Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29451.4171

Integral inequalities involving fractional moments for continuous random variables

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(Communicated by Mugur Alexandru Acu)

Abstract

In the present work, fractional calculus is used to establish new integral inequalities for the fractional moments of continuous random variables. Generalizations of some classical integral inequalities are also obtained.

Keywords: Integral inequalities, Riemann-Liouville integral, random variable, fractional moment
2020 MSC: Primary 26D15; Secondary 26A33, 60E15

1 Introduction

It is well known that integral and fractional integral type inequalities have been used in probability theory and statistical problems for a long time and remains to be an attractive subject. In fact, in the last years several inequalities were discovered and a great deal of applications was found, like in the field of statistical mechanics, the optimal transportation problems and the applications in cellular automata, see [6] and [25]. For the reader's convenience, we also refer to [3, 4, 5, 15, 16, 20, 23]. We particularly emphasize the interesting work [23], where P. Kumar has established several new interesting results involving the higher moments for continuous random variables; furthermore, he presented some estimates for the central moments.

We also mention that using the Grüss-type inequalities and the truncated exponential distributions useful for obtaining some other types of fractional integral inequalities. In [22], the Ostrowski-type integral inequalities involving moments of a continuous random variable defined on a finite interval are established. On the other hand, Z. Dahmani [13] has proposed several inequalities for the fractional dispersion and the fractional variance functions of continuous random variables. In [14], the same author has established several new results and applications of fractional calculus for continuous random variables. It is also worth noticing that M. Houas, Z. Dahmani and M. Z. Sarikaya have established, in a very recent paper [19], several new results for $(r, \alpha)-$fractional moments of continuous random variables.

Motivated by [19], in the present work, we develop some new integral inequalities for the fractional moments of a continuous random variable X having the probability density function p.d.f f. The main novelty of this work is to generalize some classical integral inequalities developed by P. Kumar [23], which can be very simply deduced from our results.

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Received: December 2022 *Accepted:* June 2023

2 Basic definitions and preliminaries

In this section, we recall some basic definitions and tools which will be used in the proofs of our main results.

Definition 2.1. [18]. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a, b]$, is defined as

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau;\ \alpha > 0, a < t \le b, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Remark 2.2. Observe here that if $t = b$, the Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be written as follows

$$J_a^\alpha [f(b)] = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} f(\tau)\, d\tau;\ \alpha > 0. \tag{2.2}$$

In the sequel, we also need the following properties of the Riemann-Liouville fractional integral operators:

$$J_a^\alpha J_a^\beta [f(t)] = J_a^{\alpha+\beta} [f(t)],\ \alpha \ge 0,\ \beta \ge 0, \tag{2.3}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)],\ \alpha \ge 0,\ \beta \ge 0. \tag{2.4}$$

Now, let us introduce the concept of the well known $(r, \alpha)-$fractional moment function for a continuous random variable X in the sense of [14, 19].

Definition 2.3. The fractional moment function of order $(r, \alpha); (r > 0, \alpha > 0)$, for a continuous random variable X with a positive probability density function f defined on $[a, b]$, is defined by

$$M_{r,\alpha}(t) = J_a^\alpha [t^r f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau^r f(\tau)\, d\tau;\ \alpha > 0,\ a < t \le b. \tag{2.5}$$

For $t = b$, we have the following particular intersting case:

Definition 2.4. The fractional moment of order $(r, \alpha); (r > 0, \alpha > 0)$, for a continuous random variable X, is defined by

$$M_{r,\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau^r f(\tau)\, d\tau;\ \alpha > 0. \tag{2.6}$$

Remark 2.5. Now, if we take $\alpha = 1$ in (2.6), we obtain the classical moment of order r, given by $M_r = \int_a^b \tau^r f(\tau)\, d\tau$.

We recall the following generalized property of the positive probability density function f of X; see [14].

Theorem 2.6. Let X be a continuous random variable having a positive probability density function $f : [a, b] \to \mathbb{R}^+$. Then for any $\alpha \ge 1$, we have

$$J_a^{\alpha+1} [f(b)] = \frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right], \tag{2.7}$$

where $n = [\alpha - 1]$ (the integer part of $\alpha - 1$).

3 Main Results

In this section, we give new fractional integral inequalities for the fractional moments of a continuous random variable X having the probability density function f. Our first main result is stated as follows:

Theorem 3.1. Let X be a continuous random variable having a positive probability density function $f:[a,b]\to\mathbb{R}^+$. Then, we have:

(I) : For $\alpha\geq 1$; $n=[\alpha-1]$,

$$\begin{aligned}&\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[(-1)^i C_n^i b^{n-i}M_{i,\alpha-n}\right]J_a^\alpha\left[b^{r-1}\left(b-E\left(X\right)\right)f\left(b\right)\right]-\left(J_a^\alpha\left[\left(b-E\left(X\right)\right)f\left(b\right)\right]\right)M_{r-1,\alpha} \\ \leq\;& \|f\|_\infty^2\left(\frac{(b-a)^\alpha}{\Gamma\left(\alpha+1\right)}J_a^\alpha\left[b^r\right]-J_a^\alpha\left[b^{r-1}\right]\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[(-1)^i C_n^i b^{n-i}M_{i,\alpha-n}\right]\right),\end{aligned}\tag{3.1}$$

where $E\left(X\right)=\int_a^b\tau f\left(\tau\right)d\tau$ and $f\in L^\infty\left[a,b\right]$.

(II) : The inequality

$$\begin{aligned}&\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[(-1)^i C_n^i b^{n-i}M_{i,\alpha-n}\right]J_a^\alpha\left[b^{r-1}\left(b-E\left(X\right)\right)f\left(b\right)\right]-J_a^\alpha\left[\left(b-E\left(X\right)\right)f\left(b\right)\right]M_{r-1,\alpha} \\ \leq\;& \frac{1}{2}\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[(-1)^i C_n^i b^{n-i}M_{i,\alpha-n}\right]\right)^2\left(b-a\right)\left(b^{r-1}-a^{r-1}\right),\end{aligned}\tag{3.2}$$

is also valid for all $\alpha\geq 1$; $n=[\alpha-1]$.

Proof . In [19], we locate the following identity

$$\begin{aligned}&\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-\tau\right)^{\alpha-1}\left(b-\rho\right)^{\alpha-1}\left(\tau-\rho\right)\left(\tau^{r-1}-\rho^{r-1}\right)f\left(\tau\right)f\left(\rho\right)\,d\tau\,d\rho \\ =\;& 2J_a^\alpha\left[p\left(b\right)\right]J_a^\alpha\left[pgh\left(b\right)\right]-2J_a^\alpha\left[pg\left(b\right)\right]J_a^\alpha\left[ph\left(b\right)\right].\end{aligned}$$

If we put $p=f,\ g\left(b\right)=b-E\left(X\right),\ h\left(b\right)=b^{r-1}$ in (3.3), we obtain

$$\begin{aligned}&\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-\tau\right)^{\alpha-1}\left(b-\rho\right)^{\alpha-1}\left(\tau-\rho\right)\left(\tau^{r-1}-\rho^{r-1}\right)f\left(\tau\right)f\left(\rho\right)\,d\tau\,d\rho \\ =\;& 2J_a^\alpha\left[f\left(b\right)\right]J_a^\alpha\left[b^{r-1}\left(b-E\left(X\right)\right)f\left(b\right)\right]-2J_a^\alpha\left[\left(b-E\left(X\right)\right)f\left(b\right)\right]J_a^\alpha\left[b^{r-1}f\left(b\right)\right] \\ =\;& 2J_a^\alpha\left[f\left(b\right)\right]J_a^\alpha\left[b^{r-1}\left(b-E\left(X\right)\right)f\left(b\right)\right]-2J_a^\alpha\left[\left(b-E\left(X\right)\right)f\left(b\right)\right]M_{r-1,\alpha}.\end{aligned}\tag{3.3}$$

Keeping in mind that $f\in L^\infty\left[a,b\right]$, we obtain

$$\begin{aligned}&\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-\tau\right)^{\alpha-1}\left(b-\rho\right)^{\alpha-1}\left(\tau-\rho\right)\left(\tau^{r-1}-\rho^{r-1}\right)f\left(\tau\right)f\left(\rho\right)\,d\tau\,d\rho \\ \leq\;& \|f\|_\infty^2\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-\tau\right)^{\alpha-1}\left(b-\rho\right)^{\alpha-1}\left(\tau-\rho\right)\left(\tau^{r-1}-\rho^{r-1}\right)\,d\tau\,d\rho \\ \leq\;& \|f\|_\infty^2\left[2\frac{(b-a)^\alpha}{\Gamma\left(\alpha+1\right)}J_a^\alpha\left[b^r\right]-2J_a^\alpha\left[b\right]J_a^\alpha\left[b^{r-1}\right]\right].\end{aligned}\tag{3.4}$$

The required result is handled by (3.3), (3.4) and Theorem 2.6.

For (II), we remark that

$$\begin{aligned}&\int_a^b\int_a^b\frac{\left(b-\tau\right)^{\alpha-1}}{\Gamma^2\left(\alpha\right)}\left(b-\rho\right)^{\alpha-1}\left(\tau-\rho\right)\left(\tau^{r-1}-\rho^{r-1}\right)f\left(\tau\right)f\left(\rho\right)\,d\tau\,d\rho \\ \leq\;& \sup_{\tau,\rho\in[a,t]}\left|\left(\tau-\rho\right)\right|\left|\left(\tau^{r-1}-\rho^{r-1}\right)\right|J_a^\alpha\left[f\left(b\right)\right]^2=\left(t-a\right)\left(t^{r-1}-a^{r-1}\right)J_a^\alpha\left[f\left(b\right)\right]^2.\end{aligned}\tag{3.5}$$

Therefore, by (3.3), (3.5) and (2.6), we get the required inequality. This ends the proof of Theorem 3.1. □

Remark 3.2. (i) Taking $\alpha = 1$ in (3.1) of Theorem 3.1, we obtain the last inequality of Theorem 3.1 in [24].

(ii) Taking $\alpha = 1$ in (3.2) of Theorem 3.1, we obtain the first inequality of Theorem 3.1 in [24].

We shall further generalize the estimate (3.1) by considering two fractional positive parameters; this generalization is formulated as follows:

Theorem 3.3. Let X be a continuous random variable having a probability density function f defined on $[a,b]$.

(I^*) : For any $\alpha \geq 1, \beta \geq 1$; $n = [\alpha - 1]$, $m = [\beta - 1]$, we have

$$\begin{aligned} & \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right] J_a^{\beta}\left[f(t)\, b^{r-1}(b-E(X))\right] \\ & + \frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} M_{i,\beta-m}\right] J_a^{\alpha}\left[f(b)\, b^{r-1}(b-E(X))\right] \\ & - J_a^{\alpha}\left[f(b)(b-E(x))\right] M_{r-1,\beta} - J_a^{\beta}\left[f(b)(b-E(X))\right] M_{r-1,\alpha} \\ \leq\ & \|f\|_\infty^2 \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} J_a^{\beta}\left[b^r\right] + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} J_a^{\alpha}\left[b^r\right] - J_a^{\beta}\left[b^{r-1}\right] \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right] \right. \\ & \left. - J_a^{\alpha}\left[b^{r-1}\right] \frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} M_{i,\beta-m}\right] \right), \end{aligned} \tag{3.6}$$

where $E(X) = \int_a^b \tau f(\tau)\, d\tau$ and $f \in L^{\infty}[a,b]$.

(II^*) : We have also

$$\begin{aligned} & \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right] J_a^{\beta}\left[f(t)\, b^{r-1}(b-E(X))\right] \\ & + \frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} M_{i,\beta-m}\right] J_a^{\alpha}\left[f(b)\, b^{r-1}(b-E(X))\right] \\ & - J_a^{\alpha}\left[f(b)(b-E(X))\right] M_{r-1,\beta} - J_a^{\beta}\left[f(b)(b-E(X))\right] M_{r-1,\alpha} \\ \leq\ & (t-a)\left(t^{r-1}-a^{r-1}\right) \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right] \times \frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} M_{i,\beta-m}\right], \end{aligned} \tag{3.7}$$

for any $\alpha \geq 1,\ \beta \geq 1$; $n = [\alpha - 1]$, $m = [\beta - 1]$.

Proof . From [19], one has

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_a^b (b-\tau)^{\alpha-1} (b-\rho)^{\beta-1} (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right) f(\tau) f(\rho)\, d\tau\, d\rho \\ =\ & J_a^{\alpha}[p(b)]\, J_a^{\beta}[pgh(b)] + J_a^{\beta}[p(b)]\, J_a^{\alpha}[pgh(b)] - J_a^{\alpha}[ph(b)]\, J_a^{\beta}[pg(b)] - J_a^{\beta}[ph(b)]\, J_a^{\alpha}[pg(b)]. \end{aligned} \tag{3.8}$$

Now, in (3.8), we put $p = f$, $g(b) = b - E(X)$ and $h(b) = b^{r-1}$. Then, one has

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_a^b (b-\tau)^{\alpha-1} (b-\rho)^{\beta-1} (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right) f(\tau) f(\rho)\, d\tau\, d\rho \\ =\ & J_a^{\alpha}[f(b)]\, J_a^{\beta}\left[b^{r-1}(b-E(X))\, f(b)\right] + J_a^{\beta}[f(b)]\, J_a^{\alpha}\left[b^{r-1}(b-E(X))\, f(b)\right] \\ & - M_{r-1,\alpha}\, J_a^{\beta}\left[f(b)(b-E(X))\right] - M_{r-1\ \beta} J_a^{\alpha}\left[f(b)(b-E(X))\right]. \end{aligned} \tag{3.9}$$

We also have

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_a^b (b-\tau)^{\alpha-1} (b-\rho)^{\beta-1} (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right) f(\tau) f(\rho)\, d\tau\, d\rho \\ \leq\ & \|f\|_\infty^2 \int_a^b \int_a^b \frac{(b-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(b-\rho)^{\beta-1}}{\Gamma(\beta)} (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right) d\tau\, d\rho. \end{aligned} \tag{3.10}$$

Then

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau\,d\rho$$
$$\leq \|f\|_\infty^2\left[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}J_a^\beta[b^r]+\frac{(b-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha[b^r]-J_a^\alpha[b]J_a^\beta\left[b^{r-1}\right]-J_a^\beta[b]J_a^\alpha\left[b^{r-1}\right]\right].$$

Using (3.9), (3.10) and (2.7), we obtain (I^*). For the part (II^*) of this theorem, we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau\,d\rho\\ &\leq \sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho|\left|\tau^{r-1}-\rho^{r-1}\right|\right]J_a^\alpha[f(t)]J_a^\beta[f(t)]\\ &=(b-a)\left(b^{r-1}-a^{r-1}\right)J_a^\alpha[f(b)]J_a^\beta[f(b)].\end{aligned}\tag{3.11}$$

This implies

$$\int_a^b\int_a^b\frac{(b-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{(b-\rho)^{\beta-1}}{\Gamma(\beta)}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau\,d\rho\leq(b-a)\left(b^{r-1}-a^{r-1}\right)J_a^\alpha[f(b)]J_a^\beta[f(b)].\tag{3.12}$$

Now the estimate (3.7) follows by using Theorem 2.6 along with (3.9) and (3.12). □

The next theorem expresses a new $(r,\alpha)-$fractional moment result formulated as follows:

Theorem 3.4. Let X be a continuous random variable having a probability density function $f:[a,b]\to\mathbb{R}^+$, and assume that there exist $\omega,\varpi\in\mathbb{R}^+$such that $\omega\leq f\leq\varpi$. Then, for $\alpha\geq1$ and $n=[\alpha-1]$, we have

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]M_{2r,\alpha}-M_{r,\alpha}^2\leq\frac14\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]\right)^2(b^r-a^r)^2.\tag{3.13}$$

Proof . The use of the fractional Grüss-type inequality from [11], we can write

$$\left|J_a^\alpha[p(b)]J_a^\alpha\left[pg^2(b)\right]-\left(J_a^\alpha[pg(b)]\right)^2\right|\leq\frac14\left(J_a^\alpha[p(b)]\right)^2(\omega-\varpi)^2.$$

If we put $p=f,\ g(b)=b^r,\ \omega=a^r,\ \varpi=b^r$ in (3.9), then we can rewrite the above inequality as follows

$$0\leq J_a^\alpha[f(b)]J_a^\alpha\left[b^{2r}f(b)\right]-\left(J_a^\alpha[b^rf(b)]\right)^2\leq\frac14\left(J_a^\alpha[f(b)]\right)^2(b^r-a^r)^2,\tag{3.14}$$

which implies that

$$J_a^\alpha[f(b)]M_{2r,\alpha}-M_{r,\alpha}^2\leq\frac14\left(J_a^\alpha[f(b)]\right)^2(b^r-a^r)^2.\tag{3.15}$$

Finally, by Theorem 2.6, we obtain (3.13). □

Remark 3.5. If we take $\alpha=1$ in (3.4), we obtain Theorem 3.2 in [24].

By the use of two fractional parameters, we can propose the following generalization of the above results:

Theorem 3.6. Let X be a continuous random variable having a probability density function $f:[a,b]\to\mathbb{R}^+$ and assume that there exist $\omega,\varpi\in\mathbb{R}^+$ such that $\omega\leq f\leq\varpi$. Then for all $\alpha\geq1,\ \beta\geq1$, we have

$$\begin{aligned}&\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]M_{2r,\beta}+\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]\\ &\times M_{2r,\alpha}+2a^rb^r\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]\times\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^n\left[(-1)^iC_n^ib^{n-i}M_{i,\alpha-n}\right]\\ \leq\ &(a^r+b^r)\left[J_a^\alpha[f(b)]M_{r,\beta}+J_a^\beta[f(b)]M_{r,\alpha}\right],\end{aligned}\tag{3.16}$$

where $n=[\alpha-1]$ and $m=[\beta-1]$.

Proof . We put $p(b) = f(b)$, $g(b) = b^r$. The use of [12, Theorem 3.4] allows us to conclude that

$$\begin{aligned} & \left[J_a^\alpha[f(b)] J_a^\beta\left[b^{2r} f(b)\right]+J_a^\beta[f(b)] J_a^\alpha\left[b^{2r} f(b)\right]-2 J_a^\alpha\left[b^r f(b)\right] J_a^\beta\left[b^r f(b)\right]\right]^2 \\ \leq & \left[\left(\varpi J_a^\alpha[f(b)]-J_a^\alpha\left[b^r f(b)\right]\right)\left(J_a^\beta\left[b^r f(b)\right]-\omega J_a^\beta[f(b)]\right)\right. \\ & \left.+\left(J_a^\alpha\left[b^r f(b)\right]-\omega J_a^\alpha[f(b)]\right)\left(\varpi J_a^\beta[f(b)]-J_a^\beta\left[b^r f(b)\right]\right)\right]^2 . \end{aligned} \tag{3.17}$$

This implies

$$\begin{aligned} J_a^\alpha[f(b)] M_{2r,\beta}+J_a^\beta[f(b)] M_{2r,\alpha}-2 M_{r,\alpha} M_{r,\beta} \leq & \left(\varpi J_a^\alpha[f(b)]-M_{r,\alpha}\right)\left(M_{r,\beta}-\omega J_a^\beta[f(b)]\right) \\ & +\left(M_{r,\alpha}-\omega J_a^\alpha[f(b)]\right)\left(\varpi J_a^\beta[f(b)]-M_{r,\beta}\right) . \end{aligned} \tag{3.18}$$

Now, substituting the values of ω and ϖ in (3.18), we obtain

$$J_a^\alpha[f(b)] M_{2r,\beta}+J_a^\beta[f(b)] M_{2r,\alpha} \leq\left(a^r+b^r\right) J_a^\alpha[f(b)] M_{r,\beta}+\left(a^r+b^r\right) J_a^\beta[f(b)] M_{r,\alpha}-2 a^r b^r J_a^\alpha[f(t)] J_a^\beta[f(t)] . \tag{3.19}$$

Then, we end the proof by using Theorem 2.6. □

We also have the following:

Theorem 3.7. Let X be a continuous random variable having a probability density function $f:[a,b] \to \mathbb{R}^+$ and assume that there exist $\omega, \varpi \in \mathbb{R}^+$ such that $\omega \leq f \leq \varpi$. Then, for all $\alpha \geq 1, n = [\alpha - 1]$, we have

$$\left|\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} M_{r,\alpha}-\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^n\left[(-1)^i C_n^i b^{n-i} M_{i,\alpha-n}\right] J_a^\alpha\left[b^r\right]\right| \leq \frac{(\varpi-\omega)}{2 \Gamma(\alpha+1)}\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha\left[b^{2r}\right]-\left(J_a^\alpha\left[b^r\right]\right)^2\right)^{\frac{1}{2}}(b-a)^\alpha . \tag{3.20}$$

Proof . To prove this theorem, it suffices to use Theorem 3.1 and Lemma 3.2 of [17]. Doing so, we have

$$\left|\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha[f g(b)]-J_a^\alpha[f(b)] J_a^\alpha[g(b)]\right| \leq \frac{(\varpi-\omega)(b-a)^\alpha}{2 \Gamma(\alpha+1)}\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha\left[g^2(b)\right]-\left(J_a^\alpha[g(b)]\right)^2\right)^{\frac{1}{2}} . \tag{3.21}$$

Then, if we put $g(b) = b^r$ in (3.21), we have

$$\left|\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha\left[b^r f(b)\right]-J_a^\alpha[f(b)] J_a^\alpha\left[b^r\right]\right| \leq \frac{1}{2 \Gamma(\alpha+1)}\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha\left[b^{2r}\right]-\left(J_a^\alpha\left[b^r\right]\right)^2\right)^{\frac{1}{2}}(\varpi-\omega)(b-a)^\alpha . \tag{3.22}$$

Then, by Theorem 2.6 and (3.22), we get the required inequality (3.20). □

We can also establish the following generalization by using two fractional parameters:

Theorem 3.8. Let X be a continuous random variable having a probability density function f defined on $[a,b]$,and assume that there exist $\omega, \varpi \in \mathbb{R}^+$ such that $\omega \leq f \leq \varpi$. Then for all $\alpha \geq 1$ and $\beta \geq 1$, we have

$$\begin{aligned} & \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} M_{r,\alpha}(t)+\frac{(t-a)^\beta}{\Gamma(\beta+1)} M_{r,\beta}(t)-J_a^\alpha[f(t)] J_a^\beta\left[t^r\right]-J_a^\beta[f(t)] J_a^\alpha\left[t^r\right] \\ \leq & {\left[\left(M \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}-J_a^\alpha[f(t)]\right)\left(J_a^\beta[f(t)]-m \frac{(t-a)^\beta}{\Gamma(\beta+1)}\right)+\left(J_a^\alpha[f(t)]-m \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\right)\left(M \frac{(t-a)^\beta}{\Gamma(\beta+1)}-J_a^\beta[f(t)]\right)\right] } \\ & \times\left[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta\left[t^{2r}\right]+\frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha\left[t^{2r}\right]-2 J_a^\alpha\left[t^r\right] J_a^\beta\left[t^r\right]\right]^{\frac{1}{2}}, \end{aligned} \tag{3.23}$$

where $n = [\alpha - 1]$ and $m = [\beta - 1]$.

Proof . We use the following inequality from [17]:

$$\left|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}[fg(b)]+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}[fg(b)]-J_a^{\alpha}[f(b)]J_a^{\beta}[g(b)]-J_a^{\beta}[f(b)]J_a^{\alpha}[g(b)]\right|$$
$$\leq\left[\left(\varpi\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}[f(b)]\right)\left(J_a^{\beta}[f(b)]-\omega\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}\right)\left(J_a^{\alpha}[f(b)]-\omega\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\varpi\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}-J_a^{\beta}[f(b)]\right)\right]$$
$$\times\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\left[g^2(b)\right]+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}\left[g^2(b)\right]-2J_a^{\alpha}[g(b)]J_a^{\beta}[g(b)]\right)^{\frac{1}{2}}. \tag{3.24}$$

Inserting $g(b)=b^r$ in (3.24), we have

$$\left|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}[t^r f(t)]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}[t^r f(t)]-J_a^{\alpha}[f(t)]J_a^{\beta}[t^r]-J_a^{\beta}[f(t)]J_a^{\alpha}[t^r]\right|$$
$$\leq\left[\left(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}[f(t)]\right)\left(J_a^{\beta}[f(t)]-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right)+\left(J_a^{\alpha}[f(t)]-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-J_a^{\beta}[f(t)]\right)\right]$$
$$\times\left(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\left[t^{2r}\right]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}\left[t^{2r}\right]-2J_a^{\alpha}[t^r]J_a^{\beta}[t^r]\right)^{\frac{1}{2}}. \tag{3.25}$$

Finally, the required inequality follows from (2.7). □

Remark 3.9. If $\alpha=\beta$, then Theorem 3.8 is reduced to Theorem 3.7.

4 Conclusions and final remarks

The main aim of this paper was to establish several new integral inequalities for the fractional moments of continuous random variables. Concerning applications of the abstract theoretical results, we would like to mention that, using the same reasoning as in [13], we can illustrate our results by considering some classical situations involving the fractional variance of order α and the fractional moment of order (r,α); we feel it is our duty to say that we have not been able to incorporate our results in the qualitative analysis of solutions for some classes of fractional integro-differential equations, unfortunately. This could be a topic of further analyses; besides that, we would like to note that we will continue the present work in our forthcoming research studies by investigating similar results for the Hadamard fractional integral operators and the Hadamard $k-$fractional integral operators. Another axis to explore, is to exploit this type of inequalities in some concrete problems like those studied in [7], [8], [9], [10].

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