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Predicting agents' investment behavior using game theory and bankruptcy problem

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Abstract

This study considers two agents, risk-neutral and risk-averse ones, and studies their investment behavior. There are two investment options-safe investments such as a bank account and a risky investment in a company. The company runs a risky project. In the case of success, its return is more than the bank's, and that is less in the case of failure. When the project fails, the company divides the left amount among the investors based on the proportional bankruptcy rule. We model the problem as a strategic game and explore its Nash equilibrium.

Keywords: Investment game, Bankruptcy problem, Game theory, Risk-neutral, Risk-averse
2020 MSC: Primary 91A10; Secondary 91A80, 91A05

1 Introduction

Investment is one of the most common problems in economic theory and real life. Almost everybody thinks of investing some of his/her wealth. Generally, there are two essential kinds of investment options: 1. risk-free investments like bank accounts, 2. risky investments like the stock market. The former has a guaranteed constant low return. The latter, on the other hand, may lead to a much greater return without any guarantee.

When the risky project fails, the company goes bankrupt; because its endowment is less than its financial obligations. The well-known bankruptcy problem, first introduced by O'Neill [10] in 1982, deals with this situation. Bankruptcy literature has been rich with different approaches like axiomatic and game-theoretic ones and a large variety of rules.

Game theory is the study of mathematical models of strategic interactions among rational decision-makers [7, 8]. Nowadays, game theory applies to various fields like social science, logic, computer science, biology, etcetera. Games are classified based on different criteria. One is the possibility or impossibility of cooperation. Based on it, games are divided into cooperative and non-cooperative ones [6, 9]. Also, non-cooperative games are in two main forms: strategic and extensive.

Undoubtedly decisions that companies running risky projects make in the case of bankruptcy influence the agents' investment behavior. [4, 5] study this impact in two different settings. [4] considers risk-neutral agents and both risk-free and risky investment options and investigates which bankruptcy rule taken by the company maximizes the total investment. The author set up an extensive game where first the company chooses a bankruptcy rule. Then the

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agents simultaneously decide how much to invest and study the subgame perfect Nash equilibrium of this sequential game. [5] explores the same question in a different setting, considering risk-averse agents having no wealth constraints and just facing a risky investment option. The assumption of risk-averse agents complicates the computations, and omitting the safe investment option reduces the complications. The authors design three strategic games based on three different bankruptcy rules and compare the total investment and total welfare in the Nash equilibria of the games.

From our point of view, it is more reasonable to consider agents from both types, risk-neutral and risk-averse. Also, the assumption of no wealth constraints and omitting the safe investment alternative are not realistic. So we consider a new setting, combining the settings of the two mentioned articles. As we mentioned earlier, considering risk-aversion and two investment alternatives makes the problem complicated. Due to this complexity, we first start with the proportional bankruptcy rule, mostly used in practice. Even at this start point, computing the equilibrium strategy of the risk-averse agent is not analytically computable, and numerical approximations are needed. We use least-squares approximation to find a degree two polynomial as an approximation for the investment function of the risk-averse agent.

The rest of the paper is organized as follows. In Section 2 we present some necessary preliminaries from game theory and bankruptcy problem. Section 3 describes the model. In Section4 we investigate the Nash equilibrium of the presented game, including numerical approximations. Section 5 concludes the paper.

2 Preliminaries

2.1 Game Theory

Game theory is a useful mathematical tool to model and investigate strategic interactions among rational and intelligent agents, called players. Rationality means that each player tries to maximize his/her utility by choosing the best strategy. Intelligence says that players know everything that the game designer knows about the game, and they can make any inferences that the designer can make [7]. Games are classified based on different criteria. One is the possibility or impossibility of cooperation. Based on it, games are divided into cooperative and non-cooperative ones. Also, non-cooperative games are in two main forms: strategic and extensive. In this study, we deal with strategic non-cooperative games.

A strategic game is formally a triple, $< N, (S_i)_{i\in N}, (U_i)_{i\in N} >$, where $N = \{1, 2, \dots, N\}$ is the set of players, S_i is the player i's strategy set, and U_i is the player i's expected utility or payoff function. Here are some basic and necessary definitions.

Definition 2.1. A strategy $s_i^\star$ strictly dominates $s_i \in S_i$ if and only if

$$U_i(s_i^\star, s_{-i}) > U_i(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i},$$

where s_{-i} denotes the other players' strategy and $S_{-i} = \prod_{j\neq i} S_j$.

Definition 2.2. Astrategy $s_i^\star$ is a strictly dominant strategy for player i if and only if $s_i^\star$ strictly dominates all the other strategies of player i.

Definition 2.3. A strategy profile $s^\star = (s_1^\star, s_2^\star, \dots, s_n^\star)$ is a strictly dominant strategy equilibrium if $s_i^\star$ is a strictly dominant strategy for all $i \in N$.

Definition 2.4. For any player i, a strategy s_i is a best response to s_{-i} if and only if

$$U_i(s_i, s_{-i}) \geq U_i(s_i', s_{-i}), \quad \forall\ s_i' \in S_i.$$

Definition 2.5. A strategy profile $s^\star = (s_1^\star, s_2^\star, \dots, s_n^\star)$ is a Nash equilibrium if $s_i^\star$ is a best response to $s_{-i}^\star$ for all $i \in N$.

2.2 Bankruptcy Problem

The bankruptcy problem refers to situations where some agents have claims on a resource that is not enough to honor all of them, like what happens when a firm goes bankrupt. In addition, many different situations can be modeled as a bankruptcy problem. Some examples in the literature are [1, 2, 3].

Formally a bankruptcy problem is a triple (N, c, E) where $N = \{1, 2, \dots, n\}$ is the set of claimants, $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n_+$ the vector of claims and $E \in \mathbb{R}_+$, $\sum_{i \in N} c_i \geq E$, the estate to be divided among claimants. Let denote the class of all bankruptcy problems by $\mathbb{B}$. An award vector for $(N, c, E) \in \mathbb{B}$ is a vector $x \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n x_i = E$, and $0 \leq x \leq c$. A rule $R : \mathbb{B} \to \mathbb{R}^n_+$ is a function that assigns to each bankruptcy problem an award vector. We refer to [11, 12] for surveys on the bankruptcy problem.

There is a variety of rules to solve the bankruptcy problem in the literature. Some classical ones are as follow:

1. Proportional rule
   The assignments are proportional to the claims. For all $i \in N$:
$$PROP(N, c, E)_i = \frac{E}{\sum_{i \in N} c_i} \times c_i.$$
2. Constrained equal awards
   It assigns equal amounts to all claimants, provided that no one receives more than his claim. For all $i \in N$:
$$CEA(N, c, E)_i = min\{c_i, \alpha\}, \quad with\ \alpha\ s.t. \quad \sum_{i \in N} CEA(N, c, E)_i = E.$$
3. Constrained equal losses
   The assignments are such that the claimants' losses are the same, provided that no one receives a negative amount. For all $i \in N$:
$$CEL(N, c, E)_i = max\{0, c_i - \beta\}, \quad with\ \beta\ s.t. \quad \sum_{i \in N} CEL(N, c, E)_i = E.$$
4. Talmud rule
   It can be viewed as a hybrid of the constrained equal awards and losses rules. For all $i \in N$:
$$TAL(N, c, E)_i = \begin{cases} CEA(N, \frac{c}{2}, E)_i & E \leq \sum_{i \in N} \frac{c_i}{2} \\ \frac{c_i}{2} + CEL(N, \frac{c}{2}, E - \sum_{i \in N} \frac{c_i}{2})_i & E > \sum_{i \in N} \frac{c_i}{2} \end{cases}.$$

The constrained equal awards rule shows favoritism toward agents with smaller claims, and the constrained equal losses rule does it toward ones with larger. The Talmud rule takes an intermediate approach. Although several bankruptcy rules are discussed in the literature, the proportional rule is the only widely used in practice. Also, the allocation prescribed by the proportional rule for each agent just depends on his/her claim, the endowment, and total claims, not on how the claims are distributed. This fact makes the computation of proportional rule easier. Therefore in this study, we focus on the proportional bankruptcy rule.

3 The Model

We assume that there are two agents, 1 and 2 both with the same wealth w to invest. Agent 1 is risk-nuetral, hence his/her utility on montary payoff x is linear, i.e. $u_1(x) = x$. Agent 2 is risk-averse. We use constant absolute risk aversion (CARA) utility function to represent agent 2' utility on montary payoff x as $u_2(x) = -e^{-ax}$ where a is his/her coefficient of risk-aversion.

There are two investment options for the agents. A safe investment, like a bank saving account and a risky project that a company runs. Depositing money in the bank brings constant return $r \in [0, 1]$. The risky project of the company succeeds with probability $p \in [0, 1]$ and brings a profit. Based on the estimated profit in the case of success, the company promises to pay its investors $r_s \in [0, 1]$, $r_s > r$. The project fails with the remaining probability and leads to a profit less than the estimated one such that the company can only pay $r_f \in [0, 1]$, $r_f < r$ to the investors. Hence, in the case of success, the risky project leads to a return greater than the bank's, and it does less in the case of failure $(0 \leq r_f < r < r_s \leq 1)$.

Let $s_1, s_2 \in [0, w]$ show the agens' invested amount in the company. When the project fails the company faces a bankruptcy problem (N, c, E), where $N = \{1, 2\}$, $c = ((1 + r_s)s_1, (1 + r_s)s_2)$, and $E = (1 + r_f)(s_1 + s_2)$. Now the question is that how much each agent invests in the company. We can model the situation as the following strategic game:

$$
\begin{aligned}
&N = \{1, 2\},\\
&S_i = [0, w], \quad i = 1, 2,\\
&U_1(s_1, s_2) = (1 + r)(w - s_1) + p(1 + r_s)s_1 + (1 - p)F_1(s_1, s_2),\\
&U_2(s_1, s_2) = -e^{-a(1+r)(w-s_2)} - pe^{-a(1+r_s)s_2} - (1 - p)e^{-aF_2(s_1,s_2)}.
\end{aligned}
$$

In the agents' expected utility functions, $U_i, i = 1, 2$, the first expression is the payoff for investment in the bank. The two others are the payoffs for investment in the company in success and failure cases, respectively. $F_1(s_1, s_2), F_2(s_1, s_2)$ are the amounts that agents 1 and 2 receive in the case of failure when the bankruptcy rule F is applied. If F is the proportional rule, then

$$F_1(s_1, s_2) = (1 + r_f)s_1, \quad F_2(s_1, s_2) = (1 + r_f)s_2.$$

So we have:

$$
\begin{aligned}
&U_1(s_1, s_2) = (1 + r)(w - s_1) + p(1 + r_s)s_1 + (1 - p)(1 + r_f)s_1,\\
&U_2(s_1, s_2) = -e^{-a(1+r)(w-s_2)} - pe^{-a(1+r_s)s_2} - (1 - p)e^{-a(1+r_f)s_2}.
\end{aligned}
$$

4 Nash equilibrium of the model

To compute the Nash equilibrium of the model, we must find the best response of the players. Each agent tries to maximize his/her expected utility function. For the player 1, the risk-neutral one, we have:

$$
\begin{aligned}
&U_1(s_1, s_2) = (1 + r)(w - s_1) + p(1 + r_s)s_1 + (1 - p)(1 + r_f)s_1,\\
&\frac{dU_1(s_1, s_2)}{ds_1} = -(1 + r) + p(1 + r_s) + (1 - p)(1 + r_f).
\end{aligned}
$$

Since $U_1(., .)$ is linear, its derriative is a constant. If this constant is positive, the best respone of player 1 is to choose the maximum possible amount of investment which is w. If otherwise, the best response is to choose the least possible amount, i.e., 0. So we have:

$$
s_1^\star = \begin{cases} w & p(1 + r_s) + (1 - p)(1 + r_f) \geq (1 + r),\\ 0 & p(1 + r_s) + (1 - p)(1 + r_f) < (1 + r). \end{cases}
$$

Now we explore the best response of the second player. For the second player we have:

$$
\begin{aligned}
&U_2(s_1, s_2) = -e^{-a(1+r)(w-s_2)} - pe^{-a(1+r_s)s_2} - (1 - p)e^{-a(1+r_f)s_2}\\
&\frac{dU_2(s_1, s_2)}{ds_2} = -a(1 + r)e^{-a(1+r)(w-s_2)} + pa(1 + r_s)e^{-a(1+r_s)s_2} + (1 - p)a(1 + r_f)e^{-a(1+r_f)s_2}\\
&\frac{d^2U_2(s_1, s_2)}{ds_2^2} = -a^2(1 + r)^2e^{-a(1+r)(w-s_2)} - pa^2(1 + r_s)^2e^{-a(1+r_s)s_2} - (1 - p)a^2(1 + r_f)^2e^{-a(1+r_f)s_2}
\end{aligned}
$$

Agent 2's best respose is the maximizer of $U_2(., .)$. Since $\frac{d^2U_2(s_1,s_2)}{ds_2^2} < 0$, the maximizer of $U_2(., .)$ is the root of $\frac{dU_2(.,.)}{ds_2}$. As we mentioned in the Introduction part, in [5], authors consider risk-aversion agents without the bank investment option. Considering just the investment in the company reduces the expressions of $U_2(., .)$ and consequently $\frac{dU_2(.,.)}{ds_2}$ to two rather than three. In that case, it is possible to find the root of $\frac{dU_2(.,.)}{ds_2}$ analytically. But in our setting, analytical computation is not possible. Hence, the investigation of agent 2's best response needs numerical computations. We use least squares estimation to find a suitable estimation for the agent 2's best response function, or equivalently the function giving the maximizer of $U_2(., .)$.

Least Squers Estimation

We perform the following procedure to obtain a least-squares approximation for the best investment strategy of agent 2, the risk-averse one. First for the parameter values shown in the Table 1, we compute the $s_2^\star = argmax\ U_2(.,.)$. Then we applied the least-squares approximation on the resulting data set to find a polynomial of degree two. Table 2 demonstrates the coefficients of approximating degree two polynomial P_2. We test it on a data set of size 5920, related to the following parameter values:

$$\begin{aligned} &w = 3, 5, 7, 9, \\ &p = 0.5, 0.6, 0.7, 0.8, 0.9, \\ &a = 0.1, 2.1, 4.1, 6.1, \\ &r_f = 0.1, 0.2, 0.3, 0.4, \\ &r = r_f + 0.1, r_f + 0.2, \ldots, 0.8, \\ &r = r + 0.1, r + 0.2, \ldots, 0.9. \end{aligned}$$

P_2 approximates the above data set with relative errors less than 2. For 95.3% of data set, relative errors are less than 1, for 3.4% of data set relative errors are between 1 and 1.5, and just for 1.3% of data set relative errors are in the interval [1.5, 2]. Fig. 1 shows the relative errors diagram. Fig. 2 and Fig. 3 demonstrate real investment amount and its least-squers approximation for the above mentioned parameter values, respectively. Fig. 4 represents real investment in percentage.

Note that each agents' expected utility function depends just on its own strategy. It is also reflected in their Nash equilibrium. It means that under the proportional rule, the Nash equilibrium strategy profile is a strictly dominant strategy equilibrium, too.

Table 1: Numeric results used for least squares approximation

w	p	a	r_f	r	r_s	$s_2^\star$	w	p	a	r_f	r	r_s	$s_2^\star$
1	0.8	2.1	0.1	0.4	0.8	0.4819	3	0.8	2.1	0.1	0.4	0.8	1.4384
1	0.8	2.1	0.1	0.4	0.9	0.4764	3	0.8	2.1	0.1	0.4	0.9	1.4211
1	0.8	2.1	0.1	0.5	0.8	0.4880	3	0.8	2.1	0.1	0.5	0.8	1.4809
1	0.8	2.1	0.1	0.5	0.9	0.4825	3	0.8	2.1	0.1	0.5	0.9	1.4642
1	0.8	2.1	0.2	0.4	0.8	0.4814	3	0.8	2.1	0.2	0.4	0.8	1.4196
1	0.8	2.1	0.2	0.4	0.9	0.4759	3	0.8	2.1	0.2	0.4	0.9	1.4008
1	0.8	2.1	0.2	0.5	0.8	0.4874	3	0.8	2.1	0.2	0.5	0.8	1.4614
1	0.8	2.1	0.2	0.5	0.9	0.4820	3	0.8	2.1	0.2	0.5	0.9	1.4431
1	0.8	4.1	0.1	0.4	0.8	0.4754	3	0.8	4.1	0.1	0.4	0.8	1.5076
1	0.8	4.1	0.1	0.4	0.9	0.4683	3	0.8	4.1	0.1	0.4	0.9	1.5042
1	0.8	4.1	0.1	0.5	0.8	0.4871	3	0.8	4.1	0.1	0.5	0.8	1.5575
1	0.8	4.1	0.1	0.5	0.9	0.4802	3	0.8	4.1	0.1	0.5	0.9	1.5542
1	0.8	4.1	0.2	0.4	0.8	0.4722	3	0.8	4.1	0.2	0.4	0.8	1.4642
1	0.8	4.1	0.2	0.4	0.9	0.4651	3	0.8	4.1	0.2	0.4	0.9	1.4586
1	0.8	4.1	0.2	0.5	0.8	0.4838	3	0.8	4.1	0.2	0.5	0.8	1.5134
1	0.8	4.1	0.2	0.5	0.9	0.4767	3	0.8	4.1	0.2	0.5	0.9	1.5084
1	0.9	2.1	0.1	0.4	0.8	0.4783	3	0.9	2.1	0.1	0.4	0.8	1.3975
1	0.9	2.1	0.1	0.4	0.9	0.4723	3	0.9	2.1	0.1	0.4	0.9	1.3735
1	0.9	2.1	0.1	0.5	0.8	0.4844	3	0.9	2.1	0.1	0.5	0.8	1.4392
1	0.9	2.1	0.1	0.5	0.9	0.4783	3	0.9	2.1	0.1	0.5	0.9	1.4157
1	0.9	2.1	0.2	0.4	0.8	0.4781	3	0.9	2.1	0.2	0.4	0.8	1.3869
1	0.9	2.1	0.2	0.4	0.9	0.4721	3	0.9	2.1	0.2	0.4	0.9	1.3620
1	0.9	2.1	0.2	0.5	0.8	0.4841	3	0.9	2.1	0.2	0.5	0.8	1.4281
1	0.9	2.1	0.2	0.5	0.9	0.4781	3	0.9	2.1	0.2	0.5	0.9	1.4035
1	0.9	4.1	0.1	0.4	0.8	0.4662	3	0.9	4.1	0.1	0.4	0.8	1.4519
1	0.9	4.1	0.1	0.4	0.9	0.4578	3	0.9	4.1	0.1	0.4	0.9	1.4435
1	0.9	4.1	0.1	0.5	0.8	0.4778	3	0.9	4.1	0.1	0.5	0.8	1.5026
1	0.9	4.1	0.1	0.5	0.9	0.4694	3	0.9	4.1	0.1	0.5	0.9	1.4959
1	0.9	4.1	0.2	0.4	0.8	0.4645	3	0.9	4.1	0.2	0.4	0.8	1.4173
1	0.9	4.1	0.2	0.4	0.9	0.4561	3	0.9	4.1	0.2	0.4	0.9	1.4060
1	0.9	4.1	0.2	0.5	0.8	0.4761	3	0.9	4.1	0.2	0.5	0.8	1.4666
1	0.9	4.1	0.2	0.5	0.9	0.4676	3	0.9	4.1	0.2	0.5	0.9	1.4563

Table 2: Approximated degree two polynomial for the best investment strategy of agent 2

term	coefficient	term	coefficient	term	coefficient
w^2	-0.0317	wr_s	-0.0354	rr_s	0.0317
a^2	0.0190	ap	-0.0462	w	0.6990
p^2	0.0997	ar_f	-0.0686	a	-0.1315
r_f^2	22.9938	ar	0.0344	p	0.3273
r^2	-0.7519	ar_s	0.0311	r_f	-6.8408
r_s^2	0.4939	pr_f	0.4553	r	0.4579
wa	0.0173	pr	-0.0070	r_s	-0.6848
wp	-0.2022	pr_s	-0.3307	1	0.5843
wr_f	-0.1343	r_fr	-0.0445		
wr	0.1862	r_fr_s	-0.1080		

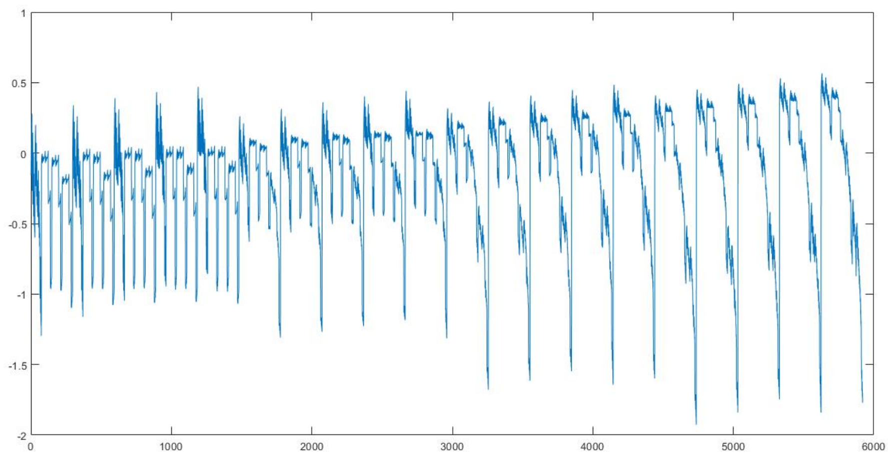

Figure 1: Relative errors diagram

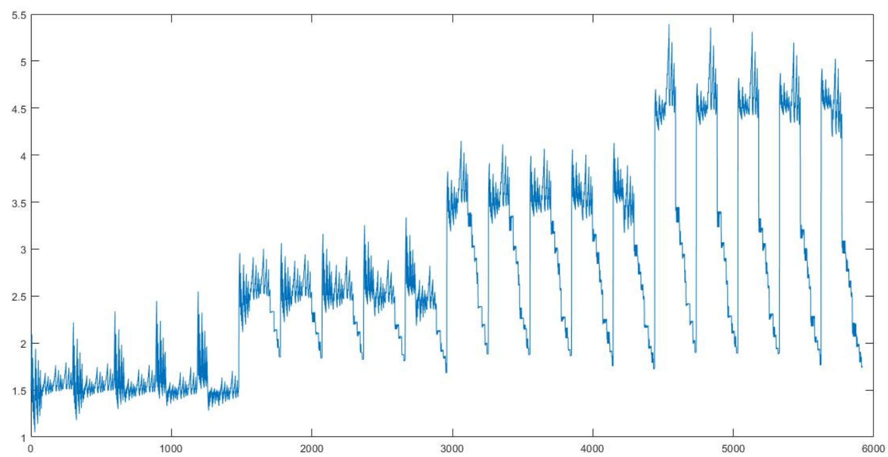

Figure 2: Exact investment

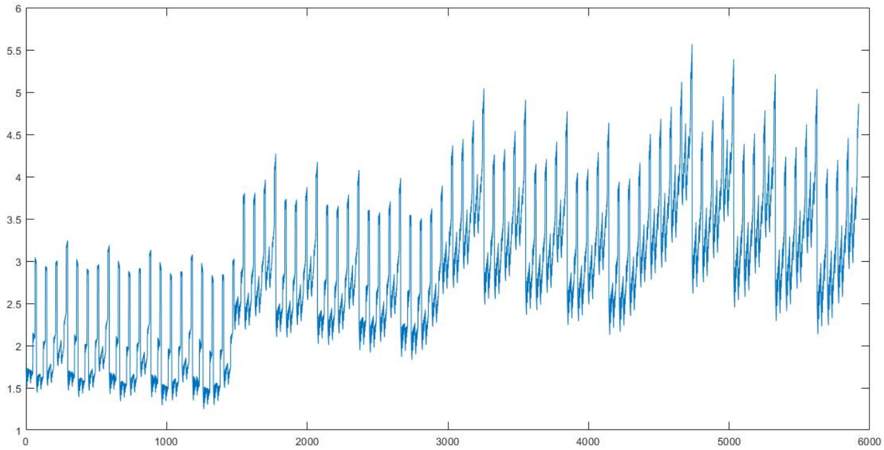

Figure 3: Least-squares approximation of the exact investment

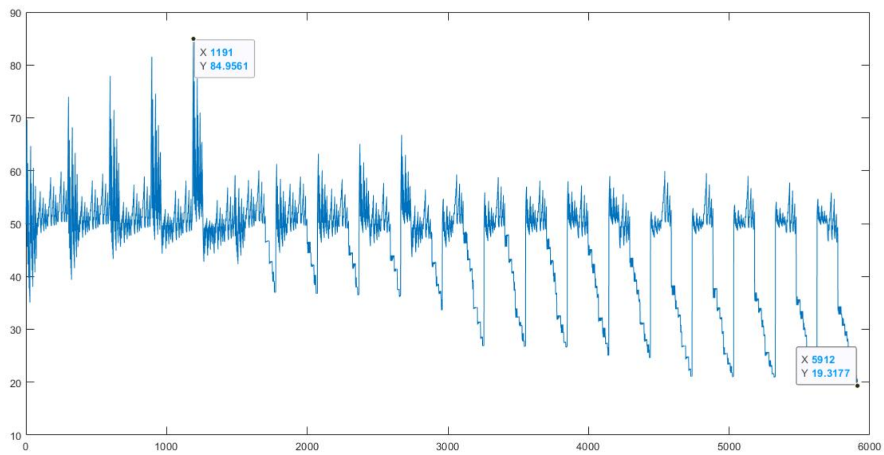

Figure 4: Investment in percentage

5 Conclusion

In this study, we investigate the investment behavior of risk-neutral and risk-averse agents when they face safe and risky investment alternatives. If the expected return in the risky investment outweighs the safe investment return, the risk-neutral agent invests all his wealth in the risky project. Otherwise, he/she puts all his/her wealth in the safe investment.

Prediction of the risk-averse agent is not as straight as the risk-neutral one. Generally, the risk-averse agent's best investment strategy can not be computed analytically, and numerical computations are needed. We present a polynomial of degree two that best approximates it based on the least-squares criteria. Our numerical test shows that the risk-averse agent invests 19.32 to 84.96 percentage of his/her wealth based on the problem parameter values.

Considering other bankruptcy rules, comparing investment behaviors under different bankruptcy rules, and generalizing the presented model for more agents remains for future works.

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