

Routh stability criterion and Lyapunov-Routh method in control theory

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Abstract

One of the most important issues in the linear control subject is to obtain eigenvalues of the system to study the stability of a system. It is needed to identify the sign of eigenvalues but not the value of it. To fulfil this, there are different methods such as Routh stability criterion, Lyapunov's method and Nyquist stability criterion. In this research, we will present the most simple one to determine the sign of eigenvalues and we shall discuss and explain different types of stability. Also, we will discuss some special cases that cross a controversial mission, and a new method is proposed to calculate the stability of systems, which we call the Lyapunov-Routh method (composition of Lyapunov and Routh method).

Keywords: Stability, Stability types, Routh-Hurwitz stability criterion, Lyapunov's method
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1 Introduction

The stability of the system is to make the system equilibrium which does not accept the major and sudden changes that leads the system to be in a state of dispersion or may cause noise, which hinders the system from well functioning. Before we give a mathematical definition of stability, let us explain this concept with examples from our daily life such as calmness after a big challenge. Mathematically, the system is considered stable when the output is within certain values, meaning and does not go to infinity, the researchers need to define the transfer function as shown in Fig. 1 [4].

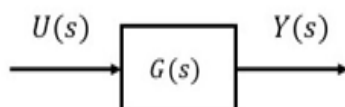


Figure 1:

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$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

when n^{th} is of the system ($n \geq m$) and $D(s)$ is characteristic polynomial when the $D(s)$. When getting close to zero ($D(s) = 0$) shown below

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0 \tag{1.1}$$

are called the characteristic equation. The roots of the numerator polynomial

$$\text{i.e., } N(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0 = 0 \tag{1.2}$$

are called the zeroes of the system. The roots of the denominator polynomial equation (1.1) are called poles of the system.



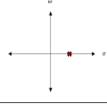
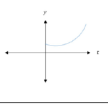

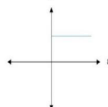
2 Stability types

There are three types of system stability [2] we will explain each of these types:

- 1- Absolutely stable system.
- 2- Conditionally stable system.
- 3- Marginally stable system.

- 1- Absolutely stable system: The control system is stable for all ranges of system component values. The system is stable if all the poles of the present transfer function are in left half of S-plan.
- 2- Conditionally stable system: If the system is stable for certain ranges of system component values.
- 3- Marginally stable system: if the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input [1].


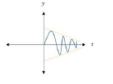
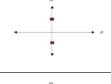
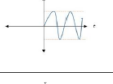
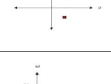
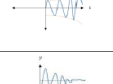
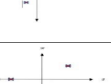
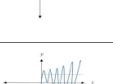

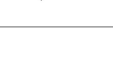
The control system is marginally stable if any two poles of the transfer function are present on the imaginary axis. Impulse responses for various root locations in the s-plan [2] :

Type of root	s - plan graph $s = \sigma + i\omega$	Response graph	type stable
Real and Negative			Absolutely stable
Real and Positive			Unstable
Zero			Marginally stable

3 Routh stability criterion:

The Routh stability criterion [10] is an analytical procedure for determining if all the roots of a polynomial have negative real parts, and it is used in the stability analysis of linear time invariants systems [5]. This stability criterion is useful in various engineering applications [3, 7].

It is important to obtain the sign of the $G(s)$ roots not the value of them .to distinguish the sign of $D(s)$ can be use Routh stability criterion. After reading the theory of the network structure [1], we can easily say that any pole of

Conjugate complex With Negative Real part			Conditionally stable
Conjugate Imaginary			Marginally stable
Conjugate With Positive real part			unstable
Complex With Negative Real part			Conditionally stable
Conjugate With Positive real part and one pole Negative			unstable

the system located on the right side of the origin of the s - plane, makes the system unstable based on this condition [8]. Routh [10] began to verify the necessary and sufficient conditions for the stability of the system.

Also known as modified Hurwitz Criterion of stability of the system. We will study this criterion in two parts. Part one will cover necessary conditions for stability of the system and part two will cover the sufficient conditions for the stability of the system. One-The necessary conditions: it includes two parts as follows All the coefficients of the characteristic of polynomial (equation (1)) should be positive and real.

I. All the coefficients of the characteristic polynomial should be non-zero.

Two-The sufficient condition:

The helpful key to identify whether a system is stable or not is to consider that all elements of the first column of the Routh table have same sign.

This method yields stability information without the need to solve closed -loop system poles.

Using this method, we can tell the number of the closed -loop system poles in the half -plan, in the right half-plan, and on the $i\omega$ -axis(notice that we say how many not where).

The method requires two steps:

1-generate a data table called a Routh table:

- i. Begin by labeling the rows with powers of s form the highest power of the denominator polynomial to s_0 .
- ii. The first row will consist of all the even terms of the characteristic equation. Arrange them from first (even term) to last (even term). The first row is written below: $a_0, a_2, a_4, a_6, \dots$
- iii. The second row will consist of all the odd terms of the characteristic equation. Arrange them from first (odd term) to last (odd term). The first row is written below: $a_1, a_3, a_5, a_7, \dots$
- iv. The elements of third row can be calculated as follows
 - a) First element: Multiply a_0 with the diagonally opposite element of next column (*i.e.* a_3) then subtract this from the product of a_1 and a_2 (where a_2 is diagonally opposite element of next column) and finally divide the result so obtain with a_1 . Mathematically we write as first element

$$b_1 = \frac{(a_1 a_2 - a_3 a_0)}{a_1}$$

- b) Second element: Multiply a_0 with the diagonally opposite element of one column after the next column (*i.e.* a_5) then subtract this from the product of a_1 and a_4 (where, a_4 is diagonally opposite element of next to next column) and then finally divide the result then obtain with a_1 . Mathematically we write second element as follows:

$$b_2 = \frac{(a_1 a_4 - a_5 a_0)}{a_1}$$

- c) Third element: Multiply a_0 with the diagonally opposite element of one column after the next column (*i.e.* a_7). then subtract this from the product of a_1 and a_6 (where, a_6 is diagonally opposite element of next-to-next column) and then finally divide the result so obtain with a_1 . Mathematically we write as second element

$$b_3 = \frac{(a_1 a_6 - a_7 a_0)}{a_1}$$

- v. The elements of the fourth row can be calculated through using the following procedure:

- a) First element: Multiply a_1 with the diagonally the opposite element of next column (*i.e.* b_2) then subtract this from the product of a_3 and b_1 (where, b_1 is diagonally the opposite element of next column) and then finally divide the result so obtain with b_1 . Mathematically we write the first element

$$c_1 = \frac{(b_1 a_3 - b_2 a_1)}{b_1}$$

- b) First element: Multiply a_1 with the diagonally the opposite element of the next column (*i.e.* b_3) then subtract this from the product of a_5 and b_3 (where, b_1 is diagonally opposite element of next column) and finally divide the result to obtain b_1 . Mathematically we write the first element as follows:

$$c_2 = \frac{(b_1 a_5 - b_3 a_1)}{b_1}$$

- c) Similarly, we can calculate all the elements of the fourth row.

- vi. Similarly, we can calculate all the elements of all rows.

We will explain this on the table below. The Characteristic equation is:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s^1 + a_n s^0 = 0.$$

s^n	a_0	a_2	a_4	a_6	...
s^{n-1}	a_1	a_3	a_5	a_7	...
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	$b_3 = \frac{a_1 a_6 - a_7 a_0}{a_1}$
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1}$	\vdots		
\vdots	\vdots	\vdots	\vdots		
s^1	\vdots	\vdots			
s^0	a_n				

We can multiply positive constant value and for the equation if not found s to power i that means it's coefficient equal 0.

Stability criteria: if all the elements of the first column are positive then the system will be stable. However, if anyone of them is negative the system will be unstable. The number of roots of polynomial that are in the RHP is equal to the number of sign changes in first column.

Example 3.1. Generate the Routh table for the characteristic equation as below,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Solution:

s^4	1	3	1
s^3	3	2	0
s^2	$7/3$	1	0
s^1	$5/7$	0	0
s^0	1	0	0

The system is stable.

Example 3.2. Consider the following characteristics equation, develop Routh array and determine the stability system.

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0.$$

Solution:

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	- 6	0	0
s^0	5	0	0

It is unstable because two poles in sign appear in the first column; we find that two roots of the characteristic equation lie on the right-hand plane of the s-plan.

4 Special cases in Routh Stability Criteria:

There are some special cases related to Routh Stability Criteria, which are discussed below:

(1) Case one: It is when the first term in any row of the array is zero while the rest of the row has at least one none zero term.

In this case we will assume a very small value (ϵ) which is tending to zero in place of zero. By replacing zero with (ϵ) we will calculate all the elements of the Routh array. After calculating all the elements, we will apply the limit at each element containing (ϵ). Through solving the limit at every element if we get positive limiting value, we will say the given system is stable otherwise in all other conditions the given system is not stable.

Example 4.1. Check whether the given system in Fig.2 is stable or unstable by Routh criterion:

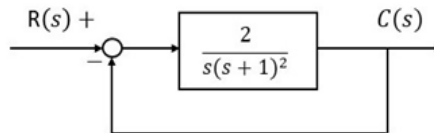


Figure 2:

Solution: Consider the characteristics equation $1 + G(s)H(s) = 0$. Then $1 + \frac{2}{s(s+1)^2} = 0$. This implies that $s^3 + s^2 + s + 2 = 0$. Let's create a Routh table

s^3	1	1
s^2	2	2
s^1	0	0
s^0	?	

We notice that the third element of the first column is zero, so we deleted the zero and put a value very close to zero $\epsilon(\epsilon \rightarrow 0)$, and then we completed the Routh table. The system is stable.

s^3	1	1
s^2	2	2
s^1	ϵ	0
s^0	2	0

Example 4.2. Determine the stability of the system having a characteristic equation given below:

$$D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Solution: Let's create a Routh table We notice that the third element of the first column is zero, so we deleted the

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	?		
s^1			
s^0			

zero and put a value very close to zero ($\epsilon \rightarrow 0$). We can calculate the element:

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \text{ and } d_1 = \frac{(6c_1 - 10\epsilon)}{c_1} \rightarrow 0$$

Then we complete the Routh table. Note that c_1 is negative i.e., the transformation from positive to negative and

s^5	1	2	11
s^4	2	4	10
s^3	ϵ	6	0
s^2	c_1	10	0
s^1	d_1	0	0
s^0	10	0	0

negative to positive, the system is unstable and this indicates the presence of two poles in the right-hand plane of the s-plan.

Example 4.3. Make the Routh table for the system as below:

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Solution: The characteristic equation for system is

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3.$$

Then we complete the Routh table

s^5	1	3	5
s^4	2	6	3
s^3	0	$\frac{7}{2}$	0
s^2	?		
s^1			
s^0			

We notice that the third element of the first column is zero, so we deleted the zero and put a value very close to zero ($\epsilon \rightarrow 0$). We can calculate the

$$c_1 = \frac{6\epsilon - 7}{\epsilon} = \frac{-12}{\epsilon} \text{ and } d_1 = \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$$

s^5	1	3	5
s^4	2	6	3
s^3	ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Label	first column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	ϵ	+	-
s^2	$\frac{6\epsilon - 7}{\epsilon}$	-	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

The ϵ It is a very close value to zero. It can be a negative value close to zero or a positive value close to zero. To analyze the stability of this system, we can suppose the ϵ positive and negative as on the table below:

If we assume that the value of ϵ is positive, then the fourth element of the first column is negative, and this indicates the presence of a change in the sign from positive to negative then to positive, and this indicates the presence of two of poles in the right-hand plane of the s-plan.

Also, if we assume that the value of ϵ is negative, we notice a change in the sign in the third element from the first column, this indicates the presence of two poles in the right - hand plane of the s-plan. This means that the system is unstable.

(2) Case second: When all the elements of any row of the Routh array are zero. In this case we can say the system has the symptoms of marginal stability. Let us first understand the physical meaning of having all the elements zero of any row. The physical meaning is that there are symmetrically located roots of the characteristic equation in the s plane. Now in order to find out the stability in this case we will first find out auxiliary equation. Auxiliary equation can be formed by using the elements of the row just above the row of zeros in the Routh array. After finding the auxiliary equation, we will differentiate the auxiliary equation to obtain elements of the zero row. If there is no sign change in the new Routh array formed by using auxiliary equation, then the given system is a limited stable. However, in all other cases we will say, the given system is unstable.

Example 4.4. Make the Routh table for the system shown and determine the pole distribution in the s-plan the auxiliary polynomial method

$$G(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}.$$

Solution: The characteristic equation for system is

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56.$$

Then we complete the Routh table Here to complete Routh table, We Derivative the equation for the second row

$$\frac{d(s^4 + 6s^2 + 8)}{ds} = 4s^3 + 12s \tag{4.1}$$

We substitute the coefficients of the equation (4.1) in the third row instead of zero, and divide the third row by 4 becomes the table below: We notice that the system is stable because the first column did not change its signal.

s^5	1	6	8
s^4	1	6	8
s^3	0	0	0
s^2	?		
s^1			
s^0			

s^5	1	6	8
s^4	1	6	8
s^3	1	3	0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

5 Lyapunov Routh stability method:

Sometimes it is difficult to check the stability of a problem by Lyapunov's method, and in this case, we can use Routh method for help. We call this method Lyapunov Routh method (composition of Lyapunov and Routh). In this method, first we use Lyapunov's method to obtain the form of matrix in state space:

$$\dot{X} = AX + BU.$$

Now we use the Routh method to write a specific sentence of the matrix A and check the stability of the system. As shown in the examples (6 and 7), we can easily check the stability of a parametric system.

Example 5.1. Determine the range of K for the system in Fig. 3 to be asymptotically stable.

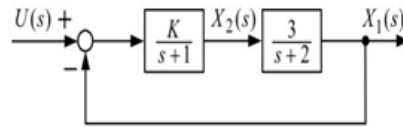


Figure 3:

Solution: using Lyapunov's method,

$$X_1 = \frac{3}{s+2}X_2 \Rightarrow \dot{x}_1 = -2x_1 + 3x_2$$

$$X_2 = \frac{K}{s+1}(-X_1 + U) \Rightarrow \dot{x}_2 = -Kx_1 - x_2 + 2Ku$$

The state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -K & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} u$$

$$A = \begin{bmatrix} -2 & 3 \\ -K & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ K \end{bmatrix}$$

Solving $A'P + PA = -I$,

$$\begin{bmatrix} -2 & -K \\ 3 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -K & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We obtain

$$p = \frac{1}{18K + 12} \begin{bmatrix} K^2 + 3K + 3 & 3 - 2K \\ 3 - 2K & 3K + 15 \end{bmatrix}$$

Here it is difficult to calculate the value K . So, we use a simpler method to know The stability of system by using Routh stability criterion:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \left\| \begin{bmatrix} -2 & 3 \\ -K & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right\| &= \begin{bmatrix} -2 - \lambda & 3 \\ -K & -1 - \lambda \end{bmatrix} \\ &= 2 + 2\lambda + \lambda + \lambda^2 + 3K = \lambda^2 + 3\lambda + 2 + 3K \end{aligned}$$

By using Routh table:

λ^2	1	$2 + 3K$
λ^1	3	0
λ^0	$2 + 3K$	0

The system is stable

$$2 + 3K > 0 \implies K > -2/3$$

Example 5.2. Determine a range of values of system parameter K for which the system in Fig. 4 is stable.

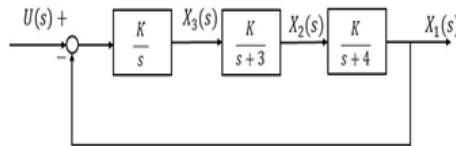


Figure 4:

Solution: using Lyapunov's method

$$\begin{aligned} X_1 &= \frac{3}{s+4} X_2 \Rightarrow \dot{x}_1 = -4x_1 + Kx_2 \\ X_2 &= \frac{K}{s+3} X_3 \Rightarrow \dot{x}_2 = -3x_2 + Kx_3 \\ X_3 &= \frac{K}{s} (-X_1 + U) \Rightarrow \dot{x}_3 = -Kx_1 + Ku \end{aligned}$$

The state equation is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & K & 0 \\ 0 & -3 & K \\ -K & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u$$

$$A = \begin{bmatrix} -4 & K & 0 \\ 0 & -3 & K \\ -K & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}$$

Solving $A'P + PA = -I$,

$$\begin{bmatrix} -4 & 0 & -K \\ K & -3 & 0 \\ 0 & K & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} -4 & K & 0 \\ 0 & -3 & K \\ -K & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Here it is difficult to calculate the value K . So, we use a simpler method to know The stability of system by using

Routh stability criterion:

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -4 & K & 0 \\ 0 & -3 & K \\ -K & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -4 - \lambda & K & 0 \\ 0 & -3 - \lambda & K \\ -K & 0 & -\lambda \end{bmatrix} \right| = 0$$

$$\lambda^3 + 7\lambda^2 + 12\lambda + K^3 = 0.$$

By using Routh table:

λ^3	1	12
λ^2	7	K^3
λ^1	$\frac{84 - K^3}{7}$	0
λ^0	K^3	0

Note that K appears twice in the first column in the third element and the fourth through $\frac{84 - K^3}{7}$ must be $84 - K^3 > 0 \implies K < 4.38$ in the fourth element K^3 must be $K > 0$ Thus, for the system to be stable, it must be $0 < K < 4.38$.

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