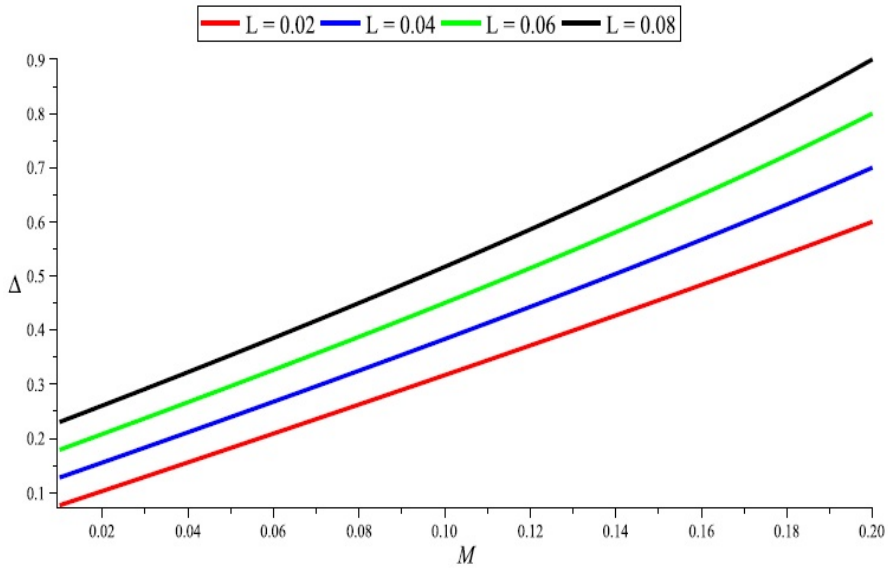

Figure 2: Impact of the change of values for the M and L on Δ.

Table 2: Impact of the change of values for the M and L on Δ.

M	$L = 0.02$	$L = 0.04$	$L = 0.06$	$L = 0.08$
	Δ			
0.01	0.07628	0.12756	0.17884	0.23012
0.03	0.12905	0.18310	0.23714	0.29121
0.05	0.18214	0.23928	0.29642	0.35357
0.07	0.23560	0.29621	0.35681	0.41742
0.09	0.28951	0.35403	0.41854	0.48306
0.11	0.34396	0.41293	0.48189	0.55086
0.13	0.39907	0.47314	0.54722	0.62129
0.15	0.45500	0.53500	0.61500	0.69500
0.17	0.51195	0.59891	0.68586	0.77282
0.19	0.57023	0.66547	0.76071	0.85595

6 Conclusion

The work established the existence results for hybrid fractional pantograph equation as an application to Darbo's fixed point theorem. The study also proved the unique solution for the problem and performed stability analysis. Examples resembling the considered generalized pantograph equation are presented and the values are assumed for the parameters such that the basic criterion are satisfied. The impact of the parameters are discussed with simulation provided as 3-dimensional and 2-dimensional portraits. The plots are simulated focusing on the value of Δ which is vital for determining the stability and existence of unique solutions. The tabulation for value of Δ at different possible states are given and it is evident that the simulations support the obtained theoretical results.

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