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Amenability properties of vector-valued function algebras

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Abstract

Let X be a compact Hausdorff space, A be a (commutative) Banach algebra and $\mathscr{A}$ be a Banach A-valued function algebra on X. Let $\mathfrak{A}$ be the function algebra on X, consisting of scalar-valued functions in $\mathscr{A}$. We study and characterize various amenability properties of the algebra $\mathscr{A}$ in terms of cohomological properties of $\mathfrak{A}$ and A. Containing some well-known examples, such as $\mathscr{C}(X, A)$ and $\mathrm{Lip}(X, A)$, the class of vector-valued function algebras also includes, in some sense, the tensor products $\mathfrak{A}\hat{\otimes}_\gamma A$. As consequences, some known results in this area are covered.

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1 Introduction

The notion of amenability of a Banach algebra was first introduced by Johnson, [10], where it is shown that, for a locally compact group G, the convolution group algebra $L^1(G)$, as a Banach algebra, is amenable, if and only if the underlying group G is amenable; however, this equivalence does not remain true for the convolution semigroup algebra $\ell^1(S)$ of a discrete semigroup S. Motivated by these considerations, the concept of left amenability for F-algebras was introduced by Lau, [15].

For arbitrary Banach algebras A, the notion of ϕ-amenability, as a generalization of left amenability of F-algebras, was introduced and studied by Kaniuth, Lau and Pym in [12], where $\phi \in \mathfrak{M}(A)$, the set of all characters on A. In fact, A is called *ϕ-amenable* if A has a *ϕ-mean*, that is an element $M \in A^{**}$ satisfying $M(\phi) = 1$ and $M(\eta \cdot a) = \phi(a)M(\eta)$ for all $a \in A$ and $\eta \in A^*$. Just as for amenability, there are many characterizations for ϕ-amenability of Banach algebras. For example, in [12] the authors characterized ϕ-amenability in two different ways; through vanishing of the cohomology groups $H^1(A, X^*) = \{0\}$, for certain Banach A-bimodules X, and in terms of the existence of a bounded net (u_α) in A such that $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for all $a \in A$. See [9], for more characterizations.

The notion of character amenability was introduced and studied, independently, by Monfared [16]. A Banach algebra A is *character amenable* if it has a bounded right approximate identity and is ϕ-amenable for all $\phi \in \mathfrak{M}(A)$. For a locally compact group G, it is shown in [16] that character amenability of $L^1(G)$ is equivalent to amenability of G, and character amenability of $M(G)$ is equivalent to discreteness and amenability of G. See also [19].

The notion of ϕ-contractibility of A, for $\phi \in \mathfrak{M}(A)$, was introduced and studied by Hu, Monfared and Tranor, [9]. The algebra A is called *ϕ-contractible* if there exists an element $m \in A$ satisfying $\phi(m) = 1$ and $am = \phi(a)m$ for all

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$a \in A$. Moreover, A is called *character contractible* if A has a right identity and is ϕ-contractible, for all $\phi \in \mathfrak{M}(A)$. The notion of ϕ-contractility has analogous properties as of ϕ-amenability. See [9] and references therein for more information.

An important class of Banach algebras is the so-called *vector-valued function algebras*. Some classical examples are $\mathscr{C}(X, A)$, $\mathrm{Lip}(X, A)$, and tensor products $\mathfrak{A}\hat{\otimes}_\gamma A$, where $\mathfrak{A}$ is a (complex) function algebra on X and γ is an algebra cross-norm. These algebras have interesting properties which recently put them in the core of attention of many authors. For example, in [7], some amenability properties of $\mathscr{C}(X, A)$ were investigated. They showed that $\mathscr{C}(X, A)$ is amenable if and only if the range algebra A is amenable. It is also proved that if A is weakly amenable and has a bounded approximate identity, then $\mathscr{C}(X, A)$ is weakly amenable.

Our main goal is investigating some amenability properties of vector-valued function algebras and present characterizations.

2 Preliminaries

Let X be a compact Hausdorff space and A be a (semisimple) commutative Banach algebra (with identity $\mathbf{1}$). The space of all A-valued continuous functions on X is denoted by $\mathscr{C}(X, A)$. Algebraic operations on $\mathscr{C}(X, A)$ are defined pointwise, and the uniform norm $\|\cdot\|_X$ on $\mathscr{C}(X, A)$ is defined in the obvious way; $\|f\|_X = \sup\{\|f(x)\| : x \in X\}$, for all $f \in \mathscr{C}(X, A)$. In this setting, $(\mathscr{C}(X, A), \|\cdot\|_X)$ is a commutative unital Banach algebra. In case $A = \mathbb{C}$, we simply get the classical algebra $\mathscr{C}(X)$ of continuous complex-valued functions on X.

An *A-valued function algebra* on X is a subalgebra $\mathscr{A}$ of $\mathscr{C}(X, A)$ that contains the constant functions and separates points on X. If $\mathscr{A}$ is endowed with some complete algebra norm $\|\cdot\|$ such that the restriction of $\|\cdot\|$ to A is equivalent to the original norm of A and, moreover, $\|f\|_X \le \|f\|$, for every $f \in \mathscr{A}$, then $\mathscr{A}$ is called a *Banach A-valued function algebra* on X. In case $A = \mathbb{C}$, the classical Banach function algebras are obtained. For a complete study of scalar-valued function algebras see [5, Chapter 4] and [13, Chapter 2]. For more information on vector-valued function algebras, see [1, 2, 18].

We consider A as the subalgebra of $\mathscr{A}$ consisting of all constant functions. Also, we identify every $\alpha \in \mathbb{C}$ with $\alpha\mathbf{1} \in A$, so that, for every $\phi \in \mathfrak{M}(A)$ and $f \in \mathscr{A}$, we have $\phi \circ f \in \mathscr{C}(X, A)$. The A-valued function algebra $\mathscr{A}$ is called *admissible*, [1], if $\phi \circ f \in \mathscr{A}$, for all $\phi \in \mathfrak{M}(A)$ and $f \in \mathscr{A}$. In this case, set

$$\mathfrak{A} = \{\phi \circ f : \phi \in \mathfrak{M}(A), f \in \mathscr{A}\},$$

which forms a (complex) function algebra on X by itself. Note that $\mathfrak{A}$ is, in fact, equal to the subalgebra of $\mathscr{A}$ consisting of all scalar-valued functions, that is $\mathfrak{A} = \mathscr{A} \cap \mathscr{C}(X)\mathbf{1}$.

Every semisimple commutative Banach algebra A can be seen, through its Gelfand transform, as a Banach function algebra on its character space $\mathfrak{M}(A)$; i.e., every element $a \in A$ is identified with the continuous function $\hat{a} : \mathfrak{M}(A) \to \mathbb{C}$. Using this fact, we note that tensor products of commutative Banach algebras can be seen as admissible vector-valued function algebras. If $\mathfrak{A}$ and A are two semisimple commutative Banach algebras, then $\mathfrak{A} \otimes A$ can be seen either as an A-valued function algebra on $\mathfrak{M}(\mathfrak{A})$ or as an $\mathfrak{A}$-valued function algebra on $\mathfrak{M}(A)$. More precisely, as we mentioned above, $\mathfrak{A}$ is regarded as a Banach function algebra on its character space $X = \mathfrak{M}(\mathfrak{A})$. Now, consider the algebraic tensor product $\mathfrak{A} \otimes A$ and identify every tensor element $g \otimes a$ in $\mathfrak{A} \otimes A$ with the A-valued function $ga : X \to A, \xi \mapsto \hat{g}(\xi)a = \xi(g)a$. Now, if γ is an algebra cross-norm on $\mathfrak{A} \otimes A$, then its completion $\mathfrak{A}\hat{\otimes}_\gamma A$ forms an admissible Banach A-valued function algebra on X; see [3].

Identifying the character space of a Banach A-valued function algebra is a problem of interest. In 1957, Hausner [8] proved that, for every character τ of $\mathscr{C}(X, A)$, there exist a point $x \in X$ and a character $\phi \in \mathfrak{M}(A)$ such that $\tau(f) = \phi(f(x))$, $f \in \mathscr{C}(X, A)$, from which we get $\mathfrak{M}(\mathscr{C}(X, A)) = X \times \mathfrak{M}(A)$. In general, for an admissible Banach A-valued function algebra $\mathscr{A}$ on X, take characters $\psi \in \mathfrak{M}(\mathfrak{A})$ and $\phi \in \mathfrak{M}(A)$ and define $\psi \diamond \phi : \mathscr{A} \to \mathbb{C}$ by $\psi \diamond \phi(f) = \psi(\phi \circ f)$. Then $\psi \diamond \phi$ is a character of $\mathscr{A}$. In [2], it is shown that, under certain conditions, the converse is also true; that is every character $\tau \in \mathscr{A}$ is of the form $\tau = \psi \diamond \phi$. In this case, $\mathfrak{M}(\mathscr{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$.

For all $x \in X$, the evaluation homomorphisms $\mathcal{E}_x : \mathscr{A} \to A$, given by $\mathcal{E}_x(f) = f(x)$ are included in the class of vector-valued character of $\mathscr{A}$. According to [1], a homomorphism $\Psi : \mathscr{A} \to A$ is called a *vector-valued character*, or an *A-character*, of $\mathscr{A}$, if $\Psi(\mathbf{1}) = \mathbf{1}$ and $\phi(\Psi f) = \Psi(\phi \circ f)$, for all $f \in \mathscr{A}$ and $\phi \in \mathfrak{M}(A)$. If $\Psi : \mathscr{A} \to A$ is an A-character of $\mathscr{A}$, then the restriction $\psi = \Psi|_{\mathfrak{A}}$ is a character of $\mathfrak{A}$. It is natural to ask whether there is, for every character $\psi : \mathfrak{A} \to \mathbb{C}$, an A-character $\Psi : \mathscr{A} \to A$ such that $\psi = \Psi|_{\mathfrak{A}}$. If this is the case, then it is said that ψ *lifts to* Ψ, [1].

Throughout the paper, unless explicitly stated, A is assumed to be a unital semisimple commutative Banach algebra and $\mathscr{A}$ is an admissible Banach A-valued function algebra on a compact space X such that every character $\psi : \mathfrak{A} \to \mathbb{C}$ lifts to some A-character $\Psi : \mathscr{A} \to A$.

3 Character Contractibility

In this section, we investigate character contractibility of vector-valued function algebras. First, a remark on contractibility is in order. Suppose that the Banach A-valued function algebra $\mathscr{A}$ is contractible (i.e., its first cohomological group $H^1(\mathscr{A}, \mathcal{X}) = \{0\}$, for all Banach $\mathscr{A}$-bimodule $\mathcal{X}$). Take a point $x \in X$ and a character $\phi \in \mathfrak{M}(A)$, and consider continuous surjective homomorphisms $\mathcal{E}_x : \mathscr{A} \to A$ defined by $\mathcal{E}_x(f) = f(x)$, and $\Phi : \mathscr{A} \to \mathfrak{A}$ defined by $\Phi(f) = \phi \circ f$, for all $f \in \mathscr{A}$. By [20, Chapter 4], we get that both A and $\mathfrak{A}$ are contractible. The converse of the above observation holds true in certain circumstances. For instance, if $\mathscr{A} = \mathfrak{A}\hat{\otimes}_\gamma A$, then $\mathscr{A}$ is contractible whenever $\mathfrak{A}$ and A are contractible. In fact, suppose that $\mathfrak{A}$ and A are contractible. Since, by [5, Proposition 2.8.64], the projective tensor product $\mathfrak{A}\hat{\otimes}_\pi A$ is contractible and the identity mapping $\mathfrak{A} \otimes A \to \mathfrak{A}\hat{\otimes}_\gamma A$ extends to a homomorphism of $\mathfrak{A}\hat{\otimes}_\pi A$ onto a dense subalgebra of $\mathfrak{A}\hat{\otimes}_\gamma A$, we get that $\mathfrak{A}\hat{\otimes}_\gamma A$ is contractible. It is well-know that the algebra $\mathscr{C}(X, A)$ is isometrically isomorphic with the injective tensor product $\mathscr{C}(X)\hat{\otimes}_\epsilon A$; [20, Theorem B.2.5]. This yields that $\mathscr{C}(X, A)$ is contractible if and only if $\mathscr{C}(X)$ and A are contractible, if and only if X is finite and A is finite dimensional, in which case $\mathscr{C}(X, A) = \mathbb{C}^n$, for some n.

A necessary condition for discreteness of the character space $\mathfrak{M}(A)$ with respect to the weak topology is presented in [11], where it is shown that $\mathfrak{M}(A)$ is discrete if A is ϕ-amenable, for all $\phi \in \mathfrak{M}(A)$. Applying Shilov idempotent theorem, we have the following result. Recall that $\phi \in \mathfrak{M}(A)$ is an isolated point, if $U = \{\phi\}$ is an open set in the Gelfand topology of $\mathfrak{M}(A)$.

Proposition 3.1. For $\phi \in \mathfrak{M}(A)$, the algebra A is ϕ-contractible if and only if ϕ is an isolated point of $\mathfrak{M}(A)$.

Proof . Suppose that A is ϕ-contractible. By [9, Theorem 6.4], $\ker\phi$ has an identity u, say. This means that $\phi(u) = 0$ and $ua = a$, for all $a \in \ker\phi$. Take $\psi \in \mathfrak{M}(A)$ with $\psi \neq \phi$. Take an element $a \in \ker\phi$ with $\psi(a) \neq 0$. Then $\psi(u)\psi(a) = \psi(ua) = \psi(a)$, so that $\psi(u) = 1$. Let

$$U = \{\psi \in \mathfrak{M}(A) : |\psi(u) - \phi(u)| < 1\}.$$

We see that $U = \{\phi\}$ is a neighborhood of ϕ, and thus ϕ is an isolated point of $\mathfrak{M}(A)$.

Conversely, assume that ϕ is an isolated point of $\mathfrak{M}(A)$, so that $\{\phi\}$ and $\mathfrak{M}(A) \setminus \{\phi\}$ are two disjoint closed sets in $\mathfrak{M}(A)$. By Shilov theorem on idempotents, e.g. [13, Section 3.5], A contains an element u such that $\hat{u}(\phi) = 0$ and $\hat{u}(\psi) = 1$, for $\psi \neq \phi$. Therefore, $\hat{u}\hat{a} = \hat{a}$, for all $a \in \ker\phi$. Since A is semisimple, $ua = a$ for all $a \in \ker\phi$, and u is an identity for $\ker\phi$. Using [9, Theorem 6.4], we conclude that A is ϕ-contractible. □

Using the above result, we study character contractibility of vector-valued function algebras in terms of isolated points.

Theorem 3.2. For the Banach A-valued function algebra $\mathscr{A}$, let $\phi \in \mathfrak{M}(A)$, $\psi \in \mathfrak{M}(\mathfrak{A})$. Then $\mathscr{A}$ is $\psi \diamond \phi$-contractible if and only if A is ϕ-contractible and $\mathfrak{A}$ is ψ-contractible.

Proof . Applying Proposition 3.1, we just need to show that $\psi \diamond \phi$ is an isolated point of $\mathfrak{M}(\mathscr{A})$ if and only if ϕ is an isolated point of $\mathfrak{M}(A)$, and ψ is an isolated point of $\mathfrak{M}(\mathfrak{A})$.

First, suppose that $\tau = \psi \diamond \phi$ is an isolated point of $\mathfrak{M}(\mathscr{A})$. There exist $f_1, \ldots, f_n \in \mathscr{A}$ and $\epsilon > 0$ such that

$$W = \{\tau' \in \mathfrak{M}(\mathscr{A}) : |\tau'(f_i) - \tau(f_i)| < \epsilon, i = 1, \ldots, n\} = \{\tau\}.$$

Set $g_i = \phi \circ f_i$, $i = 1, \ldots, n$, and

$$V = \{\psi' \in \mathfrak{M}(\mathfrak{A}) : |\psi'(g_i) - \psi(g_i)| < \epsilon, i = 1, \ldots, n\}.$$

If $\psi' \neq \psi$, then $\psi' \diamond \phi \neq \tau$ and thus, for some $i = 1, \ldots, n$,

$$|\psi'(g_i) - \psi(g_i)| = |\psi' \diamond \phi(f_i) - \tau(f_i)| \geq \epsilon.$$

Therefore, $\psi' \notin V$ and $V = \{\psi\}$. To show that ϕ is an isolated point in $\mathfrak{M}(A)$, assume that ψ lifts to the A-character $\Psi : \mathscr{A} \to A$. Set $a_i = \Psi(f_i)$, for $i = 1, \ldots, n$ and

$$U = \{\phi' \in \mathfrak{M}(A) : |\phi'(a_i) - \phi(a_i)| < \epsilon, i = 1, \ldots, n\}.$$

If $\phi' \neq \phi$, then $|\phi'(a_i) - \phi(a_i)| \geq \epsilon$, and therefore, $U = \{\phi\}$.

Conversely, suppose that $\phi \in \mathfrak{M}(A)$ and $\psi \in \mathfrak{M}(\mathfrak{A})$ are isolated points. There exist $a_1, \ldots, a_n \in A$ and $g_1, \ldots, g_n \in \mathfrak{A}$ and $\epsilon > 0$ such that

$$U = \{\phi' \in \mathfrak{M}(A) : |\phi'(a_i) - \phi(a_i)| < \epsilon, i = 1, \ldots, n\} = \{\phi\},$$
$$V = \{\psi' \in \mathfrak{M}(\mathfrak{A}) : |\psi'(g_i) - \psi(g_i)| < \epsilon, i = 1, \ldots, n\} = \{\psi\}.$$

Set $S = \{a_1, \ldots, a_n, g_1, \ldots, g_n\}$, and

$$W = \{\tau' \in \mathfrak{M}(\mathscr{A}) : |\tau'(f) - \tau(f)| < \epsilon, f \in S\}.$$

Then $W = \{\tau\}$. □

Since every tensor product $\mathfrak{A}\hat{\otimes}\gamma A$ can be seen as a Banach A-valued function algebra on $\mathfrak{M}(\mathfrak{A})$, we have the following.

Corollary 3.3. Let γ be an algebra cross-norm on $\mathfrak{A} \otimes A$. For $\phi \in \mathfrak{M}(A)$ and $\psi \in \mathfrak{M}(\mathfrak{A})$, the algebra $\mathfrak{A}\hat{\otimes}\gamma A$ is $\psi \otimes \phi$-contractible if and only if A is ϕ-contractible and $\mathfrak{A}$ is ψ-contractible.

Special case of the above corollary is the algebra $\mathscr{C}(X, A) = \mathscr{C}(X)\hat{\otimes}_\epsilon A$. Therefore, $\mathscr{C}(X, A)$ is $\epsilon_x \diamond \phi$-contractible if and only if $x \in X$ is an isolated point and A is ϕ-contractible.

By Proposition 3.1, the notions of contractibility and character contractibility coincide, for the class of unital commutative Banach algebras.

Corollary 3.4. Suppose that A has an identity. The Banach A-valued function algebra $\mathscr{A}$ is (character) contractible if and only if A and $\mathfrak{A}$ are (character) contractible.

4 Character Amenability

In this section, the character amenability of vector-valued function algebras are investigated. Recall, from [12], that a Banach algebra A is ϕ_0-amenable, $\phi_0 \in \mathfrak{M}(A)$, if A has a ϕ_0-mean in A^{**}; that is, an element $M \in A^{**}$ such that $a \cdot M = \phi_0(a)M$, for all $a \in A$. Here, the module action of A on A^{**} is given by $(a \cdot M)(\phi) = M(\phi \cdot a)$, for all $\phi \in A^*$, where $(\phi \cdot a)(x) = \phi(ax)$ for all $x \in A$.

Proposition 4.1. For the Banach A-valued function algebra $\mathscr{A}$, let $\tau_0 \in \mathfrak{M}(\mathscr{A})$ and set $\phi_0 = \tau_0|_A$ and $\psi_0 = \tau_0|_{\mathfrak{A}}$. If $\mathscr{A}$ is τ_0-amenable, then A is ϕ_0-amenable and $\mathfrak{A}$ is ψ_0-amenable.

Proof . Suppose that $\mathscr{A}$ is τ_0-amenable and that $\mathbf{M} : \mathscr{A}^* \to \mathbb{C}$ is a τ_0-mean for $\mathscr{A}$. Define the linear map $M : A^* \to \mathbb{C}$ by $M(\phi) = \mathbf{M}(\psi_0 \diamond \phi)$ for all $\phi \in A^*$. Now, $M(\phi_0) = \mathbf{M}(\tau_0) = 1$, and, for any $a \in A$, $\phi \in A^*$, we have

$$\begin{aligned} M(\phi \cdot a) &= \mathbf{M}(\psi_0 \diamond (\phi \cdot a)) = \mathbf{M}((\psi_0 \diamond \phi) \cdot a) \\ &= \tau_0(a)\mathbf{M}(\psi_0 \diamond \phi) = \phi_0(a)M(\phi). \end{aligned}$$

Therefore, M is a ϕ_0-mean for A, showing that it is ϕ_0-amenable. Similarly, we see that $\mathfrak{A}$ is ψ_0-amenable. □

It is natural to ask whether the converse of the above result holds; that is, whether A being ϕ-amenable and $\mathfrak{A}$ being ψ-amenable imply that $\mathscr{A}$ is $\psi \diamond \phi$-amenable. Under certain conditions, affirmative answers to this question are presented in the sequel.

The following is a generalization of [12, Theorem 3.3].

Proposition 4.2. Suppose that $\mathfrak{A}A$, the subalgebra of $\mathscr{A}$ generated by $\mathfrak{A}\cup A$, is dense in $\mathscr{A}$. If A is ϕ-amenable and $\mathfrak{A}$ is ψ-amenable, then $\mathscr{A}$ is $\psi\diamond\phi$-amenable. In particular, $\mathfrak{A}\hat{\otimes}_\gamma A$, for an algebra cross-norm γ, is $\psi\otimes\phi$-amenable if and only if $\mathfrak{A}$ is ψ-amenable and A is ϕ-amenable.

Proof . Assume that A is ϕ-amenable and that $\mathfrak{A}$ is ψ-amenable. By [12, Theorem 1.4], there exist bounded nets $(g_\alpha)_{\alpha\in I}$ in $\mathfrak{A}$ and $(a_\beta)_{\beta\in J}$ in A such that,

$$\psi(g_\alpha)=1, \qquad \|gg_\alpha-\psi(g)g_\alpha\|\to 0, \qquad (g\in\mathfrak{A}),$$
$$\phi(a_\beta)=1, \qquad \|aa_\beta-\phi(a)a_\beta\|\to 0, \qquad (a\in A).$$

Make $I\times J$ a directed set in a traditional way. For every $\gamma=(\alpha,\beta)$ in $I\times J$, define $f_\gamma(x)=g_\alpha(x)a_\beta$ $(x\in X)$. Also set $\tau=\psi\diamond\phi$. Then

$$\tau(f_\gamma)=(\psi\diamond\phi)(g_\alpha a_\beta)=\psi(g_\alpha)\phi(a_\beta)=1.$$

We just need to show that $\|ff_\gamma-\tau(f)f_\gamma\|\to 0$, for all $f\in\mathscr{A}$. First of all, assume that $f=ga$, with $g\in\mathfrak{A}$ and $a\in A$.

$$\begin{aligned}\|ff_\gamma-\tau(f)f_\gamma\| &= \|fg_\alpha aa_\beta-\psi(f)\phi(a)g_\alpha a_\beta\|\\ &= \|(fg_\alpha-\psi(f)g_\alpha)aa_\beta-\psi(f)g_\alpha(aa_\beta-\phi(a)a_\beta)\|\\ &\le \|(fg_\alpha-\psi(f)g_\alpha)aa_\beta\|+\|\psi(f)g_\alpha(aa_\beta-\phi(a)a_\beta)\|\\ &\le \|fg_\alpha-\psi(f)g_\alpha\|\|aa_\beta\| \;+\|fg_\alpha\|\|aa_\beta-\phi(a)a_\beta\|\\ &\to 0.\end{aligned}$$

Next, assume that $g\in\mathfrak{A}A$ is a finite sum of the form $g=\sum_{i=1}^n f_ia_i$. It is easily verified that $\|gf_\gamma-\tau(g)f_\gamma\|\to 0$. Finally, assume that $f\in\mathscr{A}$ is arbitrary. Given $\epsilon>0$, there exist $f_1,\dots,f_n$ in $\mathfrak{A}$ and $a_1,\dots,a_n$ in A such that, for $g=\sum_{i=1}^n f_ia_i$, we have $\|f-g\|<\epsilon$. Therefore,

$$\begin{aligned}\|ff_\gamma-\tau(f)f_\gamma\| &\le \|(f-g)f_\gamma-\tau(f-g)f_\gamma\|+\|gf_\gamma-\tau(g)f_\gamma\|\\ &\le 2\|f-g\|\|f_\gamma\|+\|gf_\gamma-\tau(g)f_\gamma\|\\ &\le 2K\epsilon+\|gf_\gamma-\tau(g)f_\gamma\|,\end{aligned}$$

where $\|f_\gamma\|\le K$, for all γ. Therefore

$$\limsup_\gamma \|ff_\gamma-\tau(f)f_\gamma\|\le\epsilon.$$

Since $\epsilon>0$ is arbitrary, we get $\lim_\gamma\|ff_\gamma-\tau(f)f_\gamma\|=0$. □

The following is also immediate from Proposition 4.2. Here, A is not assumed to have an identity.

Corollary 4.3. For $x\in X$ and $\phi\in\mathfrak{M}(A)$, the algebra $\mathscr{C}(X,A)$ is $\epsilon_x\diamond\phi$-amenable if and only if A is ϕ-amenable. Consequently, $\mathscr{C}(X,A)$ is character amenable if and only if A is character amenable.

Proof . We just need to prove that if A has a bounded right approximate identity (e_α) then so does $\mathscr{C}(X,A)$. Considering each e_α as a constant function $X\to A$, we show that

$$\|f-fe_\alpha\|_X\to 0,\;\; f\in\mathscr{C}(X,A). \tag{4.1}$$

Towards a contradiction, suppose that (4.1) fails, for some $f\in\mathscr{C}(X,A)$. Then, for some $\epsilon>0$ there exist a subnet (e_β) of (e_α) and a net (x_β) of points in X such that $\|f(x_\beta)-f(x_\beta)e_\beta\|\ge\epsilon$. Since X is compact, without loss of generality, we may assume that $x_\beta\to x_0$, for some $x_0\in X$. Now, we get

$$\begin{aligned}\epsilon\le\|f(x_\beta)-f(x_\beta)e_\beta\| &\le \|f(x_\beta)-f(x_0)\|+\|f(x_0)-f(x_0)e_\beta\|\\ &\quad+\|f(x_0)e_\beta-f(x_\beta)e_\beta\|\to 0,\end{aligned}$$

which is absurd. Therefore, (e_α) is a bounded right approximate identity for $\mathscr{C}(X,A)$. □

To continue our investigation, let us present a result for arbitrary Banach algebras. In the following, A may not be commutative nor unital.

Theorem 4.4. Suppose that A has a bounded right approximate identity. Then, for every $\phi \in \mathfrak{M}(A)$, the following are equivalent.

1. There is a bounded net (u_α) in $\ker\phi$ such that $\hat{u}_\alpha \to 1$ on $\mathfrak{M}(A) \setminus \{\phi\}$.
2. There is $M \in \overline{\ker\phi}^{\mathrm{w}^*}$ such that $a \cdot M = a$ on $\mathfrak{M}(A)$, for all $a \in \ker\phi$.

If either $\mathfrak{M}(A)$ generates A^*, or A is an ideal of A^{**}, then the above statements are equivalent to

3. The algebra A is ϕ-amenable.

Proof . Assume that (1) holds. Since (u_α) is bounded, we may assume, without loss of generality, that $u_\alpha \to M$ in the weak* topology, for some $M \in A^{**}$. We show that $a \cdot M = a$ on $\mathfrak{M}(A)$, for all $a \in A$. If $\psi \in \mathfrak{M}(A)$ and $\psi \neq \phi$, then

$$\begin{aligned}(a \cdot M)(\psi) = M(\psi \cdot a) &= \lim_\alpha \hat{u}_\alpha(\psi \cdot a)\\ &= \lim_\alpha \psi(a)\hat{u}_\alpha(\psi) = \psi(a)\lim_\alpha \hat{u}_\alpha(\psi) = \psi(a).\end{aligned}$$

Also, $(a \cdot M)(\phi) = M(\phi \cdot a) = \phi(a)M(\phi)$. We conclude that $a \cdot M = a$ on $\mathfrak{M}(A)$, for all $a \in A$. Therefore (2) holds.

Conversely, assume that (2) holds. Take a bounded net (u_α) in $\ker\phi$ such that $u_\alpha \to M$ in the weak* topology (Goldstein theorem). Let $\psi \in \mathfrak{M}(A) \setminus \{\phi\}$. Then, there is $a \in \ker\phi$ such that $\psi(a) = 1$. Since $au_\alpha \to a$ in the weak topology of A, we see that $\psi(a)\psi(u_\alpha) \to \psi(a)$ so that $\psi(u_\alpha) \to 1$. This means that $\hat{u}_\alpha \to 1$ on $\mathfrak{M}(A) \setminus \{\phi\}$ and thus (1) holds.

Notice that (3) $\Rightarrow$ (2) always holds. Indeed, if A is ϕ-amenable then, by [12, Proposition 2.2], $\ker\phi$ has a bounded right approximate identity (u_α). We may assume, without loss of generality, that $u_\alpha \to M$ in the weak* topology, for some $M \in A^{**}$. Therefore, $M \in \overline{\ker\phi}^{\mathrm{w}^*}$ and $a \cdot M = a$, for all $a \in \ker\phi$.

Finally, to complete the proof, assume that (2) holds. In the case that $\mathfrak{M}(A)$ generates A^*, then $a \cdot M = a$ on A^*, for all $a \in A$. In other case where A is an ideal of A^{**}, then $a \cdot M \in A$ and, since A is semisimple, we get $a \cdot M = a$ in A, so that $a \cdot M = a$ on A^*. Now, take a bounded net (u_α) in $\ker\phi$ such that $u_\alpha \to M$ in the weak* topology (Goldstein Theorem). For every $a \in \ker\phi$ and $\psi \in A^*$, we have

$$\lim_\alpha \psi(au_\alpha) = \lim_\alpha (\psi \cdot a)(u_\alpha) = M(\psi \cdot a) = (a \cdot M)(\psi) = \psi(a).$$

This means that (u_α), is a bounded weak right approximate identity for $\ker\phi$. By [5, Proposition 2.9.14], $\ker\phi$ has a bounded right approximate identity. We conclude, from [12, Proposition 2.2], that A is ϕ-amenable. □

Remark 4.5. Obviously, ϕ-amenability implies ϕ-contractibility. The converse is also true if A is an ideal of A^{**}. In fact, if $M \in A^{**}$ is an invariant ϕ-mean and a is an element of A with $\phi(a) = 1$, then $a \cdot M$ is an invariant ϕ-mean in A. Also note that, if A has an identity, then A is character contractible if and only if A is character amenable.

The above discussion leads to the following theorem as another answer to our question.

Theorem 4.6. Suppose that either $\mathfrak{M}(\mathscr{A})$ generates $\mathscr{A}^*$ or $\mathscr{A}$ is an ideal of $\mathscr{A}^{**}$. For $\phi \in \mathfrak{M}(A)$ and $\psi \in \mathfrak{M}(\mathfrak{A})$, if A is ϕ-amenable and $\mathfrak{A}$ is ψ-amenable, then $\mathscr{A}$ is $\psi \diamond \phi$-amenable.

Proof . The algebra A being ϕ-amenable implies that there is a net $(u_\alpha)_{\alpha \in I}$ in A such that $\hat{u}_\alpha(\phi) = 0$ and $\hat{u}_\alpha \to 1$ on $\mathfrak{M}(A) \setminus \{\phi\}$. Similarly, for $\mathfrak{A}$, there is a net $(v_\beta)_{\beta \in J}$ in $\mathfrak{A}$ such that $\hat{v}_\beta(\psi) = 0$ and $\hat{v}_\beta \to 1$ on $\mathfrak{M}(\mathfrak{A}) \setminus \{\psi\}$. Set $U_\lambda = u_\alpha + v_\beta - u_\alpha v_\beta$, where $\lambda = (\alpha, \beta)$ runs over the suitably directed set $I \times J$. It is easily verified that $\tau(U_\lambda) = 0$ and that $\tau'(U_\lambda) \to 1$, for $\tau' \neq \tau$. □

We provide some examples to support our results.

Example 4.7. Let S be any nonempty set, and, for $1 < p < \infty$, let $A = \ell^p(S)$. It is well-know that $A^* = \ell^q(S)$ with $1/p + 1/q = 1$ and $A^{**} = A$. Define multiplication on A pointwise; that is, $(fg)(s) = f(s)g(s)$, for all $s \in S$. It is a matter of calculation to verify that $(A, \|\cdot\|_p)$ is a commutative Banach algebra;

$$\sum_{s \in S} |f(s)|^p |(g(s)|^p \leq \sum_{s \in S} |f(s)|^p \sum_{s \in S} |g(s)|^p.$$

For every $s \in S$, the evaluation homomorphism $\phi_s : f \mapsto f(s)$ is a character of A. Conversely, let $\phi : A \to \mathbb{C}$ be a character of A. Note that the set $\{\chi_s : s \in S\}$ of all the characteristic functions generates A; that is,

$$f = \sum_{s \in S} f(s)\chi_s \quad (f \in A).$$

Therefore, $\phi(f) = \sum_{s\in S} f(s)\phi(\chi_s)$. Since $\phi \neq 0$, there must be a point $s_0 \in S$ such that $\phi(\chi_{s_0}) \neq 0$. If $s \neq s_0$, then $\phi(\chi_s)\phi(\chi_{s_0}) = \phi(\chi_s\chi_{s_0}) = 0$ so that $\phi(\chi_s) = 0$. Also $\phi(\chi_{s_0})^2 = \phi(\chi_{s_0})$ and thus $\phi(\chi_{s_0}) = 1$. We therefore have

$$\phi(f) = \sum_{s \in S} f(s)\phi(\chi_s) = f(s_0) = \phi_{s_0}(f), \quad (f \in A).$$

We conclude that $\mathfrak{M}(A)$ is homeomorphic with S. Since $\{\phi_s : s \in S\}$ separates the points of $A^{**} = A$, one can see that $\mathfrak{M}(A)$ generates A^*.

Our next example concerns vector-valued function algebras.

Example 4.8. Let X be a compact Hausdorff space, A be a unital Banach algebra, and $\mathscr{A} = \mathscr{C}(X, A)$. It is well-known that the dual space $\mathscr{A}^*$ is the space $M(X, A^*)$ of all regular Borel vector measures μ on X to A^*, with finite variation $|\mu|$; see [4] and references therein. If X is dispersed, in the sense that any non-empty subset of X contains at least one isolated point, then $M(X) \times A^*$ generates $\mathscr{A}^*$, [4, Theorem 1]. If, in addition, $\mathfrak{M}(A)$ generates A^*, we see that $\mathfrak{M}(\mathscr{A}) = X \times \mathfrak{M}(A)$ generates $\mathscr{A}^*$.

For instance, take $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $A = \ell^p(S)$ from Example 4.7. Then $\mathfrak{M}(\mathscr{A})$ generates $\mathscr{A}^*$, for $\mathscr{A} = \mathscr{C}(X, A)$.

For the sake of completeness, at the end of this section, let us bring a result analogous with Theorem 4.4, related to pseudo ϕ-amenable Banach algebras. Recall, from [9, 17], that a Banach algebra A is *pseudo ϕ-amenable*, $\phi \in \mathfrak{M}(A)$, if A has an approximate ϕ-diagonal; that is, a net (u_α) in $A\hat{\otimes}_\pi A$ such that

$$a \cdot u_\alpha - \phi(a)u_\alpha \to 0,\ \phi(\pi(u_\alpha)) \to 1,\ a \in A.$$

Proposition 4.9. Suppose that A is an arbitrary Banach algebra with a right approximate identity. Take $\phi \in \mathfrak{M}(A)$ and consider the following statements.

1. The algebra A is pseudo ϕ-amenable.
2. $\ker\phi$ has a right approximate identity.
3. There is a net (u_α) in $\ker\phi$ such that $\hat{u}_\alpha \to 1$ on $\mathfrak{M}(A) \setminus \{\phi\}$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof . (1) $\Rightarrow$ (2): Let (e_β) be a right approximate identity for A. By [17], A has an approximate ϕ-mean (a_α). For $\gamma = (\alpha, \beta)$, let $u_\gamma = e_\beta - \phi(e_\beta)a_\alpha$. Then

$$\phi(u_\gamma) = \phi(e_\beta) - \phi(e_\beta)\phi(a_\alpha) = \phi(e_\beta) - \phi(e_\beta) = 0,$$

and thus $u_\gamma \in \ker\phi$, for all γ. Also, for $u \in \ker\phi$, we have

$$\begin{aligned} uu_\gamma - u &= u(e_\beta - \phi(e_\beta)a_\alpha) - u \\ &= ue_\beta - \phi(e_\beta)ua_\alpha - u \\ &= ue_\beta - \phi(e_\beta)ua_\alpha - u + \phi(e_\beta)\phi(u)a_\alpha \\ &= ue_\beta - u - \phi(e_\beta)(ua_\alpha - \phi(u)a_\alpha) \to 0. \end{aligned}$$

(2) $\Rightarrow$ (1): Let $(u_\alpha) \subset \ker\phi$ be a right approximate identity for $\ker\phi$. Take an element $a_0 \in A$ with $\phi(a_0) = 1$ and set $a_\alpha = a_0 - a_0u_\alpha$. Then $\phi(a_\alpha) = 1$, and

$$\begin{aligned} aa_\alpha - \phi(a)a_\alpha &= a(a_0 - a_0u_\alpha) - \phi(a)(a_0 - a_0u_\alpha) \\ &= aa_0 - \phi(a)a_0 - aa_0u_\alpha + \phi(a)a_0u_\alpha \\ &= (aa_0 - \phi(a)a_0) - (aa_0 - \phi(a)a_0)u_\alpha \\ &= b - bu_\alpha \to 0. \end{aligned}$$

(2) ⇒ (3): Let (u_α) in $\ker\phi$ be a right approximate identity for $\ker\phi$. Take an element $\psi \in \mathfrak{M}(A)$ with $\psi \neq \phi$. Then, there is $a \in \ker\phi$ such that $\psi(a) = 1$. Since $au_\alpha \to a$ in A, we have $\psi(a)\psi(u_\alpha) \to \psi(a)$, so that $\psi(u_\alpha) \to 1$. □

Similar to Theorem 4.6, we conjecture that if either $\mathfrak{M}(A)$ generates A^*, or A is an ideal of A^{**}, then all the statements in the above proposition are equivalent. However, at this point, we do not have any evidence to verify it.

We conclude the paper by presenting an example of an A-valued function algebra satisfying conditions in Proportion 4.9.

Example 4.10. In Example 4.7, let $S = \mathbb{N}$ and $p = 2$ and consider the Banach algebra $A = \ell^2(\mathbb{N})^\#$, the unitization of $\ell^2(\mathbb{N})$. Then A is pseudo ϕ^∞-amenable, where $\phi^\infty(f, \lambda) = \lambda$, for all $(f, \lambda) \in A$; see [17, Example 2.7]. Let X be a compact Hausdorff space having an isolated point x_0, and let $\mathscr{A} = \mathscr{C}(X, A)$. We show that $\mathscr{A}$ is pseudo τ_0-amenable, where $\tau_0 = \epsilon_{x_0} \diamond \phi^\infty$. Since A is pseudo ϕ^∞-amenable, by Proposition 4.9, $\ker\phi^\infty$ has a right approximate identity $\{u_\alpha\}$. Define, for every α, a function $f_\alpha : X \to A$, as follows;

$$f_\alpha(x) = \begin{cases} u_\alpha, & x = x_0; \\ \mathbf{1}_A, & x \neq x_0. \end{cases}$$

Then $f_\alpha \in \mathscr{C}(X, A)$ and $\tau_0(f_\alpha) = \phi^\infty(u_\alpha) = 0$ so that (f_α) is a net in $\ker\tau_0$. Moreover, for every $f \in \ker\tau_0$, we have $f(x_0) \in \ker\phi^\infty$ and thus

$$\|ff_\alpha - f\|_X = \|f(x_0)u_\alpha - f(x_0)\| \to 0.$$

Therefore, (f_α) is a right approximate identity for $\ker\tau_0$. By Proposition 4.9, $\mathscr{A}$ is pseudo τ_0-amenable

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References

[1] M. Abtahi, *Vector-valued characters on vector-valued function algebras*, Banach J. Math. Anal. **10** (2016), no. 3, 608–620.

[2] M. Abtahi, *On the character space of Banach vector-valued function algebras*, Bull. Iran. Math. Soc. **43** (2017), no. 5, 1195–1207.

[3] M. Abtahi and S. Farhangi. *Vector-valued spectra of Banach algebra valued continuous functions*, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A. Mat. **112** (2018), no. 1, 103–115.

[4] M. Cambern and P. Greim. *The bidual of $C(X, E)$*, Proc. Amer. Math. Soc. **85** (1982), no. 1, 53–58.

[5] H.G. Dales, *Banach Algebras and Automatic Continuity*, LMS Monographs 24, Clarenden Press, Oxford, 2000.

[6] M. Dashti, R. Nasr-Isfahani, and S. Soltani Renani, *Vector-valued invariant means on spaces of bounded linear maps*, Colloq. Math. **1** (2013), no. 132, 1–11.

[7] R. Ghamarshoushtari and Y .Zhang, *Amenability properties of Banach algebra valued continuous functions*, J. Math. Anal. Appl., **422** (2014), 1335–1341.

[8] A. Hausner, *Ideal in a certain Banach algebra*, Proc. Amer Math. Soc. **8** (1957), 246–249.

[9] Z. Hu, M. S. Monfared, and T. Tranor. *On Character amenability of Banach algebras*, **344** (2008), no. 2, 942–955.

[10] B.E. Johnson, *Cohomology in Banach Algebras*, Amer. Math. Soc., 1970.

[11] E. Kaniuth, A. Lau, and J. Pym. *On character amenability of Banach algebras*, J. Math. Anal. App.**344** (2008), 942–955.

[12] E. Kaniuth, A.T. Lau, and J. Pym. *On ϕ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc. **144** (2008), 85–96.

[13] E. Kaniuth, *A Course in Commutative Banach Algebras*, Graduate Texts in Mathematics, 246, Springer, 2009.

[14] A.T.-M. Lau, *The second conjugate algebra of the Fourier algebra of a locally compact group*. Trans. Amer. Math. Soc. **267** (1981), 53–63.

[15] A.T.-M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups.* Fund. Math. **118** (1983), no. 3, 161–175.

[16] M.S. Monfared, *Character amenability of Banach algebras,*. Math. Proc. Cambridge Philos. Soc. **144** (2008), 697–706.

[17] R. Nasr-Isfahani and M. Nemati, *Character pseudo-amenability of Banach algebras.* Colloq. Math, **132** (2013), 177–193.

[18] A. Nikou and A.G. O'Farrell. *Banach algebras of vector-valued functions.* Glasgow Math. J., **56** (2014), no. 2, 419–426.

[19] M. Ramezanpour, N. Tavallaei, and B. Olfatian Gillan. *Character amenability and contractibility of some Banach algebras on left coset spaces.* Annal. Func. Anal., **7** (2016), no. 4, 564–572.

[20] V. Runde, *Lectures on amenability.* Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2002.