Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 179–198
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29432.4163

Construction a novel generating functions for the products of certain numbers and orthogonal polynomials in several variables using the symmetric functions technique

Sana Krioui[a], Ali Boussayoud[b,*], Abdelaziz Bellagh[a]

[a]Mathematics Faculty, Department of Algebra and Number Theory, LATN Laboratory, USTHB, BP32, Bab Ezzouar, Algiers, Algeria

[b]LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria

(Communicated by Ali Jabbari)

Abstract

In this paper, we introduce novel generating functions of the products of k-Fibonacci numbers, k-Lucas numbers, k-Pell numbers, k-Jacobsthal numbers, k-Mersenne numbers and symmetric functions in multiple variables. Accordingly, the novel generating functions are assigned to the other orthogonal Chebyshev polynomials with symmetric functions in multiple variables.

Keywords: Symmetric functions, Generating functions, k-Fibonacci numbers, k-Lucas numbers, k-Pell numbers, k-Mersenne numbers, Chebishev polynomials
2020 MSC: Primary 05E05; Secondary 11B39

1 Introduction

Different applications of the generalization functions are assigned to several branches of mathematics and mathematical physics. Previous studies show that the most commonly used classical orthogonal polynomials are Fibonacci polynomials, Lucas polynomials, and Chebyshev polynomials of first, second kinds given by the following recurrences relations respectively [7, 9, 24].

$$\begin{aligned}
F_n(x) &= xF_{n-1}(x) + F_{n-2}(x) \quad n > 2,\ F_0(x) = 1,\ F_1(x) = x,\\
L_n(x) &= xL_{n-1}(x) + L_{n-2}(x) \quad n > 2,\ L_0(x) = 2,\ L_1(x) = x,\\
T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \quad n > 2,\ T_0(x) = 1,\ T_1(x) = x,\\
U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x) \quad n > 2,\ U_0(x) = 1,\ U_1(x) = 2x.
\end{aligned}$$

In [3, 15], The third and fourth kinds chebyshev polynomials are defined by

$$\begin{aligned}
V_n(x) &= 2xV_{n-1}(x) - V_{n-2}(x) \quad n > 2,\ V_0(x) = 1,\ V_1(x) = 2x - 1,\\
W_n(x) &= 2xW_{n-1}(x) - W_{n-2}(x) \quad n > 2,\ W_0(x) = 1,\ W_1(x) = 2x + 1.
\end{aligned}$$

*Corresponding author
Email addresses: sannakr92@gmail.com (Sana Krioui), aboussayoud@yahoo.fr (Ali Boussayoud), abellagh@hotmail.com (Abdelaziz Bellagh)

Received: January 2023 *Accepted:* July 2023

Now, the Jacobsthal polynomials $J_n(x)$ are defined in [2], by the following recurrence relation as

$$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x) \qquad n > 2, \;\; J_0(x) = 0, \;\; J_1(x) = 1.$$

Similarly, previous studies presented several kinds of generalizations of Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, and Mersenne numbers. One of the most recognised generalizations of these numbers is k-Fibonacci numbers $\{F_{k,n}\}_{n\in\mathbb{N}}$, k-Lucas numbers $\{L_{k,n}\}_{n\in\mathbb{N}}$, k-Pell numbers $\{P_{k,n}\}_{n\in\mathbb{N}}$, k-Jacobsthal numbers $\{J_{k,n}\}_{n\in\mathbb{N}}$, and k-Mersenne numbers $\{M_{k,n}\}_{n\in\mathbb{N}}$ given by the following recurrence [14, 16, 17, 18, 26].

$$\begin{aligned}
F_{k,n} &= kF_{k,n-1} + F_{k,n-2} \quad n > 2, \;\; F_{k,0} = 1, \;\; F_{k,1} = k,\\
L_{k,n} &= kL_{k,n-1} + L_{k,n-2} \quad n > 2, \;\; L_{k,0} = 2, \;\; L_{k,1} = k,\\
P_{k,n} &= 2P_{k,n-1} + kP_{k,n-2} \quad n > 2, \;\; P_{k,0} = 0, \;\; P_{k,1} = 1,\\
J_{k,n} &= kJ_{k,n-1} + 2J_{k,n-2} \quad n > 2, \;\; J_{k,0} = 0, \;\; J_{k,1} = 1,\\
M_{k,n} &= 3kM_{k,n-1} - 2M_{k,n-2} \quad n > 2, \;\; M_{k,0} = 0, \;\; M_{k,1} = 1.
\end{aligned}$$

In 1991, A. Lascoux and A. Abderrezzek defined the $\delta^1_{p_1p_2}$ operator, which was applied starting in 2013 by many researchers, as it enabled them to retrieve many identities and famous generating functions, using the symmetric functions technique. For more information see [5, 10, 11, 12, 13].

In 2022, [27] H.Zerrouk et all, applied this operator to the invertible formal series $\sum\limits_{n=0}^{\infty} h_n(E)p_1^n z^n$ of the alphabet E of cardinal four, where they came up with new results about the generating functions of the products of certain orthogonal numbers and polynomials.

In this paper, we generalize the last results by applying the same operator to the invertible formal series with a change in the cardinal of the alphabet to five so that we can obtain other generating functions for the products of certain orthogonal numbers and polynomials.

The rest of this paper is divided as follows: In section 2 we give a few basic of symmetric functions. In section 3, our main result that associate the symmetric function defined in the previous section with the symmetrizing operator is proved. Moreover, several previously recognized results on generating functions were unified due to this main theorem which is used to find out the products of the symmetric functions in various variables with k-Fibonacci, k-Lucas, k-Pell, and k-Jacobsthal numbers, and k-Mersenne numbers identities presented in section 4. Accordingly, section 5 represents the new generating functions for the products of some well-known polynomials with the symmetric function in multiple variables. In section 6, We present some theory applications and we end up with a conclusion in section 7.

2 Preliminaries

In this section, we introduce some necessary definitions on the symmetric functions which will be used throughout the paper. For the details see [6, 22, 23, 25, 28].

Definition 2.1. [20] Assume that k and n be two integers and $\{x_1, x_2, ..., x_n\}$ are set of given variable. The k-th elementary symmetric function and The k-th complete symmetric function are defined respectively as follows:

$$e_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1}x_2^{i_2}...x_n^{i_n} \qquad (0 \le k \le n),$$

with $i_j = 0$ or 1, $j = 1 \ldots n$.

$$h_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1}x_2^{i_2}...x_n^{i_n} \qquad (k \ge 0),$$

with $i_j \ge 0, j = 1 \ldots n$.

Remark 2.2. For $k > n$ or $k < 0$, we set $e_k(x_1, x_2, ..., x_n) = 0$ and $h_k(x_1, x_2, ..., x_n) = 0$.

Definition 2.3. [1] Considering two sets of undefined alphabets B and P, we define $S_n(B - P)$ as

$$\frac{\prod\limits_{p\in P}(1 - pz)}{\prod\limits_{b\in B}(1 - bz)} = \sum_{n=0}^{\infty} S_n(B - P)z^n, \tag{2.1}$$

with $S_n(B-P)=0$, for $n<0$.

Remark 2.4. Puting $B=\emptyset$ in (2.1), we get

$$\prod_{p\in P}(1-pz)=\sum_{n=0}^{\infty}S_n(-P)z^n.$$

In fact, the summation is specifically restricted to a finite number of non-zero terms.

$$\prod_{p\in P}(x-p)=S_n(x-P)=x^nS_0(-P)+x^{n-1}S_1(-P)+x^{n-2}S_2(-P)+\dots. \tag{2.2}$$

For $0\leq j\leq n$, the $S_j(-P)$ are the coefficients of the polynomials $S_n(x-P)$. These are therefore at the sign near the elementary symmetric functions of the alphabet P; who are void for $j\geq n$. Specially, when $P=\{p,p,p,...,p\}$ (we note $P=np$), we have $S_n(x-np)=(x-p)^n$.

The binomial coefficients are determined by the specialisation $p=1$, i.e $P=\{1,1,1,\dots\}$ (we note $P=n$):

$$S_j(-n)=(-1)^j\left(\begin{array}{c}n\\ j\end{array}\right)\text{ and }S_j(n)=\left(\begin{array}{c}n+j-1\\ j\end{array}\right),$$

and

$$S_j(-np)=(-p)^j\left(\begin{array}{c}n\\ j\end{array}\right).$$

Definition 2.5. [8] Given a function f on $\mathbb{R}^n$; the divided difference operator is defined by

$$\partial_{x_ix_{i+1}}f(x_1,...,x_i,x_{i+1},...,x_n)=\frac{f(x_1,...,x_{i-1},x_i,x_{i+1},...,x_n)-f(x_1,...,x_{i-1},x_{i+1},x_i,...,x_n)}{x_i-x_{i+1}}.$$

Definition 2.6. [21] Assume that n be a positive integer and $P=\{p_1,p_2\}$ are set of given variables, then, the n-th symmetric function $h_n(p_1,p_2)$ is defined by

$$h_n(p_1,p_2)=\frac{p_1^{n+1}-p_2^{n+1}}{p_1-p_2},$$

with

$$\begin{aligned}h_0(p_1,p_2)&=1,\\ h_1(p_1,p_2)&=p_1+p_2,\\ h_2(p_1,p_2)&=p_1^2+p_1p_2+p_2^2,\\ &\vdots\end{aligned}$$

Definition 2.7. [4] The symmetrizing operator $\delta^k_{p_1p_2}$ is defined as

$$\delta^k_{p_1p_2}f(p_1)=\frac{p_1^kf(p_1)-p_2^kf(p_2)}{p_1-p_2},\ \text{ for all } k\in\mathbb{N}_0. \tag{2.3}$$

If $f(p_1)=p_1$, then operator (2.3) gives

$$\delta^k_{p_1p_2}f(p_1)=\frac{p_1^{k+1}-p_2^{k+2}}{p_1-p_2}=S_k(p_1+p_2)=h_k(p_1,p_2).$$

Proposition 2.8. Assuming $P=\{p_1,p_2\}$ is an alphabet, we have the operator $\delta^k_{p_1p_2}$ as

$$\delta^k_{p_1p_2}f(p_1)=h_{k-1}(p_1,p_2)f(p_1)+p_2^k\partial_{p_1p_2}(f).$$

3 Main theorems

In this section, we state and prove our new main theorems on the complete symmetric functions.

Theorem 3.1. Given two alphabets P=$\{p_1, p_2\}$ and A=$\{a_1, a_2, a_3, a_4, a_5\}$, we obtain

$$\sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5) h_n(p_1, p_2) z^n = \frac{\left[\begin{array}{c} e_0^{(5)} - p_1p_2e_2^{(5)}z^2 + p_1p_2(p_1+p_2)e_3^{(5)}z^3 \\ -p_1p_2\left[(p_1+p_2)^2 - p_1p_2\right]e_4^{(5)}z^4 \\ +p_1p_2(p_1+p_2)\left[(p_1+p_2)^2 - 2p_1p_2\right]e_5^{(5)}z^5 \end{array}\right]}{\prod_{i=1}^{5}(1-a_ip_1z)\prod_{i=1}^{5}(1-a_ip_2z)}. \tag{3.1}$$

Proof . Using the operator $\delta^1_{p_1p_2}$ to the series $f(p_1z) = \sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n$, we get

$$\begin{aligned} \delta^1_{p_1p_2} f(p_1z) &= \frac{p_1 \sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n - p_2 \sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n}{p_1 - p_2} \\ &= \sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5)\frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2} z^n \\ &= \sum_{n=0}^{\infty} h_n(a_1, a_2, a_3, a_4, a_5)h_n(p_1, p_2)z^n. \end{aligned}$$

On the other hand, since

$$f(p_1z) = \frac{1}{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n},$$

we have

$$\begin{aligned} \partial_{p_1p_2} f(p_1z) &= \frac{1}{p_1 - p_2}\left[\begin{array}{c} \frac{1}{\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_1^n z^n} \\ -\frac{1}{\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_2^n z^n} \end{array}\right] \\ &= \frac{\frac{1}{p_1-p_2}\left[\begin{array}{c} \sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n \\ -\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n \end{array}\right]}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n\right)} \\ &= \frac{\frac{1}{p_1-p_2}\left[\begin{array}{c} \sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n \\ -\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n \end{array}\right]}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n\right)} \\ &= \frac{-\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)h_{n-1}(p_1, p_2)z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1, a_2, a_3, a_4, a_5)p_2^n z^n\right)}. \end{aligned}$$

It follows that, according to Proposition 2.8,

$$\delta^1_{p_1p_2}f(p_1z)=\left[\begin{array}{c}\dfrac{h_0(p_1,p_2)}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)}\\ -p_2\dfrac{\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n\right)}\end{array}\right]$$

$$=\frac{\left[\begin{array}{c}h_0(p_1,p_2)\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_2z^n\\ -p_2\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n\end{array}\right]}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n\right)}$$

$$=\frac{\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,a_3,a_4,a_5)\left[h_0(p_1,p_2)p_2^n-p_2h_{n-1}(p_1,p_2)\right]z^n}{\prod\limits_{i=1}^{5}(1-a_ip_1z)\prod\limits_{i=1}^{5}(1-a_ip_2z)}$$

$$=\frac{\left[\begin{array}{c}e_0^{(5)}\left[h_0(p_1,p_2)-p_2h_{-1}(p_1,p_2)\right]\\ -e_1^{(5)}\left[h_0(p_1,p_2)p_2-p_2h_0(p_1,p_2)\right]z\\ +e_2^{(5)}\left[h_0(p_1,p_2)p_2^2-p_2h_1(p_1,p_2)\right]z^2\\ -e_3^{(5)}\left[h_0(p_1,p_2)p_2^3-p_2h_2(p_1,p_2)\right]z^3\\ +e_4^{(5)}\left[h_0(p_1,p_2)p_2^4-p_2h_3(p_1,p_2)\right]z^4\\ -e_5^{(5)}\left[h_0(p_1,p_2)p_2^4-p_2h_4(p_1,p_2)\right]z^5\end{array}\right]}{\prod\limits_{i=1}^{5}(1-a_ip_1z)\prod\limits_{i=1}^{5}(1-a_ip_2z)}$$

$$=\frac{\left[\begin{array}{c}e_0^{(5)}+e_2^{(5)}\left[p_2^2-p_2(p_1+p_2)\right]z^2\\ -e_3^{(5)}\left[p_2^3-p_2(p_1^2+p_1p_2+p_2^2)\right]z^3\\ +e_4^{(5)}\left[p_2^4-p_2(p_1^3+p_1^2p_2+p_1p_2^2+p_2^3)\right]z^4\\ -e_5^{(5)}\left[p_2^4-p_2(p_1^4+p_1^3p_2+p_1^2p_2^2+p_1p_2^3+p_2^4)\right]z^5\end{array}\right]}{\prod\limits_{i=1}^{5}(1-a_ip_1z)\prod\limits_{i=1}^{5}(1-a_ip_2z)}$$

$$=\frac{\left[\begin{array}{c}1-p_1p_2e_2^{(5)}z^2+p_1p_2(p_1+p_2)e_3^{(5)}z^3\\ -p_1p_2\left[(p_1+p_2)-p_1p_2\right]e_4^{(5)}z^4\\ +p_1p_2\left(p_1+p_2\right)\left[(p_1+p_2)^2-2p_1p_2\right]e_5^{(5)}z^5\end{array}\right]}{\prod\limits_{i=1}^{5}(1-a_ip_1z)\prod\limits_{i=1}^{5}(1-a_ip_2z)}.$$

As a result we have

$$\sum_{n=0}^{\infty}h_n(a_1,a_2,a_3,a_4,a_5)h_n(p_1,p_2)z^n=\frac{\left[\begin{array}{c}1-p_1p_2e_2^{(5)}z^2+p_1p_2(p_1+p_2)e_3^{(5)}z^3\\ -p_1p_2\left[(p_1+p_2)-p_1p_2\right]e_4^{(5)}z^4\\ +p_1p_2\left(p_1+p_2\right)\left[(p_1+p_2)^2-2p_1p_2\right]e_5^{(5)}z^5\end{array}\right]}{\prod\limits_{i=0}^{5}(1-a_ip_1z)\prod\limits_{i=1}^{5}(1-a_ip_2z)}.$$

□

We can conclude the following proposition from Theorem 3.1.

Proposition 3.2. Assume that $\{p_1,p_2\}$ and $\{a_1,a_2,a_3,a_4,a_5\}$ be two alphabets, then we have

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n = \frac{\left[\begin{array}{c} z - p_1p_2e_2^{(5)}z^3 + p_1p_2(p_1+p_2)e_3^{(5)}z^4 \\ -p_1p_2\left[(p_1+p_2)^2 - p_1p_2\right]e_4^{(5)}z^5 \\ +p_1p_2\,(p_1+p_2)\left[(p_1+p_2)^2 - 2p_1p_2\right]e_5^{(5)}z^6 \end{array}\right]}{\prod_{i=1}^{5}(1-a_ip_1z)\prod_{i=1}^{5}(1-a_ip_2z)}. \tag{3.2}$$

Theorem 3.3. Assume that $\{p_1,p_2\}$ and $\{a_1,a_2,a_3,a_4,a_5\}$ be two alphabets, then we have

$$\sum_{n=0}^{\infty} h_{n}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n = \frac{\left[\begin{array}{c} e_1^{(5)}z - (p_1+p_2)e_2^{(5)}z^2 + \left[(p_1+p_2)^2 - p_1p_2\right]e_3^{(5)}z^3 \\ -(p_1+p_2)\left[(p_1+p_2)^2 - 2p_1p_2\right]e_4^{(5)}z^4 \\ +[(p_1+p_2)^4 - p_1p_2\left[3\,(p_1+p_2)^2 - p_1p_2\right]\right]e_5^{(5)}z^5 \end{array}\right]}{\prod_{i=1}^{5}(1-a_ip_1z)\prod_{i=1}^{5}(1-a_ip_2z)}. \tag{3.3}$$

Proof . Using the divided difference operator $\partial_{p_1p_2}$ on the series $f(p_1z) = \sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n$, we get

$$\begin{aligned}
\partial_{p_1p_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right) &= \frac{\sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n - \sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n}{p_1-p_2} \\
&= \sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)\frac{p_1^n-p_2^n}{p_1-p_2}z^n \\
&= \sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n.
\end{aligned}$$

On the other hand,

$$\partial_{p_1p_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n}\right) = \frac{\left[\begin{array}{c} \frac{1}{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n} \\ -\frac{1}{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n} \end{array}\right]}{p_1-p_2}$$

$$= \frac{\frac{1}{p_1-p_2}\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n - \sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n\right)}$$

$$= -\frac{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)\frac{p_1^n-p_2^n}{p_1-p_2}z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n\right)}$$

$$= -\frac{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,p_2)z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2,a_3,a_4,a_5)p_2^nz^n\right)}$$

$$= - \frac{\begin{bmatrix} e_0(a_1,a_2,a_3,a_4,a_5)h_{-1}(p_1,p_2) \\ -e_1(a_1,a_2,a_3,a_4,a_5)h_0(p_1,p_2)z \\ +e_2(a_1,a_2,a_3,a_4,a_5)h_1(p_1,p_2)z^2 \\ -e_3(a_1,a_2,a_3,a_4,a_5)h_2(p_1,p_2)z^3 \\ +e_4(a_1,a_2,a_3,a_4,a_5)h_3(p_1,p_2)z^4 \\ -e_5(a_1,a_2,a_3,a_4,a_5)h_4(p_1,p_2)z^5 \end{bmatrix}}{\prod_{i=1}^{5}(1-a_ip_1z)\prod_{i=1}^{5}(1-a_ip_2z)}$$

$$= \frac{\begin{bmatrix} e_1(a_1,a_2,a_3,a_4,a_5)z-(p_1+p_2)\,e_2(a_1,a_2,a_3,a_4,a_5)z^2 \\ +\left[(p_1+p_2)^2-p_1p_2\right]e_3(a_1,a_2,a_3,a_4,a_5)z^3 \\ -(p_1+p_2)\left[(p_1+p_2)^2-2p_1p_2\right]e_4(a_1,a_2,a_3,a_4,a_5)z^4 \\ +\left[(p_1+p_2)^4-p_1p_2\left[3\,(p_1+p_2)^2-p_1p_2\right]\right]e_5(a_1,a_2,a_3,a_4,a_5)z^5 \end{bmatrix}}{\prod_{i=1}^{5}(1-a_ip_1z)\prod_{i=1}^{5}(1-a_ip_2z)}.$$

□

4 On the generating functions of some numbers

In this section, we take into account the previous theorems to create a novel generating functions for the products of the symmetric functions in multiple variables with k-Fibonacci, k-Lucas, k-Pell,k-Jacobsthal and k Mersenne numbers.

In (3.1), (3.2) and (3.3), we get p_2 replaced by $(-p_2)$. Then, we have

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(p_1,[-p_2])z^n = \frac{\begin{bmatrix} 1+p_1p_2e_2^{(5)}z^2-p_1p_2(p_1-p_2)e_3^{(5)}z^3 \\ +p_1p_2\left[(p_1-p_2)^2+p_1p_2\right]e_4^{(5)}z^4 \\ -p_1p_2(p_1-p_2)\left[(p_1-p_2)^2+2p_1p_2\right]e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_i\,(p_1-p_2)\,z-a_i^2p_1p_2z^2\right)}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n = \frac{\begin{bmatrix} z+p_1p_2e_2^{(5)}z^3-p_1p_2(p_1-p_2)e_3^{(5)}z^4 \\ +p_1p_2\left[(p_1-p_2)^2+p_1p_2\right]e_4^{(5)}z^5 \\ -p_1p_2\,(p_1-p_2)\left[(p_1-p_2)^2+2p_1p_2\right]e_5^{(5)}z^6 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_i\,(p_1-p_2)\,z-a_i^2p_1p_2z^2\right)}, \tag{4.2}$$

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n = \frac{\begin{bmatrix} e_1^{(5)}z-(p_1-p_2)e_2^{(5)}z^2+\left[(p_1-p_2)^2+p_1p_2\right]e_3^{(5)}z^3 \\ -(p_1-p_2)\left[(p_1-p_2)^2+2p_1p_2\right]e_4^{(5)}z^4 \\ +[(p_1-p_2)^4+p_1p_2\left[3\,(p_1-p_2)^2+p_1p_2]\right]e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_i\,(p_1-p_2)\,z-a_i^2p_1p_2z^2\right)}. \tag{4.3}$$

This case involves four related parts:

Part 1: By replacing $p_1 - p_2 = k$ et $p_1 p_2 = 1$ in (4.1), we present

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(p_1,[-p_2])z^n = \frac{1+e_2^{(5)}z^2 - ke_3^{(5)}z^3 + \left(k^2+1\right)e_4^{(5)}z^4 - k\left(k^2+2\right)e_5^{(5)}z^5}{\prod_{i=1}^{5}\left(1-ka_iz-a_i^2z^2\right)}$$
$$= \sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)F_{k,n}z^n.$$

Consequently, The following Theorems are introduced.

Theorem 4.1. Provided that $n \in \mathbb{N}$, the novel producing function for the sum of k-Fibonacci numbers and symmetric function in multiple variables is

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)F_{k,n}z^n = \frac{1+e_2^{(5)}z^2 - ke_3^{(5)}z^3 + \left(k^2+1\right)e_4^{(5)}z^4 - k\left(k^2+2\right)e_5^{(5)}z^5}{\prod_{i=1}^{5}\left(1-ka_iz-a_i^2z^2\right)}.$$

Theorem 4.2. Provided that $n \in \mathbb{N}$, the novel generating function for the product of k-Lucas numbers and symmetric function in multiple variables is

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)L_{k,n}z^n = \frac{\left[\begin{array}{c} 2-ke_1^{(5)}z+\left(k^2+2\right)e_2^{(5)}z^2 - k\left(k^2+3\right)e_3^{(5)}z^3 \\ +\left(k^4+4k^2+2\right)e_4^{(5)}z^4 - k\left(k^4+5k^2+5\right)e_5^{(5)}z^5 \end{array}\right]}{\prod_{i=1}^{5}\left(1-ka_iz-a_i^2z^2\right)}.$$

Proof . Observe that

$$L_{k,n} = 2h_n(p_1,[-p_2]) - kh_{n-1}\left(p_1,[-p_2]\right), \quad \text{(see [13])},$$

then

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)L_{k,n}z^n$$
$$= 2\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n\left(p_1,[-p_2]\right)z^n - k\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n$$
$$= \frac{2\left[\begin{array}{c} 1+e_2^{(5)}z^2 - ke_3^{(5)}z^3 \\ +\left(k^2+1\right)e_4^{(5)}z^4 - k\left(k^2+2\right)e_5^{(5)}z^5 \end{array}\right] - k\left[\begin{array}{c} e_1^{(5)}z - ke_2^{(5)}z^2 + \left(k^2+1\right)e_3^{(5)}z^3 \\ -k\left(k^2+2\right)e_4^{(5)}z^4 + \left(k^4+3k^2+1\right)e_5^{(5)}z^5 \end{array}\right]}{\prod_{i=1}^{5}\left(1-ka_iz-a_i^2z^2\right)}$$
$$= \frac{\left[\begin{array}{c} 2-ke_1^{(5)}z+\left(k^2+2\right)e_2^{(5)}z^2 - k\left(k^2+3\right)e_3^{(5)}z^3 \\ +\left(k^4+4k^2+2\right)e_4^{(5)}z^4 - k\left(k^4+5k^2+5\right)e_5^{(5)}z^5 \end{array}\right]}{\prod_{i=1}^{5}\left(1-ka_iz-a_i^2z^2\right)}.$$

□

Part 2: If we assume that $p_1 - p_2 = 2$ and $p_1 p_2 = k$ in (4.2), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n = \frac{z+ke_2^{(5)}z^3 - 2ke_3^{(5)}z^4 + k\left(k+4\right)e_4^{(5)}z^5 - 4k\left(2+k\right)e_5^{(5)}z^6}{\prod_{i=1}^{5}\left(1-2a_iz-ka_i^2z^2\right)}$$
$$= \sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)P_{k,n}z^n.$$

where the next Theorem is derived.

Theorem 4.3. Provided that $n \in \mathbb{N}$, the novel generating function for the product of k-Pell numbers and symmetric function in multiple variables is given by

$$\sum_{i=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)P_{k,n}z^n = \frac{z+ke_2^{(5)}z^3-2ke_3^{(5)}z^4+k\left(k+4\right)e_4^{(5)}z^5-4k\left(2+k\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-2a_iz-ka_i^2z^2\right)}.$$

Part 3: Puting $p_1 - p_2 = 3k$ and $p_1p_2 = -2$ in (4.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n &= \frac{z-2e_2^{(5)}z^3+6ke_3^{(5)}z^4-2\left(9k^2-2\right)e_4^{(5)}z^5+6k\left(9k^2-4\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-3ka_iz+2a_i^2z^2\right)}\\ &= \sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)M_{k,n}z^n.\end{aligned}$$

Thus, we get the following Theorem.

Theorem 4.4. Given $n \in \mathbb{N}$, the novel generating function for the product of k-Mersenne numbers and symmetric function in multiple variables is obtained by

$$\sum_{i=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)M_{k,n}z^n = \frac{z-2e_2^{(5)}z^3+6ke_3^{(5)}z^4-2\left(9k^2-2\right)e_4^{(5)}z^5+6k\left(9k^2-4\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-3ka_iz+2a_i^2z^2\right)}.$$

Part 4: Selecting p_1 and p_2 such that $p_1 - p_2 = k$ and $p_1p_2 = 2$ and replacing in (4.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n &= \frac{z+2e_2^{(5)}z^3-2ke_3^{(5)}z^4+2\left(k^2+2\right)e_4^{(5)}z^5-2k\left(k^2+4\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-ka_iz-2a_i^2z^2\right)}\\ &= \sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)J_{k,n}z^n.\end{aligned}$$

As a result, we have the following Theorem.

Theorem 4.5. Given $n \in \mathbb{N}$, the novel generating function for the product of k-Jacobsthal numbers and symmetric function in multiple variables is given by

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)J_{k,n}z^n = \frac{z+2e_2^{(5)}z^3-2ke_3^{(5)}z^4+2\left(k^2+2\right)e_4^{(5)}z^5-2k\left(k^2+4\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-ka_iz-2a_i^2z^2\right)}.$$

Set $a_5 = 0$, we get the following generating functions. The acquired results are presented in Table 1.

Table 1: Generating functions of some numbers

$p_1 - p_2$	p_1p_2	Coefficient of z^n	Generating function
k	1	$h_n(a_1,a_2,a_3,a_4)F_{k,n}$	$\frac{e_1^{(4)}z-ke_2^{(4)}z^2+\left(k^2+1\right)e_3^{(4)}z^3-k(k+2)e_4^{(4)}z^4}{\prod\limits_{i=1}^{4}\left(1-ka_iz-2a_i^2z^2\right)}$
2	k	$h_{n-1}(a_1,a_2,a_3,a_4)P_{k,n}$	$\frac{z+ke_2^{(4)}z^3-2ke_3^{(4)}z^4+k(k+4)e_4^{(4)}z^5}{\prod_{i=1}^{4}\left(1-ka_iz-2a_i^2z^2\right)}$
$3k$	-2	$h_{n-1}(a_1,a_2,a_3,a_4)M_{k,n}$	$\frac{z-2e_2^{(4)}z^3-6ke_3^{(4)}z^4-2\left(9k^2-2\right)e_4^{(4)}z^5}{\prod\limits_{i=1}^{4}\left(1-ka_iz-2a_i^2z^2\right)}$
k	2	$h_{n-1}(a_1,a_2,a_3,a_4)J_{k,n}$	$\frac{z+2e_2^{(4)}z^3-2ke_3^{(4)}z^4+2\left(k^2+2\right)e_4^{(4)}z^5}{\prod\limits_{i=1}^{4}\left(1-ka_iz-2a_i^2z^2\right)}$

Taking $a_4 = a_5 = 0$, we get the following generating functions. The acquired results are presented in Table 2.

Table 2: Generating functions of some numbers

$p_1 - p_2$	p_1p_2	Coefficient of z^n	Generating function
k	1	$h_n(a_1,a_2,a_3)F_{k,n}$	$\frac{1+e_2^{(3)}z^2+ke_3^{(3)}z^3}{\prod\limits_{i=1}^{3}\left(1-ka_iz-a_i^2z^2\right)}$
2	k	$h_{n-1}(a_1,a_2,a_3)P_{k,n}$	$\frac{z+ke_2^{(3)}z^3+2ke_3^{(3)}z^4}{\prod\limits_{i=1}^{3}\left(1-2a_iz-ka_i^2z^2\right)}$
k	1	$h_n(a_1,a_2,a_3)L_{k,n}$	$\frac{2-ke_1^{(3)}z+\left(2+k^2\right)e_2^{(3)}z^2+k\left(1-k^2\right)e_3^{(3)}z^3}{\prod\limits_{i=1}^{3}\left(1-ka_iz-a_i^2z^2\right)}$
$3k$	-2	$h_{n-1}(a_1,a_2,a_3)M_{k,n}$	$\frac{z-2e_2^{(3)}z^3-6ke_3^{(3)}z^4}{\prod\limits_{i=1}^{3}\left(1-3ka_iz+2a_i^2z^2\right)}$
k	2	$h_{n-1}(a_1,a_2,a_3)J_{k,n}$	$\frac{z+2e_2^{(3)}z^3+2ke_3^{(3)}z^4}{\prod\limits_{i=1}^{3}\left(1-ka_iz-2a_i^2z^2\right)}$

Taking $a_3 = a_4 = a_5 = 0$, we get the Table 3, which represents the generating functions of various numbers obtained in previous works.

Table 3: Generating functions of some numbers

$p_1 - p_2$	p_1p_2	Coefficient of z^n	Generating function
k	1	$h_n(a_1,a_2)F_{k,n}$	$\frac{1+e_2(a_1,a_2)z^2}{\prod\limits_{i=1}^{2}\left(1-ka_iz-a_i^2z^2\right)}$
2	k	$h_{n-1}(a_1,a_2)P_{k,n}$	$\frac{z+ke_2(a_1,a_2)z^3}{\prod\limits_{i=1}^{2}\left(1-2a_iz-ka_i^2z^2\right)}$
k	1	$h_n(a_1,a_2)L_{k,n}$	$\frac{2+ke_1(a_1,a_2)z+\left(2-k^2\right)e_2(a_1,a_2)z^2}{\prod\limits_{i=1}^{2}\left(1-ka_iz-a_i^2z^2\right)}$
$3k$	-2	$h_{n-1}(a_1,a_2)M_{k,n}$	$\frac{z-2e_2(a_1,a_2)z^3}{\prod\limits_{i=1}^{2}\left(1-3ka_iz+2a_i^2z^2\right)}$
k	2	$h_{n-1}(a_1,a_2)J_{k,n}$	$\frac{z+2e_2(a_1,a_2)z^3}{\prod\limits_{i=1}^{2}\left(1-ka_iz-2a_i^2z^2\right)}$

5 On the generating functions of some polynomials

In this section, we take into account the previous theorems to create a novel generating functions for the products of the symmetric functions in many variables with Fibonacci polynomials, Jacobsthal polynomials and Chebyshev polynomials of the first, second, third and fourth kinds.

- **Case 1:** In (3.1) and (3.2), we get p_2 replaced by $(-p_2)$, this case involves three related parts:

Part1: By replacing $p_1 - p_2 = x$ and $p_1 p_2 = 1$ in (4.1), we present

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) h_n(p_1,[-p_2]) z^n = \frac{1 + e_2^{(5)} z^2 - e_3^{(5)} x z^3 + \left(x^2+1\right) e_4^{(5)} z^4 - x\left(x^2+2\right) e_5^{(5)} z^5}{\prod\limits_{i=1}^{5}\left(1 - a_i x z - a_i^2 z^2\right)}.$$

Consequently, the following Theorems are introduced.

Theorem 5.1. Provided that $n \in \mathbb{N}$, the novel generating function for the product of Fibonacci polynomials and symmetric function in multiple variables is

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) F_n(x) z^n = \frac{1 + e_2^{(5)} z^2 - e_3^{(5)} x z^3 + \left(x^2+1\right) e_4^{(5)} z^4 - x\left(x^2+2\right) e_5^{(5)} z^5}{\prod\limits_{i=1}^{5}\left(1 - a_i x z - a_i^2 z^2\right)}.$$

Theorem 5.2. Provided that $n \in \mathbb{N}$, the novel generating function for the product of Lucas polynomials and symmetric function in multiple variables is

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) L_n(x) z^n = \frac{\left[\begin{array}{c} 2 - xz + \left(2+x^2\right) e_2^{(5)} z^2 - x\left(3+x^2\right) e_3^{(5)} z^3 \\ + \left(x^4+4x^2+2\right) e_4^{(5)} z^4 - x\left(x^4+5x^2+5\right) e_5^{(5)} z^5 \end{array}\right]}{\prod\limits_{i=1}^{5}\left(1 - a_i x z - a_i^2 z^2\right)}.$$

Proof . Consider that

$$L_n(x) = 2 h_n\left(p_1,[-p_2]\right) - x h_{n-1}\left(p_1,[-p_2]\right), \quad \text{(see [9])},$$

then can be written by

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) L_n(x) z^n &= \left[\begin{array}{c} 2\sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) h_n\left(p_1,[-p_2]\right) z^n \\ -x \sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5) h_{n-1}\left(p_1,[-p_2]\right) z^n \end{array}\right] \\
&= \frac{2\left[\begin{array}{c} 1 + e_2^{(5)} z^2 - e_3^{(5)} x z^3 \\ + \left(x^2+1\right) e_4^{(5)} z^4 \\ -x\left(x^2+2\right) e_5^{(5)} z^5 \end{array}\right] - x\left[\begin{array}{c} z - e_2^{(5)} x z^2 + \left(x^2+1\right) e_3^{(5)} z^3 \\ -x\left(x^2+2\right) e_4^{(5)} z^4 \\ + \left(x^4+3x^2+1\right) e_5^{(5)} z^5 \end{array}\right]}{\prod\limits_{i=1}^{5}\left(1 - a_i x z - a_i^2 z^2\right)} \\
&= \frac{\left[\begin{array}{c} 2 - xz + \left(2+x^2\right) e_2^{(5)} z^2 - x\left(x^2+3\right) e_3^{(5)} z^3 \\ + \left(x^4+4x^2+2\right) e_4^{(5)} z^4 - x\left(x^4+5x^2+5\right) e_5^{(5)} z^5 \end{array}\right]}{\prod\limits_{i=1}^{5}\left(1 - a_i x z - a_i^2 z^2\right)}.
\end{aligned}$$

□

Part 2: By replacing $p_1 - p_2 = 2x$ and $p_1 p_2 = 1$ in (4.2), we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5) h_{n-1}(p_1,[-p_2]) z^n = \frac{z + e_2^{(5)} z^3 - 2x e_3^{(5)} z^4 + \left(4x^2+1\right) e_4^{(5)} z^5 - 4x\left(2x^2+1\right) e_5^{(5)} z^6}{\prod_{i=1}^{5}\left(1 - 2a_i x z - a_i^2 z^2\right)},$$

where the next Theorem is derived.

Theorem 5.3. Provided that $n \in \mathbb{N}$, The novel generating function for the product of Pell polynomials and symmetric function in multiple variables is given by

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)P_n(x)z^n = \frac{z+e_2^{(5)}z^3-2xe_3^{(5)}z^4+\left(4x^2+1\right)e_4^{(5)}z^5-4x\left(2x^2+1\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-2a_ixz-a_i^2z^2\right)}.$$

Part 3: Selecting p_1 and p_2 such that $p_1p_2 = 2x$ and $p_1 - p_2 = 1$ and replacing in (4.2), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)h_{n-1}(p_1,[-p_2])z^n = \frac{z+2e_2^{(5)}xz^3-2xe_3^{(5)}z^4+2x\left(2x+1\right)e_4^{(5)}z^5-2x\left(4x+1\right)e_5^{(5)}z^6}{\prod_{i=1}^{5}\left(1-a_iz-2a_i^2xz^2\right)}.$$

Hence, we arrive at the Theorem bellow.

Theorem 5.4. Given $n \in \mathbb{N}$. The novel generating function for the Jacobsthal polynomials and symmetric function in multiple variables is obtained by

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2,a_3,a_4,a_5)J_n(x)z^n = \frac{z+2e_2^{(5)}xz^3-2xe_3^{(5)}z^4+2x\left(2x+1\right)e_4^{(5)}z^5-2x\left(4x+1\right)e_5^{(5)}z^6}{\prod\limits_{i=1}^{5}\left(1-a_iz-2a_i^2xz^2\right)}.$$

• **Case 2:** In (3.1), (3.3) and under the condition $4p_1p_2 = -1$, we get p_1 replaceb by $2p_1$ and p_2 by $[-2p_2]$ then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(2p_1,[-2p_2])z^n &= \frac{\begin{bmatrix} 1+4p_1p_2e_2^{(5)}z^2-8p_1p_2\left(p_1-p_2\right)e_3^{(5)}z^3 \\ +4p_1p_2\left[4\left(p_1-p_2\right)^2+4p_1p_2\right]e_4^{(5)}z^4 \\ -8p_1p_2\left(p_1-p_2\right)\left[4\left(p_1-p_2\right)^2+8p_1p_2\right]e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2p_1a_iz\right)\left(1+2p_2a_iz\right)} \\ &= \frac{\begin{bmatrix} 1-e_2^{(5)}z^2+2xe_3^{(5)}z^3 \\ -\left(4x^2-1\right)e_4^{(5)}z^4+4x\left(2x^2-1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)},\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(2p_1,[-2p_2])z^n &= \frac{\begin{bmatrix} e_1^{(5)}z-2\left(p_1-p_2\right)e_2^{(5)}z^2+4\left[p_1p_2+\left(p_1-p_2\right)^2\right]e_3^{(5)}z^3 \\ -2\left(p_1-p_2\right)\left[4\left(p_1-p_2\right)^2+8p_1p_2\right]e_4^{(5)}z^4 \\ +\left[16\left(p_1-p_2\right)^4+4p_1p_2\left[12\left(p_1-p_2\right)^2+4p_1p_2\right]\right]e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2\left(p_1-p_2\right)a_iz-4p_1p_2a_i^2z^2\right)} \\ &= \frac{\begin{bmatrix} e_1^{(5)}z-2xe_2^{(5)}z^2+\left(4x^2-1\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4+\left(16x^4-12x^2+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2xa_iz+a_i^2z^2\right)}.\end{aligned}$$

Consequently, the following results are provided.

Theorem 5.5. Given $n \in \mathbb{N}$, the new generating function for the product of second kind Chebyshev polynomials and symmetric function in multiple variables is obtained by

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)U_n(x)z^n = \frac{\begin{bmatrix} 1-e_2^{(5)}z^2+2xe_3^{(5)}z^3 \\ -\left(4x^2-1\right)e_4^{(5)}z^4+4x\left(2x^2-1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

Theorem 5.6. Given $n \in \mathbb{N}$, the novel generating function for the product of first kind Chebyshev polynomials and symmetric function in multiple variables is obtained by

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)T_n(x)z^n = \frac{\begin{bmatrix} 1-xe_1^{(5)}z-\left(1-2x^2\right)e_2^{(5)}z^2+x\left(3-4x^2\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4+x\left(20x^2-16x^4-5\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

Proof . We observe that

$$T_n(x) = h_n(2p_1,[-2p_2]) - xh_{n-1}(2p_1,[-2p_2]), \quad \text{(see [3])}$$

then,

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)T_n(x)z^n &= \begin{bmatrix} \sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(2p_1,[-2p_2])z^n \\ -x\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(2p_1,[-2p_2])z^n \end{bmatrix} \\
&= \frac{\begin{bmatrix} 1-e_2^{(5)}z^2+2xe_3^{(5)}z^3 \\ -\left(4x^2-1\right)e_4^{(5)}z^4 \\ +4x\left(2x^2-1\right)e_5^{(5)}z^5 \end{bmatrix} - x\begin{bmatrix} e_1^{(5)}z-2xe_2^{(5)}z^2+\left(4x^2-1\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4 \\ +\left(16x^4-12x^2+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)} \\
&= \frac{\begin{bmatrix} 1-xe_1^{(5)}z-\left(1-2x^2\right)e_2^{(5)}z^2+x\left(2-4x^2+1\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4 \\ +x\left(8x^2-4-16x^4+12x^2-1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)} \\
&= \frac{\begin{bmatrix} 1-xe_1^{(5)}z-\left(1-2x^2\right)e_2^{(5)}z^2+x\left(3-4x^2\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4 \\ +x\left(20x^2-16x^4-5\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.
\end{aligned}$$

□

Theorem 5.7. Given $n \in \mathbb{N}$, the novel generating function for the product of third kind Chebyshev polynomials and symmetric function in multiple variables is obtained by

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)V_n(x)z^n = \frac{\begin{bmatrix} 1-e_1^{(5)}z-(1-2x)e_2^{(5)}z^2+\left(1+2x-4x^2\right)e_3^{(5)}z^3 \\ +\left(8x^3-4x^2-4x+1\right)e_4^{(5)}z^4 \\ -\left(16x^4-8x^3-12x^2+4x+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

Proof . We observe that

$$V_n(x) = h_n(2p_1, [-2p_2]) - h_{n-1}(2p_1, [-2p_2]), \quad \text{(see [3])},$$

then

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)V_n(x)z^n = \begin{bmatrix} \sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(2p_1,[-2p_2])z^n \\ -\sum\limits_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(2p_1,[-2p_2])z^n \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 - e_2^{(5)}z^2 + 2xe_3^{(5)}z^3 \\ -\left(4x^2-1\right)e_4^{(5)}z^4 \\ +4x\left(2x^2-1\right)e_5^{(5)}z^5 \end{bmatrix} - \begin{bmatrix} e_1^{(5)}z - 2xe_2^{(5)}z^2 + \left(4x^2-1\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4 \\ +\left(16x^4-12x^2+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}$$

$$= \frac{\begin{bmatrix} 1 - e_1^{(5)}z - (1-2x)e_2^{(5)}z^2 - \left(4x^2-2x-1\right)e_3^{(5)}z^3 \\ +\left(8x^3-4x^2-4x+1\right)e_4^{(5)}z^4 \\ -\left(16x^4-8x^3-12x^2+4x+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

□

Theorem 5.8. Given $n \in \mathbb{N}$, the new generating function for the product of fourth kind Chebyshev polynomials and symmetric function in multiple variables is obtained by

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)W_n(x)z^n = \frac{\begin{bmatrix} 1 + e_1^{(5)}z - (1+2x)e_2^{(5)}z^2 + \left(4x^2+2x-1\right)e_3^{(5)}z^3 \\ -\left(8x^3+4x^2-4x-1\right)e_4^{(5)}z^4 \\ +\left(16x^4+8x^3-12x^2-4x+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

Proof . We observe that

$$W_n(x) = h_n(2p_1, [-2p_2]) + h_{n-1}(2p_1, [-2p_2]), \quad \text{(see [3])}$$

then

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,a_3,a_4,a_5)W_n(x)z^n = \begin{bmatrix} \sum\limits_{n=0}^{+\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_n(2p_1,[-2p_2])z^n \\ +\sum\limits_{n=0}^{+\infty} h_n(a_1,a_2,a_3,a_4,a_5)h_{n-1}(2p_1,[-2p_2])z^n \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 - e_2^{(5)}z^2 + 2xe_3^{(5)}z^3 \\ -\left(4x^2-1\right)e_4^{(5)}z^4 \\ +4x\left(2x^2-1\right)e_5^{(5)}z^5 \end{bmatrix} + \begin{bmatrix} e_1^{(5)}z - 2xe_2^{(5)}z^2 + \left(4x^2-1\right)e_3^{(5)}z^3 \\ -4x\left(2x^2-1\right)e_4^{(5)}z^4 \\ +\left(16x^4-12x^2+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}$$

$$= \frac{\begin{bmatrix} 1 + e_1^{(5)}z - (1+2x)e_2^{(5)}z^2 + \left(4x^2+2x-1\right)e_3^{(5)}z^3 \\ -\left(8x^3+4x^2-4x-1\right)e_4^{(5)}z^4 \\ +\left(16x^4+8x^3-12x^2-4x+1\right)e_5^{(5)}z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-2a_ixz+a_i^2z^2\right)}.$$

□

Theorem (4.1), (4.2), (4.3), (4.5) and (4.6), where $a_4 = a_5 = 0$ produce the generationg function of various polynomial products and symmetric functions in multiple variables, respectively. The acquired results are presented in Table 4.

Table 4: Generating functions of some polynomials

$p_1 - p_2$	$p_1 p_2$	Coefficient de z^n	Generating Function
x	1	$h_n(a_1, a_2, a_3, a_4)F_n(x)$	$\frac{1-e_2^{(4)}z^2+xe_3^{(4)}z^3-\left(x^2-1\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-a_i xz-a_i^2 z^2\right)}$
x	1	$h_n(a_1, a_2, a_3, a_4)L_n(x)$	$\frac{2+xe_1^{(4)}z+\left(x^2-2\right)e_2^{(4)}z^2-x\left(x^2-1\right)e_3^{(4)}z^3+\left(x^4+2\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-a_i xz-a_i^2 z^2\right)}$
$2x$	1	$h_{n-1}(a_1, a_2, a_3, a_4)P_n(x)$	$\frac{z+e_2^{(4)}z^3-2xe_3^{(4)}z^4+\left(4x^2+1\right)e_4^{(4)}z^5}{\prod_{i=1}^{4}\left(1-2a_i xz-a_i^2 z^2\right)}$
1	$2x$	$h_{n-1}(a_1, a_2, a_3, a_4)J_n(x)$	$\frac{z+2xe_2^{(4)}z^3-2xe_3^{(4)}z^4+2x(2x+1)e_4^{(4)}z^5}{\prod_{i=1}^{4}\left(1-a_i z-2xa_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2, a_3, a_4)U_n(x)$	$\frac{1-e_2^{(4)}z^2+2xe_3^{(4)}z^3-\left(4x^2-1\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2, a_3, a_4)T_n(x)$	$\frac{1-xe_1^{(4)}z+\left(2x^2-1\right)e_2^{(4)}z^2-x\left(4x^2-3\right)e_3^{(4)}z^3+\left(8x^3+1\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2, a_3, a_4)V_n(x)$	$\frac{1-e_1^{(4)}z-(2x-1)e_2^{(4)}z^2-\left(4x^2+2x+1\right)e_3^{(4)}z^3+\left(8x^3-4x^2-4x+1\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2, a_3, a_4)W_n(x)$	$\frac{1+e_1^{(4)}z-(1+2x)e_2^{(4)}z^2+\left(4x^2+2x-1\right)e_3^{(4)}z^3-\left(8x^3+4x^2-4x-1\right)e_4^{(4)}z^4}{\prod_{i=1}^{4}\left(1-2a_i xz+a_i^2 z^2\right)}$

For $a_3 = a_4 = a_5 = 0$, we derive a novel generating functions of some polynomials attained in former works, and present them in the Table 5.

Table 5: Generating functions of some polynomials

$p_1 - p_2$	$p_1 p_2$	Coefficient de z^n	Generating Function
x	1	$h_n(a_1, a_2)F_n(x)$	$\frac{1+e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-a_i xz-a_i^2 z^2\right)}$
x	1	$h_n(a_1, a_2)L_n(x)$	$\frac{2+xe_1(a_1,a_2)z-\left(x^2-2\right)e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-a_i xz-a_i^2 z^2\right)}$
$2x$	1	$h_{n-1}(a_1, a_2)P_n(x)$	$\frac{z+e_2(a_1,a_2)z^3}{\prod_{i=1}^{2}\left(1-2a_i xz-a_i^2 z^2\right)}$
1	x	$h_{n-1}(a_1, a_2)J_n(x)$	$\frac{z+2xe_2(a_1,a_2)z^3}{\prod_{i=1}^{2}\left(1-a_i z-2xa_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2)U_n(x)$	$\frac{1-e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2)T_n(x)$	$\frac{1+e_1(a_1,a_2)xz-\left(2x^2+1\right)e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2)V_n(x)$	$\frac{1+e_1(a_1,a_2)z-(2x+1)e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-2a_i xz+a_i^2 z^2\right)}$
x	$-\frac{1}{4}$	$h_n(a_1, a_2)W_n(x)$	$\frac{1-e_1(a_1,a_2)z+(2x-1)e_2(a_1,a_2)z^2}{\prod_{i=1}^{2}\left(1-2a_i xz+a_i^2 z^2\right)}$

6 Theory applications

Definition 6.1. For all $n \in \mathbb{N}$, we define

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+...+q^{n-1}, \ \ (q \neq 1).$$

Definition 6.2. Strling numbers of two kind $s(n,k)$ are coefficients of polynomial $(x)_n$ defined by

$$(x)_n = x(x-1)(x-2)\dots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k.$$

Theorem 6.3. Let $A = \left\{1, q, q^2, q^3, q^4\right\}$ be an alphabet, we have

$$S_n(A) = \begin{bmatrix} n+4 \\ n \end{bmatrix}_q .$$

Proof . we have $\left\{1, q, q^2, ..., q^{n-1}\right\} := [n]_q$ (see [1]), Gaussian polynomial is given by:

$$S_j\left([n]_q\right) = \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q ,$$

so

$$S_j\left([5]_q\right) = \begin{bmatrix} 5+j-1 \\ j \end{bmatrix}_q = \begin{bmatrix} 4+j \\ j \end{bmatrix}_q ,$$

that's to say

$$S_n\left(A\right) = \begin{bmatrix} n+4 \\ n \end{bmatrix}_q .$$

that completes the proof. □

Theorem 6.4. Let $A = \{1, 2, 3, 4, 5\}$ be an alphabet, we have

$$S_n\left(A\right) = \frac{5^{n+2}+1}{24} - \frac{2^{n+2}+4^{n+2}}{6} + \frac{3^{n+2}}{4}$$

Proof . We have

$$\partial_{e_{n+1}e_n}\partial_{e_n e_{n-1}}...\partial_{e_2 e_1} f\left(e_1\right) = \sum_{i=0}^{n} \frac{f\left(e_{i+1}\right)}{R\left(e_{i+1}, E_i\right) R\left(e_{i+1}, E_{n+1} \backslash E_{i+1}\right)} \ \ \text{(see [19])},$$

In the case of the alphabet: $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $f\left(e_1\right) = e_1^{n+2}$ the lagrange form is:

$$\begin{aligned}
\partial_{e_5 e_4}\partial_{e_4 e_3}\partial_{e_3 e_2}\partial_{e_2 e_1} e_1^{n+2} &= \sum_{k=0}^{4} \frac{e_{k+1}^{n+2}}{R\left(e_{k+1}-E_k\right) R\left(e_{k+1}, E_{4+1} \backslash E_{k+1}\right)} \\
&= \frac{e_1^{n+2}}{R\left(e_1-E_0\right) R\left(e_1, E_5 \backslash E_1\right)} + \frac{e_2^{n+2}}{R\left(e_2-E_1\right) R\left(e_2, E_5 \backslash E_2\right)} \\
&+ \frac{e_3^{n+2}}{R\left(e_3-E_2\right) R\left(e_3, E_5 \backslash E_3\right)} + \frac{e_4^{n+2}}{R\left(e_4-E_3\right) R\left(e_4, E_5 \backslash E_4\right)} + \frac{e_5^{n+2}}{R\left(e_5-E_4\right) R\left(e_5, E_5 \backslash E_5\right)} \\
&= \frac{e_1^{n+2}}{\left(e_1-e_2\right)\left(e_1-e_3\right)\left(e_1-e_4\right)\left(e_1-e_5\right)} + \frac{e_2^{n+2}}{\left(e_2-e_1\right)\left(e_2-e_3\right)\left(e_2-e_4\right)\left(e_2-e_5\right)} \\
&+ \frac{e_3^{n+2}}{\left(e_3-e_1\right)\left(e_3-e_2\right)\left(e_3-e_4\right)\left(e_3-e_5\right)} + \frac{e_4^{n+2}}{\left(e_4-e_1\right)\left(e_4-e_2\right)\left(e_4-e_3\right)\left(e_4-e_5\right)} \\
&+ \frac{e_5^{n+2}}{\left(e_5-e_1\right)\left(e_5-e_2\right)\left(e_5-e_3\right)\left(e_5-e_4\right)}
\end{aligned}$$

$$\begin{aligned}\partial_{e_5e_4}\partial_{e_4e_3}\partial_{e_3e_2}\partial_{e_2e_1}e_1^{n+2} =& \frac{1^{n+2}}{(1-2)(1-3)(1-4)(1-5)} + \frac{2^{n+2}}{(2-1)(2-3)(2-4)(2-5)} \\ &+ \frac{3^{n+2}}{(3-1)(3-2)(3-4)(3-5)} + \frac{4^{n+2}}{(4-1)(4-2)(4-3)(4-5)} \\ &+ \frac{5^{n+2}}{(5-1)(5-2)(5-3)(5-4)} \\ &= \frac{1}{24} - \frac{2^{n+2}}{6} + \frac{3^{n+2}}{4} - \frac{4^{n+2}}{6} + \frac{5^{n+2}}{24} \\ &= \frac{5^{n+2}+1}{24} - \frac{2^{n+2}+4^{n+2}}{6} + \frac{3^{n+2}}{4}.\end{aligned}$$

□

- **Case 01:** $A = \{1,2,3,4,5\}$ and $P = \{1,q\} := [2]_q$

We pose:

$$\begin{aligned} e_0^{(5)} &= 1, \\ e_1^{(5)} &= 15, \\ e_2^{(5)} &= 85, \\ e_3^{(5)} &= 225, \\ e_4^{(5)} &= 274, \\ e_5^{(5)} &= 120. \end{aligned}$$

The following propositions are found in the relationships (3.1), (3.2), and (3.3):

Proposition 6.5.

$$\sum_{n=0}^{+\infty}\left[\frac{5^{n+2}+1}{24} - \frac{2^{n+2}+4^{n+2}}{6} + \frac{3^{n+2}}{4}\right]\begin{bmatrix} n+1 \\ n \end{bmatrix}_q z^n = \frac{\begin{bmatrix} 1-85qz^2+225q(1+q)z^3 \\ -274q\left[(1+q)^2-q\right]z^4 \\ +120q(1+q)\left[(1+q)^2-2q\right]z^5 \end{bmatrix}}{\prod_{i=1}^{5}(1-a_iz)(1-a_iqz)},$$

$$\sum_{n=0}^{+\infty}\left[\frac{5^{n+1}+1}{24} - \frac{2^{n+1}+4^{n+1}}{6} + \frac{3^{n+1}}{4}\right]\begin{bmatrix} n \\ n-1 \end{bmatrix}_q z^n = \frac{\begin{bmatrix} z-85qz^3+225q(1+q)z^4 \\ -274q\left[(1+q)^2-q\right]z^5 \\ +120q(1+q)\left[(1+q)^2-2q\right]z^6 \end{bmatrix}}{\prod_{i=1}^{5}(1-a_iz)(1-a_iqz)},$$

$$\sum_{n=0}^{+\infty}\left[\frac{5^{n+2}+1}{24} - \frac{2^{n+2}+4^{n+2}}{6} + \frac{3^{n+2}}{4}\right]\begin{bmatrix} n \\ n-1 \end{bmatrix}_q z^n = \frac{\begin{bmatrix} 15z-85(1+q)z^2+225\left[(1+q)^2-q\right]z^3 \\ -274(1+q)\left[(1+q)^2-2q\right]z^4 \\ +120\left[(1+q)^4-q\left[3(1+q)^2-q\right]\right]z^5 \end{bmatrix}}{\prod\limits_{i=1}^{5}(1-a_iz)(1-a_iqz)}.$$

- **Case 2:** $A=\{1,q,q^2,q^3,q^4\}$ and $P=\{1,2\}$. We pose:

$$\begin{aligned}
e_0^{(5)} &= 1,\\
e_1^{(5)} &= 1+q+q^2+q^3+q^4,\\
e_2^{(5)} &= q\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right),\\
e_3^{(5)} &= q^3\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right),\\
e_4^{(5)} &= q^6\left(1+q+q^2+q^3+q^{46}\right),\\
e_5^{(5)} &= q^{10}.
\end{aligned}$$

The following statements are found in the relationships (3.1), (3.2), and (3.3):

Proposition 6.6.

$$\sum_{n=0}^{+\infty}\begin{bmatrix}n+4\\n\end{bmatrix}_q\left[2^{n+1}-1\right]z^n=\frac{\begin{bmatrix}1-2q\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^2\\+6q^3\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\-14q^6\left(1+q+q^2+q^3+q^4\right)z^4+30q^{10}z^5\end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_iz\right)\left(1-2a_iz\right)},$$

$$\sum_{n=0}^{+\infty}\begin{bmatrix}n+3\\n-1\end{bmatrix}_q\left[2^{n}-1\right]z^n=\frac{\begin{bmatrix}z-2q\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\+6q^3\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^4\\-14q^6\left(1+q+q^2+q^3+q^4\right)z^5+30q^{10}z^6\end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_iz\right)\left(1-2a_iz\right)},$$

$$\sum_{n=0}^{+\infty}\begin{bmatrix}n+4\\n\end{bmatrix}_q\left[2^{n}-1\right]z^n=\frac{\begin{bmatrix}\left(1+q+q^2+q^3+q^4\right)z-3q\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^2\\+7q^3\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\-15q^6\left(1+q+q^2+q^3+q^4\right)z^4+31q^{10}z^5\end{bmatrix}}{\prod_{i=1}^{5}\left(1-a_iz\right)\left(1-2a_iz\right)}.$$

The following propositions are found in the relationships (3.1), (3.2), and (3.3):

Proposition 6.7.

$$\sum_{n=0}^{\infty}\begin{bmatrix}n+4\\n\end{bmatrix}_q\begin{bmatrix}n+1\\n\end{bmatrix}_q z^n=\frac{\begin{bmatrix}1-q^2\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^2\\+q^4\left(1+q\right)\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\-q^7\left[\left(1+q\right)^2-q\right]\left(1+q+q^2+q^3+q^4\right)z^4+q^{11}\left(1+q\right)\left[\left(1+q\right)^2-2q\right]z^5\end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-a_iz\right)\left(1-a_iqz\right)},$$

$$\sum_{n=0}^{\infty}\begin{bmatrix}n+3\\n-1\end{bmatrix}_q\begin{bmatrix}n\\n-1\end{bmatrix}_q z^n=\frac{\begin{bmatrix}z-q^2\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\+q^4\left(1+q\right)\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^4\\-q^7\left[\left(1+q\right)^2-q\right]\left(1+q+q^2+q^3+q^4\right)z^5+q^{11}\left(1+q\right)\left[\left(1+q\right)^2-2q\right]z^6\end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-a_iz\right)\left(1-a_iqz\right)},$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\begin{bmatrix}n+4\\n\end{bmatrix}_q\begin{bmatrix}n\\n-1\end{bmatrix}_q z^n\\
&=\frac{\begin{bmatrix}\left(1+q+q^2+q^3+q^4\right)z-q\left(1+q\right)\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^2\\+q^3\left[\left(1+q\right)^2-q\right]\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3\\-q^6\left(1+q\right)\left[\left(1+q\right)^2-2q\right]\left(1+q+q^2+q^3+q^4\right)z^4+q^{10}\left[\left(1+q\right)^4-q\left[3\left(1+q\right)^2-q\right]\right]z^5\end{bmatrix}}{\prod\limits_{i=1}^{5}\left(1-a_iz\right)\left(1-a_iqz\right)}.\end{aligned}$$

7 Conclusion

In this paper, we generalize the previous main results [27] by extending the elements of the alphabet that enabled us to derive new theorems by which we obtain many generating functions for the products of certain orthogonal numbers and polynomials.

References

[1] A. Abderrezzek, *Généralisation de la transformation d'Euler d'une série formelle*, Adv, Math. **103** (1994), 180–195.

[2] M. Asci and E. Gurel, *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials*, Notes Number Theory Discrete Math. **19** (2013), 25–36.

[3] A. Baghdadi and A. Boussayoud, *Generating functions of the product of the k-Fibonacci and k-Pell numbers and Chebyshev polynomials of the third and fourth kind*, Math. Eng. Sci. Aerosp. MESA. **12** (2021), 245–257.

[4] Kh. Boubellouta, A. Boussayoud, and M. Kerada, *Symmetric functions for second-order recurrence sequences*, Tbil. Math. J. **13** (2020), 225–237.

[5] S. Boughaba and A. Boussayoud, *On some identities and generating function of both k-Jacobsthal numbers and symmetric functions in several variables*, Konualp J. Math. **7** (2019), no. 2, 235–242.

[6] S. Boughabaa, A. Boussayouda, S. Aracib, M. Keradaa, and M. Acikgozc, *Construction of a new class of generating functions of binary products of some special numbers and polynomials*, Filomat **35** (2021), no. 3, 1001—1013.

[7] S. Boughaba, A. Boussayoud, and M. Kerada, *Construction of symmetric functions of generalized Fibonacci numbers*, Tamap J. Math. Statist. **3** (2019), 1–7.

[8] A. Boussayoud and A. Abderrezzak, *Complete homogeneous symmetric functions and Hadamard product*, Ars Comb. **144** (2019), 81–90.

[9] A. Boussayoud, M. Bolyer, and M. Kerada, *On some identities and symmetric functions for Lucas and Pell numbers*, Electron. J. Math. Anal. Appl. **5** (2017), 202–207.

[10] A. Boussayoud, M. Chelgham, and S. Boughaba, *On some identities and generating functions for Mersenne numbers and polynomials*, Turk. J. Anal. Number Theory **6** (2018), 93–97.

[11] A. Boussayoud and M. Kerada, *Symmetric and generating functions*, Int. Electron. J. Pure Appl. Math. **7** (2014), 195–203.

[12] A. Boussayoud, M. Kerada, and A. Abderrezzak, *A generalization of some orthogonal polynomials*, Proc. Math. Stat. **41** (2013), 229–235.

[13] A. Boussayoud, M. Kerada, and N. Harrouche, *On the k-Lucas numbers and Luca polynomials*, Turk. J. Anal. Number Theory **5** (2017), no. 3, 121–125.

[14] P. Catarino, *On some identities and generating functions for k-Pell numbers*, Int. J. Math. Anal. *7* (2013), 1877–1884.

[15] M.R. Eslahchi, M. Dehghan, and S. Amani, *The third and fourth kind of Chebyshev polynomials and best uniform approximation*, Math. Comput. Modell. **55** (2012), 1746–1762.

[16] S. Falcon, *On the k-Lucas numbers of arithmetic indexes*, Appl Math. **3** (2012), 1202–1206.

[17] S. Falcon and A. Plaza, *The k-Fibonacci sequence and the Pascal* 2*-triangle*, Chaos Solutions Fractals **33** (2008), 38–49.

[18] D. Jhala, K. Sisodiya, and G.P.S. Rathor, *On some identities for k-Jacobsthal*, Int. J. Math. Anal. 7 (2013), 551–556.

[19] M. Kerada, A.Boussayoud, and A. Abderrezzak, *Generalization of some hypergeometric functions*, Indag. Math. **28** (2017), no. 3, 711–720.

[20] M. Merca, *A generalization of the symmetry between complete and elementary symmetric functions*, Indian J. Pure Appl. Math. **45** (2014), 75–90.

[21] H. Merzouk, A. Boussayoud, and A. Abderrezzak, *Ordinary generating functions of binary products of third-order recurrence relations and 2-orthogonal polynomials*, Math. Slovaca **72** (2022), no. 1, 11–34.

[22] N. Saba and A. Boussayoud, *Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials*, Open J. Math. Sci. **4** (2020), 179–185.

[23] N. Saba, A. Boussayoud, and A. Abderrezzak, *Symmetric and generating functions of generalized* (p, q)*-numbers*, Kuwait J. Sci. **48** (2021), 1–15.

[24] N. Saba, A. Boussayoud, S. Araci, M. Kerada, and M. Acikgoz, *Construction of a new class of symmetric function of binary products of* (p, q)*-numbers with 2-orthogonal Chebyshev polynomials*, Bol. Soc. Math. Mex. **27** (2021), 1–26.

[25] N. Saba, A. Boussayoud, and K.V.V. Kanuri, *Mersenne Lucas numbers and complete homogeneous symmetric functions*, J. Math. Comput. Sci. **24** (2022), 127–139.

[26] K. Uslu and V. Deniz, *Some identities of k-Mersenne numbers*, Adv. Appl. Discrete Math. **18** (2017), 413–423.

[27] H. Zerrouk, A. Boussayoud, A. Abderrezzak, and M. Kerada, *Complete and elementary symmetric functions in several variables and orthogonal polynomials*, Nonlinear Studies 29 (2022), no. 1, 329–346.

[28] H. Zerroug, A. Boussayoud, B. Aloui, and M. Kerada, *Gaussian Padovan, Gaussian Pell-Padovan numbers and new generating functions with some numbers and polynomials*, J. Inf. Optim. Sci. **43** (2022), no. 4, 745-–760.