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A criterion for the monotonicity of the ratio of two Abelian integrals in piecewise-smooth differential systems

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Abstract

In this paper, we present a new criterion function for investigating the monotonicity of the ratio of two Abelian integrals in piecewise-smooth differential systems, and then, apply it to deal with some examples. More precisely, we consider the Abelian integrals of the form

$$I_k(h) = \oint_{\Gamma_h} f_k(x) y dx, \qquad k = 0, 1,$$

with $\Gamma_h = \Gamma_h^L + \Gamma_h^R$, where $\Gamma_h^L = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2}y^2 + \Psi_2(x) = h,\ x < 0\}$ and $\Gamma_h^R = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2}y^2 + \Psi_1(x) = h,\ x > 0\}$. We prove that the monotonicity of the presented criterion function implies the monotonicity of the ratio $\frac{I_1(h)}{I_0(h)}$ and provide a few examples to explain the application of this criterion.

Keywords: Piecewise-smooth differential systems, Melnikov function, Monotonicity, Abelian integral, Limit cycle
2020 MSC: 37G15, 34C07, 34C05

1 Introduction

The Hilbert 16th problem was set by the German scientist David Hilbert as one of the 23 problems in mathematics at the International Mathematics Conference held in France in 1900. The second part of this problem asks about the number of limit cycles in polynomial systems of degree n and their relative positions [5]. The mathematician Arnold simplified Hilbert's 16th problem and turned it into an easier one that researches the number of roots of the Abelian integrals. This simplified problem is closely related to the number of limit cycles of these systems and is known as the weakened Hilbert 16th problem [1]. The statement of this problem is as follows. Consider the perturbed Hamiltonian system

$$\dot{x} = H_y + \varepsilon\, p(x,y), \qquad \dot{y} = -H_x + \varepsilon\, q(x,y), \tag{1.1}$$

where p and q are polynomials in x, y, and ε is a small positive parameter. Suppose that system (1.1) has a continuous family of periodic orbits Γ_h continuously depending on the parameter $h \in (h_1, h_2)$ defined by $H(x,y) = h$. Then, the

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Abelian integral associated with this system has the form

$$I(h) = \oint_{\Gamma_h} q(x,y)dx - p(x,y)dy. \tag{1.2}$$

The weak form of Hilbert's 16th problem is to find the maximum number of isolated zeros of the Abelian integral $I(h)$ given in (1.2), and related to estimate the number of limit cycles of system (1.1). Assume that the Hamiltonian system has the form

$$H(x,y) = \frac{y^2}{2} + \Psi(x),$$

where $\Psi \in C^2(a,b)$, with $a, b \in \mathbb{R}$. Assume that the Abelian integral (1.2) can be written as

$$\sum_{k=0}^{n} \alpha_k I_k,$$

where α_k are real constants, and the I_k have the form

$$I_k = \int_{\Gamma_h} f_k(x)ydx, \qquad k = 0, 1, \ldots, m-1,$$

with $f_k \in C^1$. Without loss of generality, suppose that $I_0(h) \neq 0$ and set $P(h) = \frac{I_1(h)}{I_0(h)}$. Then, the monotonicity of $P(h)$ implies that if $m = 2$, then the Abelian integral (1.2) has at most one zero. There have been many works on the monotonicity of $P(h)$ for differential systems with smooth functions. Here, we recall some of them. In [11], the authors obtained a criterion for investigating the monotonicity of two Abelian integrals having the form

$$I_k(h) = \int_{\Gamma_h} f_k(x)g(y)dx,$$

where Γ_h is a compact component of the level curve $\{(x,y) : H(x,y) = h, h \in (h_1, h_2)\}$ and $H(x,y)$ has the form $H(x,y) = \Psi(x) + \Phi(y)$. In [4], the authors generalized the idea of [11] for codimension n case, and gave an efficient algebraic criterion that provides a sufficient condition to study the Chebyshev property of the collection $\{I_1(h), I_2(h), \ldots, I_n(h)\}$ of Abelian integrals, in which $H(x,y) = \phi(x) + \psi(y)$ or $H(x,y) = A(x) + B(x)y^{2m}$. For applicability and some results obtained by this criterion, we refer the reader to [12, 13, 14, 16, 17] and references therein. In [9], the authors studied the monotonicity of the ratio of two Abelian integrals $I_0(h) = \int_{\Gamma_h} ydx$ and $I_1(h) = \int_{\Gamma_h} xydx$, where Γ_h is a compact component of the level curve

$$\{(x,y) : H(x,y) = h, h \in (h_1, h_2)\},$$

with $H(x,y) = \Psi(x) + y^2$. They gave a new criterion for investigating the monotonicity of the ratio of the above two Abelian integrals, and obtained some new Hamiltonian functions $H(x,y)$ where $\Psi(x)$ is a polynomial of degree 5 in x so that the ratio of the associated two Abelian integrals is monotone. They obtained the sufficient and necessary conditions that the ratio of two Abelian integrals is monotone (see also [3, 6, 15]). In [7], the authors obtained new criteria to determine the monotonicity of the ratio of two Abelian integrals having the forms $\int_{\Gamma_h} f_1(x)ydx$ and $\int_{\Gamma_h} f_2(x)ydx$ or the forms $\int_{\Gamma_h} \frac{f_1(x)}{y}dx$ and $\int_{\Gamma_h} \frac{f_2(x)}{y}dx$ and Γ_h are the compact components of the level curves $\{(x,y) : H(x,y) = h,\ h \in (h_1, h_2)\}$, where $H(x,y)$ has the form $\frac{y^2}{2} + \Psi(x)$ or $\phi(x)\frac{y^2}{2} + \Psi(x)$. They gave new criteria defined directly by the functions appearing in the above-mentioned Abelian integrals, and proved that the monotonicity of the criteria implies the monotonicity of the ratio of the Abelian integrals. The weakened Hilbert 16th problem has been extended to apply to systems of differential equations with piecewise-smooth functions. In [8], the authors considered the general form of a piecewise near-Hamiltonian planar system as

$$\left\{ \begin{array}{l} \dot{x} = H_y(x,y) + \varepsilon\, p(x,y), \\ \dot{y} = -H_x(x,y) + \varepsilon\, q(x,y), \end{array} \right. \tag{1.3}$$

where ε is a small parameter, and H, p, q are piecewise-smooth functions defined respectively by

$$H(x,y) = \left\{ \begin{array}{ll} H^+(x,y), & x > 0, \\ H^-(x,y), & x \leqslant 0, \end{array} \right. \quad p(x,y) = \left\{ \begin{array}{ll} p^+(x,y), & x > 0, \\ p^-(x,y), & x \leqslant 0, \end{array} \right. \quad q(x,y) = \left\{ \begin{array}{ll} q^+(x,y), & x > 0, \\ q^-(x,y), & x \leqslant 0. \end{array} \right.$$

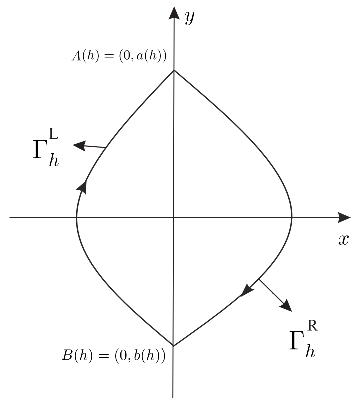

Figure 1: Periodic orbits of system $(1.3)|_{\varepsilon=0}$.

Here, $H^{\pm}$, $p^{\pm}$, and $q^{\pm}$ are assumed to be analytic functions. Therefore, system (1.3) has two analytic subsystems,

$$\begin{cases} \dot{x} = H_y^+(x,y) + \varepsilon\, p^+(x,y), \\ \dot{y} = -H_x^+(x,y) + \varepsilon\, q^+(x,y) \end{cases} \tag{1a}$$

and

$$\begin{cases} \dot{x} = H_y^-(x,y) + \varepsilon\, p^-(x,y), \\ \dot{y} = -H_x^-(x,y) + \varepsilon\, q^-(x,y). \end{cases} \tag{1b}$$

Assume that system $(1.3)|_{\varepsilon=0}$ satisfies the following assumptions:

Assumption (I). It has a continuous family of periodic orbits around the origin.

Assumption (II). There exists an interval $\Omega = (h_1, h_2)$ and two points $A(h) = (0, a(h))$ and $B(h) = (0, b(h))$ such that for $h \in \Omega$, $H^+(A(h)) = H^+(B(h)) = h$ and $H^-(A(h)) = H^-(B(h)) = h$, where $b(h) < 0 < a(h)$.

Assumption (III). The right subsystem $(1a)|_{\varepsilon=0}$ has an orbital arc Γ_h^R starting from $A(h)$ and ending at $B(h)$ defined as $H^+(x,y) = h$, $x > 0$; the left subsystem $(1b)|_{\varepsilon=0}$ has an orbital arc Γ_h^L starting from $B(h)$ and ending at $A(h)$ defined as $H^-(x,y) = H^-(B(h)) = h$, $x \leqslant 0$.

Under the above assumptions, the unperturbed system $(1.3)|_{\varepsilon=0}$ has a continuous family of non-smooth periodic orbits $\Gamma_h = \Gamma_h^R \cup \Gamma_h^L$, $h \in \Omega$. For definiteness, we let that the orbits Γ_h for $h \in \Omega$ orientate clockwise; see Fig. 1.

The authors of [8], by defining a bifurcation function, obtained the following theorem.

Theorem 1.1. Under the Assumptions (I)-(III), the first-order Melnikov function of system (1.3) takes the form

$$I(h) := \frac{H_y^+(A)}{H_y^-(A)} \left[\frac{H_y^-(B)}{H_y^+(B)} \int_{\Gamma_h^R} (q^+ dx - p^+ dy) + \int_{\Gamma_h^L} (q^- dx - p^- dy) \right], \qquad h \in \Omega. \tag{1.4}$$

Further,

(i) if $I(h)$ has k zeros in h on the interval Ω with each having an odd multiplicity, then (1.3) has at least k limit cycles bifurcating from the period annulus for $0 < \varepsilon \ll 1$;
(ii) if $I(h)$ has at most k zeros in h on the interval Ω, taking into account the multiplicity, then there are at most k limit cycles of (1.3) bifurcating from the period annulus.

Inspired by [2], we consider the classical Hamiltonian function given by

$$H(x,y) := \frac{y^2}{2} + \Psi(x) := \begin{cases} H^+(x,y) = \frac{y^2}{2} + \Psi_1(x), & x > 0, \\ H^-(x,y) = \frac{y^2}{2} + \Psi_2(x), & x \leqslant 0, \end{cases} \tag{1.5}$$

where Ψ_1 and Ψ_2 are analytic in some open intervals (α_1, A_1) and (α_2, A_2), respectively, and zero belongs to $(\alpha_1, A_1) \cap (\alpha_2, A_2)$ satisfying $\Psi_1(0) = \Psi_2(0) = 0$. We also assume that the following hypothesis is satisfied

$$\begin{array}{lll} x\Psi_1'(x) > 0, & \text{for all} & x \in (0, A_1), \\ x\Psi_2'(x) > 0, & \text{for all} & x \in (\alpha_2, 0). \end{array} \tag{H1}$$

Under the above hypothesis (H1), $(0,0)$ is a local minimum of H. Therefore, there exists a punctured neighborhood of the origin foliated by periodic orbits and these assumptions on H imply the Assumptions (II) and (III). On the other hand, $H_y^+(x,y) = H_y^-(x,y) = y$, then formula (1.4) can be written as

$$I(h) = \int_{\Gamma_h^R} (q^+ dx - p^+ dy) + \int_{\Gamma_h^L} (q^- dx - p^- dy), \qquad h \in \Omega, \tag{1.6}$$

with $\Gamma_h^R = \{(x,y) \in \mathbb{R}^2 \mid H^+(x,y) = h,\ x > 0\}$ and $\Gamma_h^L = \{(x,y) \in \mathbb{R}^2 \mid H^-(x,y) = h,\ x \leq 0\}$. The coefficients of $p^\pm$ and $q^\pm$ are considered as parameters of the problem. If we take $p^\pm = yp_+(x) = y \sum_{i=0}^{n} a_i x^i$ and $q^\pm = yq_+(x) = y \sum_{i=0}^{n} b_i x^i$, then, by rescaling, the integral (1.6) splits as a combination of

$$c_0 I_0(h) + c_1 I_1(h) + \ldots + c_{n-1} I_{n-1}(h),$$

where c_i depends on the initial parameters, and

$$I_i(h) = \int_{\Gamma_h^L} x^i y dx + \int_{\Gamma_h^R} x^i y dx, \qquad i = 0, 1, 2, \ldots, n-1.$$

Here, we suppose that

$$I_i(h) = \int_{\Gamma_h^R} f_i(x) y dx + \int_{\Gamma_h^L} f_i(x) y dx, \qquad i = 0, 1, \ldots, n-1,$$

where $f_i(x)$ are analytic functions. Now, we assume one of $\{I_i(h) \mid i = 0, 1, \ldots, n-1\}$ is non-vanishing, say $I_0(h)$, by reindexing if necessary. Then, we can put

$$P(h) := \frac{I_1(h)}{I_0(h)}, \qquad Q(h) := \frac{I_2(h)}{I_0(h)}.$$

Therefore, in the case $n = 2$, the monotonicity of the ratio of $P(h)$ implies that $I(h)$ has a unique zero. Also, in the case $n = 3$, if one of $P(h)$ and $Q(h)$, for instance $P(h)$, is monotonic, then the number of zeros of $I(h)$ is equal to the number of intersection points of the straight line $\{(P,Q)|Q = a_0 + a_1 P\}$ and the curve $\{(P,Q)|Q = -a_2 Q(h(P))\}$ on PQ-plane, where $h = h(P)$ is the inverse function of $P = P(h)$. This shows that the study on the monotonicity of the ratio of two Abelian integrals is very important to determine the number of zeros of the Melnikov function $I(h)$. It follows from (H1) that there exists an analytic function σ such that for all $x \in (\alpha_2, 0)$, $\Psi_2(x) = \Psi_1(\sigma(x))$. Note that $\sigma(0) = 0$ and $x\sigma(x) < 0$ for all $x \in (\alpha_2, 0)$. Along the curve Γ_h (see Fig. 2), we show the branches of Γ_h^{LR} (Γ_h^R and Γ_h^L) by

$$\begin{aligned} y_+^R(x,h) &:= \sqrt{2(h - \Psi_1(x))}, \qquad & y_-^R(x,h) &:= -\sqrt{2(h - \Psi_1(x))}, \\ y_+^L(x,h) &:= \sqrt{2(h - \Psi_2(x))}, \qquad & y_-^L(x,h) &:= -\sqrt{2(h - \Psi_2(x))}. \end{aligned}$$

Also, we have the following properties.

Lemma 1.2. Suppose that the functions H^+ and H^- have the form (2) and the condition (H1) holds. Then we have $y_-^L(x) = y_-^R(\sigma(x)) < 0$ and $y_+^L(x) = y_+^R(\sigma(x)) > 0$ for all $x \in (\alpha_2, 0)$ where $\alpha_2 < x < 0 < \sigma(x) < A_1$.

Here, we consider the ratio of two Abelian integrals

$$P(h) = \frac{I_1(h)}{I_0(h)}, \tag{1.7}$$

with

$$I_0(h) = \int_{\Gamma_h^R} f_0(x) y dx + \int_{\Gamma_h^L} f_0(x) y dx, \qquad I_1(h) = \int_{\Gamma_h^R} f_1(x) y dx + \int_{\Gamma_h^L} f_1(x) y dx,$$

where $f_i(x) \in C^1(\alpha_2, A_1)$ for $i = 0, 1$. To ensure that the denominator of $P(h)$ is nonzero, we assume the following hypothesis:

$$f_0(x) f_0(\sigma(x)) > 0, \quad \text{for all} \quad x \in (\alpha_2, 0), \tag{1.8}$$

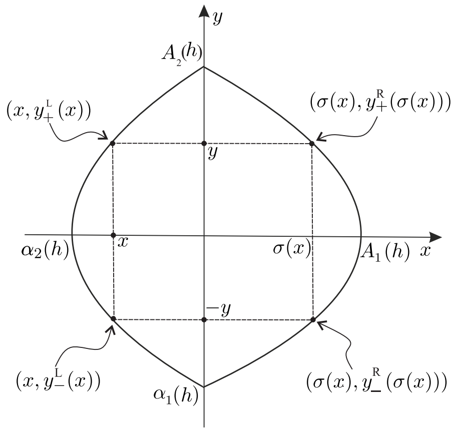

Figure 2: Notation related to piecewise smooth closed curve.

which implies that

$$I_0(h) = \int_{\alpha_2(h)}^0 f_0(x)(y_+^L(x) - y_-^L(x))dx - \int_{\alpha_2(h)}^0 f_0(\sigma(x))(y_+^R(\sigma(x)) - y_+^R(\sigma(x)))\frac{\Psi_2'(x)}{\Psi_1'(\sigma(x))}dx \neq 0.$$

The authors of [2] generalized the criterion obtained in [9] for piecewise-smooth differential systems and studied the monotonicity of the ratio of two integrals in piecewise-smooth differential systems. We give a summary of the results obtained there in the following theorem.

Theorem 1.3. Assume that $H(x, y)$ has the form (1.5), and the hypotheses (H1) and (H2) hold. Then $\xi'(x) > 0$ (resp. $\xi'(x) < 0$) for $x \in (\alpha_2, 0)$ implies that $P'(h) < 0$ (resp. $P'(h) > 0$) for $h \in (h_1, h_2)$, where

$$\xi(x) = \frac{f_1(x)\Psi_1'(\sigma(x)) - f_1(\sigma(x))\Psi_2'(x)}{f_0(x)\Psi_1'(\sigma(x)) - f_0(\sigma(x))\Psi_2'(x)}. \tag{1.9}$$

In this paper, we generalize some of the results obtained in [7] for piecewise-smooth differential systems. Our main results are stated in the next section.

2 The main results and proofs

In this section, inspired by [10], we deduce a criterion on the monotonicity of the ratio of two Abelian integrals in piecewise-smooth differential systems. Our main results are the following.

Lemma 2.1. Let $\Gamma_h = \Gamma_h^L + \Gamma_h^R$, where

$$\Gamma_h^L = \left\{(x, y) \in \mathbb{R}^2 : \Psi_2(x) + \frac{y^2}{2} = h, \quad x \leq 0\right\},$$
$$\Gamma_h^R = \left\{(x, y) \in \mathbb{R}^2 : \Psi_1(x) + \frac{y^2}{2} = h, \quad x > 0\right\}.$$

Then, for $k = 0, 1$, we have

$$\oint_{\Gamma_h} f_k(x)ydx = \int_{\Gamma_h^L} \frac{g_k^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_k^+(x)}{y}dx,$$

where

$$g_k^-(x) = \Psi_2'(x)\int_0^x f_k(t)dt, \quad g_k^+(x) = \Psi_1'(x)\int_0^x f_k(t)dt.$$

Proof . We can write

$$\begin{aligned}
\oint_{\Gamma_h} f_k(x)ydx &= \int_{\Gamma_h^L} f_k(x)ydx + \int_{\Gamma_h^R} f_k(x)ydx \\
&= \int_{\Gamma_h^L+\overrightarrow{A_2\alpha_1}} f_k(x)ydx + \int_{\Gamma_h^R+\overrightarrow{\alpha_1 A_2}} f_k(x)ydx \\
&= \oint_{\Gamma_h^L+\overrightarrow{A_2\alpha_1}} yd\left(\int_0^x f_k(t)dt\right) + \oint_{\Gamma_h^R+\overrightarrow{\alpha_1 A_2}} yd\left(\int_0^x f_k(t)dt\right) \\
&= -\oint_{\Gamma_h^L+\overrightarrow{A_2\alpha_1}} \left(\int_0^x f_k(t)dt\right)dy - \oint_{\Gamma_h^R+\overrightarrow{\alpha_1 A_2}} \left(\int_0^x f_k(t)dt\right)dy \\
&= -\oint_{\Gamma_h^L} \left(\int_0^x f_k(t)dt\right)\left(-\frac{\psi_2'(x)}{y}dx\right) - \oint_{\Gamma_h^R} \left(\int_0^x f_k(t)dt\right)\left(-\frac{\psi_1'(x)}{y}dx\right) \\
&= \oint_{\Gamma_h^L} \frac{\psi_2'(x)\int_0^x f_k(t)dt}{y}dx + \oint_{\Gamma_h^R} \frac{\psi_1'(x)\int_0^x f_k(t)dt}{y}dx \\
&= \int_{\Gamma_h^L} \frac{g_k^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_k^+(x)}{y}dx.
\end{aligned}$$

□

In the next theorem, we introduce a criterion function for studying the monotonicity of the ratio of two Abelian integrals as follows

$$J_k(h) = \int_{\Gamma_h^L} \frac{g_k^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_k^+(x)}{y}dx, \qquad k = 0, 1,$$

where $g_k^- \in C^1((\alpha_2, 0), \mathbb{R})$ and $g_k^+ \in C^1((0, A_1), \mathbb{R})$. Let

$$G_k(x) = \frac{g_k^-(x)}{\psi_2'(x)} - \frac{g_k^+(\sigma(x))}{\psi_1'(\sigma(x))}, \qquad k = 0, 1, \quad x \in (\alpha_2, 0).$$

Theorem 2.2. Assume that (H1) and the following hypothesis are satisfied:

$$G_0(x) < 0, \qquad G_0'(x) > 0, \quad \forall x \in (\alpha_2, 0). \tag{H2}$$

Let $S(h) = \frac{J_1(h)}{J_0(h)}$ and $\tau(x) = \frac{G_1(x)}{G_0(x)}$. Then $\tau'(x) > 0$ (resp. $\tau'(x) < 0$) in $(\alpha_2, 0)$ implies $S'(h) < 0$(resp. $S'(x) > 0$) in $(0, h_2)$.

Proof . If $x \in (\alpha_2, 0)$ then $\sigma(x) \in (A_1, 0)$ and $\psi_2(x) = \psi_1(\sigma(x))$ for any $x \in (\alpha_2, 0)$. Hence, $\sigma'(x) = \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}$. Note that $\sigma'(x) < 0$ for $x \in (\alpha_2, 0)$. Assume $\alpha_2(h)$ and $A_1(h)$ be the intersection of the curves Γ_h^L and Γ_h^R with the x-axis. Then $\alpha_2 < \alpha_2(h) < 0 < A_1(h) < A_1$. For $k = 0, 1$ we have

$$J_k(h) = \int_{\alpha_2(h)}^0 \frac{g_k^-(x)}{y_+^L(x,h)}dx + \int_0^{A_1(h)} \frac{g_k^+(x)}{y_+^R(x,h)}dx + \int_{A_1(h)}^0 \frac{g_k^+(x)}{y_-^R(x,h)}dx + \int_0^{\alpha_2(h)} \frac{g_k^-(x)}{y_-^L(x,h)}dx.$$

By the change of variable $x \mapsto \sigma(x)$, we can write

$$J_k(h) = \int_{\alpha_2(h)}^0 \frac{g_k^-(x)}{y_+^L(x,h)}dx + \int_0^{\alpha_2(h)} \frac{g_k^+(x)}{y_+^R(x,h)} \times \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx + \int_{\alpha_2(h)}^0 \frac{g_k^+(x)}{y_-^R(x,h)} \times \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx + \int_0^{\alpha_h(h)} \frac{g_k^-(x)}{y_-^L(x,h)}dx.$$

Since

$$\begin{aligned}
y_+^R(\sigma(x),h) &:= \sqrt{2(h-\Psi_1(\sigma(x)))} = \sqrt{2(h-\Psi_2(x))} = y_+^L(x,h), \\
y_-^R(\sigma(x),h) &:= -\sqrt{2(h-\Psi_1(\sigma(x)))} = -\sqrt{2(h-\Psi_2(x))} = -y_+^L(x,h),
\end{aligned}$$

we can express $J_k(h)$ as follows:

$$\begin{aligned}
J_k(h) &= \int_{\alpha_2(h)}^{0} \frac{g_k^-(x)}{y_+^L(x,h)}dx + \int_0^{\alpha_2(h)} \frac{g_k^+(x)}{y_+^L(x,h)} \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx + \int_{\alpha_2(h)}^{0} \frac{g_k^+(x)}{-y_+^L(x,h)} \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx + \int_0^{\alpha_h(h)} \frac{g_k^-(x)}{-y_+^L(x,h)}dx \\
&= \int_{\alpha_2(h)}^{0} \frac{g_k^-(x)}{y_+^L(x,h)}dx - \int_{\alpha_2(h)}^{0} \frac{g_k^+(x)}{y_+^L(x,h)} \frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx - \int_{\alpha_2(h)}^{0} \frac{g_k^+(x)}{y_+^L(x,h)} (\frac{\psi_2'(x)}{\psi_1'(\sigma(x))}dx + \int_{\alpha_2(h)}^{0} \frac{g_k^-(x)}{y_+^L(x,h)}dx \\
&= 2\int_{\alpha_2(h)}^{0} \left[g_k^-(x) - g_k^+(x)\frac{\psi_2'(x)}{\psi_1'(\sigma(x))}\right] \frac{1}{y_+^L(x,h)}dx \\
&= 2\int_{\alpha_2(h)}^{0} \frac{\psi_2'(x)G_k(x)}{y_+^L(x,h)}dx,
\end{aligned}$$

where

$$G_k(x) = \frac{g_k^-(x)}{\psi_2'(x)} - \frac{g_k^+(\sigma(x))}{\psi_1'(\sigma(x))}, \qquad k = 0, 1.$$

Noting that for any $x \in (\alpha_2, 0)$, $y_+^L(x,h) > 0$, $\psi_2'(x) < 0$, and $G_0(x) < 0$ for any $x \in (\alpha_2, 0)$, we have that

$$J_0(h) = 2\int_{\alpha_2(h)}^{0} \frac{\psi_2'(x)G_0(x)}{y_+^L(x,h)}dx > 0.$$

To prove the monotonicity of $S(h)$, we only need to show that for any constant c, the integral

$$K(h) = J_1(h) - cJ_0(h) = 2\int_{\alpha_2(h)}^{0} \frac{\psi_2'(x)\,(G_1(x) - cG_0(x))}{y_+^L(x,h)}dx, \tag{2.1}$$

has at most one zero for $h \in (0, h_2)$. Without loss of generality, we may assume that $\tau'(x) > 0$. This assumption implies that $\frac{G_1(x)}{G_0(x)}$ is monotonically increasing, thus $G_1(x) - cG_0(x)$ has at most one zero in $(\alpha_2, 0)$. If the function $G_1(x) - cG_0(x)$ has no zero in $(\alpha_2, 0)$, then $K(h)$ in (2.1) has no zero, and the proof is finished. Hence, we may assume that the function $G_1(x) - cG_0(x)$ has exactly one zero in $(\alpha_2, 0)$. Denote this zero by $c^* \in (\alpha_2, 0)$, then

$$G_1(x) - cG_0(x) \begin{cases} = 0, & x = c^*, \\ < 0, & x \in (c^*, 0), \\ > 0, & x \in (\alpha_2, c^*). \end{cases}$$

Let $h^* = \psi_2(c^*) \in (0, h_2)$. Then, for $h \in (0, h^*]$, we have $c^* \le \alpha_2(h) < 0$ and $K(h) > 0$, that is, the function $K(h)$ has no zero in $h \in (0, h^*]$. When $h > h^*$, then $\alpha_2 < \alpha_2(h) < c^* < 0$ and we can write $K(h)$ in (2.1) as follows:

$$\begin{aligned}
K(h) &= 2\int_{\alpha_2(h)}^{c^*} \frac{\psi_2'(x)(G_1(x) - cG_0(x))}{y_+^L(x,h)}dx + 2\int_{c^*}^{0} \frac{\psi_2'(x)(G_1(x) - cG_0(x))}{y_+^L(x,h)}dx \\
&= -2\int_0^{y_+^L(c^*,h)} (G_1(x^L(y,h)) - cG_0(x^L(y,h)))dy + 2\int_{c^*}^{0} \frac{\psi_2'(x)(G_1(x) - cG_0(x))}{y_+^L(x,h)}dx.
\end{aligned}$$

Differentiating $K(h)$ with respect to h gives that

$$\begin{aligned}
K'(h) =& 2\int_{y_+^L(c^*,h)}^{0} \left[G_1'(x^L(y,h)) - cG_0'(x^L(y,h))\right] \times \frac{\partial x^L(y,h)}{\partial h}dy \\
&- 2\int_{c^*}^{0} \frac{\psi_2'(x)[G_1(x) - cG_0(x))}{(y_+^L(x,h))^2} \times \frac{\partial y_+^L(x,h)}{\partial h}dx - 2[G_1(c^*) - cG_0(c^*)] \times \frac{\partial y_+^L(c^*,h)}{\partial h} \\
=& 2\int_{y_+^L(c^*,h)}^{0} \frac{G_1'(x) - cG_0'(x)}{\psi_2'(x)}dy + 2\int_{c^*}^{0} \frac{-\psi_2'(x)(G_1(x) - cG_0(x))}{(y_+^L(x,h))^3}dx \\
=& 2\int_{c^*}^{\alpha_2(h)} \frac{G_1'(x) - cG_0'(x)}{y_+^L(x,h)}dx + 2\int_{c^*}^{0} \frac{-\psi_2'(x)(G_1(x) - cG_0(x))}{(y_+^L(x,h))^3}dx.
\end{aligned}$$

We will prove that $K'(h) < 0$ for $h \in (0, h_2)$. Since $\left(\frac{G_1(x)}{G_0(x)}\right)' > 0$, $G_1'G_0 - G_0'G_1 > 0$, thus $G_1'G_0 > G_0'G_1$. This implies that $G_1' < \frac{G_0'G_1}{G_0}$ since $G_0 < 0$ by the assumption. Now, we can write

$$G_1'(x) - cG_0'(x) < \frac{G_1(x)}{G_0(x)}G_0'(x) - cG_0'(x) = G_0'(x)\left[\frac{G_1(x)}{G_0(x)} - c\right] < 0, \qquad \forall x \in (\alpha_2, 0).$$

We conclude that $K'(h) < 0$, that is, $K(h)$ has at most one zero in $h \in (0, h_2)$. This completes the proof of Theorem 2.2. □

Remark 2.3. By Lemma 2.1, the monotonicity of $S(h)$ implies that the function $P(h)$ defined in (1.7) is monotone in $(0, h_2)$.

3 Applications

In this section, we provide some examples to show the application of our main results.

Example 3.1. Consider

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{2}\,y^2 + \frac{1}{2}\,x^2 - \frac{1}{3}\,x^3, & x > 0, \\ H^-(x,y) = \frac{1}{2}\,y^2 + \frac{1}{2}\,x^2, & x \leqslant 0. \end{cases} \tag{3.1}$$

Applying Theorem 2.2, we will study the monotonicity of the function

$$S(h) = \frac{J_1(h)}{J_0(h)} = \frac{\int_{\Gamma_h^L} \frac{g_1^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_1^+(x)}{y}dx}{\int_{\Gamma_h^L} \frac{g_0^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_0^+(x)}{y}dx},$$

where the semi-orbits Γ_h^L and Γ_h^R are defined for $h \in (0, \frac{1}{6})$; see Fig. 3. It is easy to see that the function $x \mapsto \sigma(x)$ defined by the implicit relation

$$\Psi_2 = \Psi_1 \circ \sigma \Longleftrightarrow \frac{1}{2}x^2 = \frac{1}{2}\sigma^2 - \frac{1}{3}\sigma^3, \tag{3.2}$$

satisfies $-\frac{1}{\sqrt{3}} < x < 0 < \sigma(x) < 1$. We also see that

$$\begin{aligned} &x\Psi_1'(x) = x^2(1-x) > 0, && \forall x \in (-\infty, 1) \setminus \{0\}, \\ &x\Psi_2'(x) = x^2 > 0, && \forall x \neq 0. \end{aligned}$$

So, the hypothesis (H1) holds. We choose $f_0(x) = 1$ and $f_1(x) = x$. Then

$$\begin{aligned} g_1^-(x) &= \psi_2'(x)\int_0^x f_1(t)dt = x\int_0^x tdt = \frac{x^3}{2}, \\ g_1^+(x) &= \psi_1'(x)\int_0^x f_1(t)dt = x(1-x)\int_0^x tdt = \frac{x^3}{2}(1-x), \\ g_0^-(x) &= \psi_2'(x)\int_0^x f_0(t)dt = x\int_0^x dt = x^2, \\ g_0^+(x) &= \psi_1'(x)\int_0^x f_0(t)dt = x(1-x)\int_0^x dt = x^2(1-x). \end{aligned}$$

Therefore,

$$\begin{aligned} &G_0(x) = \frac{g_0^-(x)}{\psi_2'(x)} - \frac{g_0^+(\sigma(x))}{\psi_1'(\sigma(x))} = \frac{x^2}{x} - \frac{\sigma^2(1-\sigma)}{\sigma(1-\sigma)} = x - \sigma < 0, \\ &G_0'(x) = 1 - \sigma' > 0, \end{aligned}$$

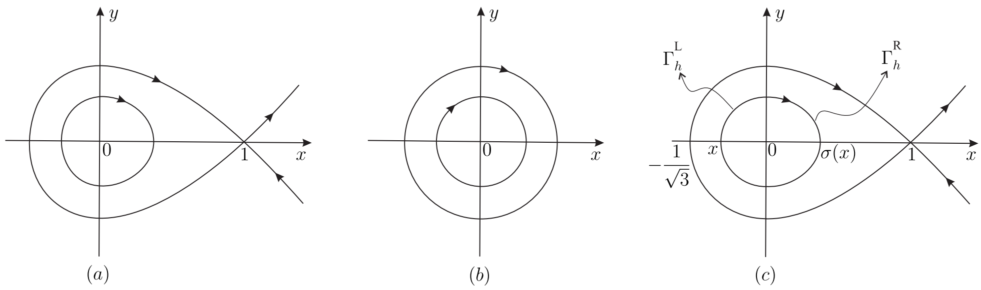

Figure 3: The level curves of Example 3.1.

for $x \in (-\frac{1}{\sqrt{3}}, 0)$ and the hypothesis (H2) holds. Since

$$G_1(x) = \frac{g_1^-(x)}{\psi_2'(x)} - \frac{g_1^+(\sigma(x))}{\psi_1'(\sigma(x))} = \frac{\frac{x^3}{2}}{x} - \frac{\frac{\sigma^3}{2}(1-\sigma)}{\sigma(1-\sigma)} = \frac{x^2}{2} - \frac{\sigma^2}{2},$$

we have

$$\tau(x) = \frac{x+\sigma}{2}, \quad \tau'(x) = \frac{1+\sigma'}{2}.$$

It follows from (3.2) that $x = -\sigma\sqrt{1-\frac{2}{3}\sigma}$. Therefore,

$$\tau'(x) = \frac{1}{2}\left(1 + \frac{x}{\sigma(1-\sigma)}\right) = \frac{(1-\sigma)+x}{2(1-\sigma} = \frac{1-\sigma-\sqrt{1-\frac{2}{3}\sigma}}{2(1-\sigma)}.$$

Consider the function

$$f(\sigma) = 1 - \sigma - \sqrt{1-\frac{2}{3}\sigma},$$

so that $f(0) = 0$ and $f(1) = -\frac{1}{\sqrt{3}} < 0$. We will investigate the sign of this function on the interval $(-\frac{1}{\sqrt{3}}, 0)$. By differentiating, we find that

$$f'(\sigma) = \frac{2(\sigma - \frac{4}{3})}{(1+3\sqrt{1-\frac{2}{3}\sigma})\sqrt{1-\frac{2}{3}\sigma}} < 0.$$

Since f is monotonically decreasing with respect to σ, we get that $f(1) = -\frac{1}{\sqrt{3}} < f(\sigma) < f(0) = 0$. Hence, $\tau'(x) < 0$ for any $x \in (-\frac{1}{\sqrt{3}}, 0)$. Using Theorem 2.2 we obtain that $S'(h) > 0$ for any $h \in (0, \frac{1}{6})$.

Example 3.2. Consider

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x, & x > 0, \\ H^-(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x^2 - x, & x \leqslant 0. \end{cases} \tag{3.3}$$

We will show that

$$S(h) = \frac{J_1(h)}{J_0(h)} = \frac{\int_{\Gamma_h^L} \frac{g_1^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_1^+(x)}{y}dx}{\int_{\Gamma_h^L} \frac{g_0^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_0^+(x)}{y}dx},$$

for $h \in (0, \frac{2}{3})$ is monotone. The semi-orbits Γ_h^L and Γ_h^R are defined for $h \in (0, \frac{2}{3})$; see Fig. 4. We have

$$\Psi_2 = \Psi_1 \circ \sigma \Longleftrightarrow N_1(x,\sigma) = -\frac{1}{3}x^3 + x^2 - x + \frac{1}{3}\sigma^3 - \sigma = 0, \tag{3.4}$$

where $1 - 3^{\frac{1}{3}} = x^* < x < 0 < \sigma(x) < 1$. Moreover,

$$x\Psi_1'(x) = x(1-x^2) > 0, \qquad \forall x \in (0,1),$$
$$x\Psi_2'(x) = -x(x-1)^2 > 0, \qquad \forall x \in (1-3^{\frac{1}{3}}, 0).$$

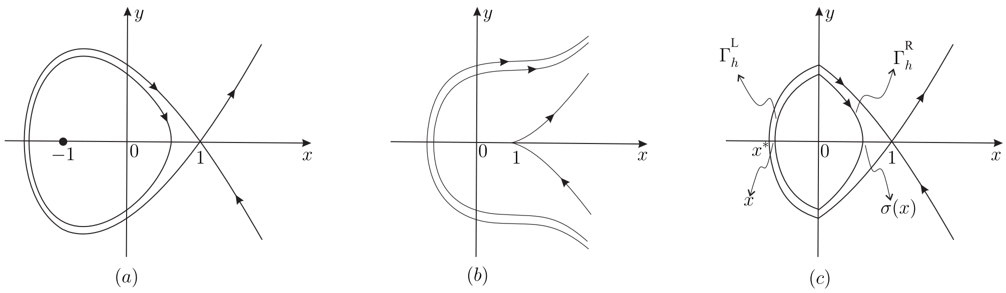

Figure 4: The level curves of Example 3.2.

and the hypothesis (H1) is satisfied. Now choose the functions $f_0(x) = 1$ and $f_1(x) = x^2$. Then,

$$\begin{aligned}
g_1^-(x) &= \psi_2'(x)\int_0^x f_1(t)dt = -(x-1)^2\int_0^x t^2dt = -\frac{x^3}{3}(x-1)^2,\\
g_1^+(x) &= \psi_1'(x)\int_0^x f_1(t)dt = (1-x^2)\int_0^x t^2dt = \frac{x^3}{3}(1-x^2),\\
g_0^-(x) &= \psi_2'(x)\int_0^x f_0(t)dt = -(x-1)^2\int_0^x dt = -x(x-1)^2,\\
g_0^+(x) &= \psi_1'(x)\int_0^x f_0(t)dt = (1-x^2)\int_0^x dt = x(1-x^2),
\end{aligned}$$

and

$$G_0(x) = \frac{g_0^-(x)}{\psi_2'(x)} - \frac{g_0^+(\sigma(x))}{\psi_1'(\sigma(x))} = x - \sigma < 0, \quad G_0'(x) = 1 - \sigma' > 0,$$

for $x \in (1 - 3^{\frac{1}{3}}, 0)$ and the hypothesis (H2) is satisfied. We now compute

$$\begin{aligned}
G_1(x) &= \frac{g_1^-(x)}{\psi_2'(x)} - \frac{g_1^+(\sigma(x))}{\psi_1'(\sigma(x))} = \frac{x^3}{3} - \frac{\sigma^3}{3},\\
\tau(x) &= \frac{G_1(x)}{G_0(x)} = \frac{1}{3}(x^2 + x\sigma + \sigma^2).
\end{aligned}$$

Since σ depends on x, we get

$$\tau'(x) = \frac{x^3 + 2(\sigma-1)x^2 - (2\sigma+1)x + \sigma^2 + \sigma}{-3(1+\sigma)} \equiv \frac{W_1(x,\sigma)}{-3(1+\sigma)}.$$

We compute the resultant between $N_1(x,\sigma)$ in (3.4) and $W_1(x,\sigma)$ with respect to σ and obtain that

$$Res(N_1, W_1, \sigma) = x^2(x^7 - \frac{20}{3}x^6 + 21x^5 - 38x^4 + 42x^3 - 27x^2 + 8x - \frac{16}{3}). \tag{3.5}$$

Using Sturm's Theorem, we deduce that the polynomial

$$x^7 - \frac{20}{3}x^6 + 21x^5 - 38x^4 + 42x^3 - 27x^2 + 8x - \frac{16}{3}, \tag{3.6}$$

has no zero in $(1 - 3^{\frac{1}{3}}, 0)$. So $W_1(x,\sigma)$ and $N_1(x,\sigma)$ have no common zeros in $(1 - 3^{\frac{1}{3}}, 0)$. Indeed, $W_1(x,\sigma) < 0$ and $\tau'(x) > 0$ for any $x \in (1 - 3^{\frac{1}{3}}, 0)$. Now, Theorem 2.2 implies that $S(h)$ is monotone for $h \in (0, \frac{1}{12})$.

Example 3.3. Consider

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{2}y^2 - \frac{1}{24}(3x^4 - 8x^2 + 6), & x > 0,\\ H^-(x,y) = \frac{1}{2}y^2 - \frac{1}{192}(24x^4 - 50x^2 + 27), & x \leqslant 0. \end{cases} \tag{3.7}$$

We show that the function

$$S(h)=\frac{J_1(h)}{J_0(h)}=\frac{\int_{\Gamma_h^L}\frac{g_1^-(x)}{y}dx+\int_{\Gamma_h^R}\frac{g_1^+(x)}{y}dx}{\int_{\Gamma_h^L}\frac{g_0^-(x)}{y}dx+\int_{\Gamma_h^R}\frac{g_0^+(x)}{y}dx},$$

is monotone in $h\in(0,\frac{1}{24})$. The semi-orbits Γ_h^L and Γ_h^R in this example are defined for $h\in(0,\frac{1}{24})$; see Fig. 5. We have

$$\Psi_2=\Psi_1\circ\sigma \Longleftrightarrow N_2(x,\sigma)=\frac{1}{8}x^8-\frac{25}{96}x^6+\frac{9}{64}x^4-\frac{1}{8}\sigma^8+\frac{1}{3}\sigma^6-\frac{1}{4}\sigma^4=0, \tag{3.8}$$

with $-\frac{3}{4}<x<0<\sigma(x)<1$. Since

$$\begin{aligned}
x\Psi_1'(x)&=x^4(x^2-1)^2>0, \qquad \forall x\in(0,1),\\
x\Psi_2'(x)&=x^4(x^2-\frac{9}{16})(x^2-1)>0, \qquad \forall x\in(-\frac{3}{4},0),
\end{aligned}$$

the hypothesis (H1) holds. Now, we choose the functions $f_0(x)=1$ and $f_1(x)=x^3$. Then

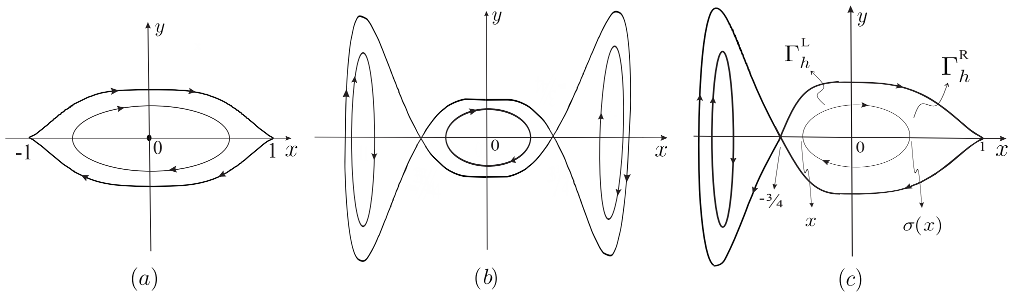

Figure 5: The level curves of Example 3.4.

$$\begin{aligned}
g_1^-(x)&=\psi_2'(x)\int_0^x f_1(t)dt=x^3(x^2-\frac{9}{16})(x^2-1)\int_0^x t^3dt=\frac{1}{4}x^7(x^2-\frac{9}{16})(x^2-1),\\
g_1^+(x)&=\psi_1'(x)\int_0^x f_1(t)dt=x^3(x^2-1)^2\int_0^x t^3dt=\frac{1}{4}x^7(x^2-1)^2,\\
g_0^-(x)&=\psi_2'(x)\int_0^x f_0(t)dt=x^3(x^2-\frac{9}{16})(x^2-1)\int_0^x dt=x^4(x^2-\frac{9}{16})(x^2-1),\\
g_0^+(x)&=\psi_1'(x)\int_0^x f_0(t)dt=x^3(x^2-1)^2\int_0^x dt=x^4(x^2-1)^2,
\end{aligned}$$

and

$$G_0(x)=\frac{g_0^-(x)}{\psi_2'(x)}-\frac{g_0^+(\sigma(x))}{\psi_1'(\sigma(x))}=x-\sigma<0, \quad G_0'(x)=1-\sigma'>0,$$

for $x\in(-\frac{3}{4},0)$ and the hypothesis (H2) holds. We have

$$G_1(x)=\frac{g_1^-(x)}{\psi_2'(x)}-\frac{g_1^+(\sigma(x))}{\psi_1'(\sigma(x))}=\frac{x^4}{3}-\frac{\sigma^4}{3}.$$

We get

$$\tau(x)=\frac{G_1(x)}{G_0(x)}=\frac{1}{4}(x+\sigma)(x^2+\sigma^2), \qquad \tau'(x)=\frac{W_2(x,\sigma)}{64\,\sigma^3\,(\sigma^2-1)^2},$$

where

$$\begin{aligned}
W_2(x,\sigma)=&\,16\,\sigma^9+32\,x\sigma^8+\left(48\,x^2-32\right)\sigma^7-64\,x\sigma^6+\left(-96\,x^2+16\right)\sigma^5+32\,x\sigma^4+48\,x^2\sigma^3\\
&+\left(48\,x^7-75\,x^5+27\,x^3\right)\sigma^2+\left(32\,x^8-50\,x^6+18\,x^4\right)\sigma+16\,x^9-25\,x^7+9\,x^5.
\end{aligned}$$

We compute the resultant between $N_2(x,\sigma)$ in (3.8) and $W_2(x,\sigma)$ with respect to σ and obtain that

$$Res(N_2, W_2, \sigma) = \frac{-7x^{20}}{169075682574336} K_2(x),$$

where

$$\begin{aligned}
K_2(x) = &- 17303797119546556416\, x^{46} + 25091138272067636 4288\, x^{44} - 17294454122813356769 28\, x^{42} \\
&+ 7500466460396266979328\, x^{40} - 22856896790542491844608\, x^{38} + 51823785315669579362304\, x^{36} \\
&- 90371050818741941649408\, x^{34} + 123581449074990259824660\, x^{32} - 133784180147841029737308\, x^{30} \\
&+ 114677307119444399894781\, x^{28} - 77045002412045809005144\, x^{26} + 39632289800414361531012\, x^{24} \\
&- 15002171537223785559456\, x^{22} + 4001793060740844419358\, x^{20} - 844949019290918206728\, x^{18} \\
&+ 279627062108940351456\, x^{16} - 126844502727089175588\, x^{14} + 32392450653602202093\, x^{12} \\
&- 2093327701236455936\, x^{10} + 327889671064596480\, x^{8} - 448111019837214720\, x^{6} \\
&+ 3083746919694336\, x^{4} + 30636835652763648\, x^{2} + 2580375195353088.
\end{aligned}$$

From Sturm's Theorem, we see that the polynomial $K(x)$ has no zero in $(-\frac{3}{4}, 0)$. So $W_2(x,\sigma)$ and $N_2(x,\sigma)$ have no common zeros in $(-\frac{3}{4}, 0)$ and by Theorem 2.2, $S(h)$ is monotone in $h \in (0, \frac{1}{24})$.

Example 3.4. Consider

$$H(x,y) = \begin{cases} \frac{1}{2}y^2 - \frac{1}{3}x^3 + \frac{1}{2}x^2, & x > 0, \\ \frac{1}{2}y^2 + \frac{1}{16807}x^2(2401x^2 + 2401x^4 - 1029x^3 - 1421x^2 + 112x + 288), & x \leqslant 0. \end{cases} \tag{3.9}$$

We show that the function

$$S(h) = \frac{J_1(h)}{J_0(h)} = \frac{\int_{\Gamma_h^L} \frac{g_1^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_1^+(x)}{y}dx}{\int_{\Gamma_h^L} \frac{g_0^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_0^+(x)}{y}dx},$$

is monotone in $h \in (0, \frac{1}{6})$. The semi-orbits Γ_h^L and Γ_h^R in this example are defined for $h \in (0, \frac{1}{6})$; see Fig. 6. Also,

$$\Psi_2 = \Psi_1 \circ \sigma \Longleftrightarrow N_3(x,\sigma) = \frac{1}{7}x^7 + \frac{1}{7}x^6 - \frac{3}{49}x^5 - \frac{29}{343}x^4 + \frac{16}{2401}x^3 + \frac{288}{16807}x^2 + \frac{1}{3}\sigma^3 - \frac{1}{2}\sigma^2 = 0, \tag{3.10}$$

where $-\frac{4}{7} < x < 0 < \sigma(x) < 1$. We have

$$\begin{aligned}
x\Psi_1'(x) &= -x^2(x-1) > 0, \qquad && \forall x \in (0,1), \\
x\Psi_2'(x) &= x^2(x+\frac{4}{7})^3(x-\frac{3}{7})^2 > 0, \qquad && \forall x \in (-\frac{4}{7}, 0).
\end{aligned}$$

Thus, the hypothesis (H1) holds. Now, choose the functions $f_0(x) = 1$ and $f_1(x) = x^2$. Then

$$\begin{aligned}
g_1^-(x) &= \psi_2'(x)\int_0^x f_1(t)dt = x^2(x-\frac{4}{7})^3(x-\frac{3}{7})^2\int_0^x t^2 dt = \frac{1}{3}x^4(x-\frac{4}{7})^3(x-\frac{3}{7})^2, \\
g_1^+(x) &= \psi_1'(x)\int_0^x f_1(t)dt = -x(x-1)\int_0^x t^2 dt = -\frac{1}{3}x^4(x-1), \\
g_0^-(x) &= \psi_2'(x)\int_0^x f_0(t)dt = x^2(x-\frac{4}{7})^3(x-\frac{3}{7})^2\int_0^x dt = x^2(x-\frac{4}{7})^3(x-\frac{3}{7})^2, \\
g_0^+(x) &= \psi_1'(x)\int_0^x f_0(t)dt = -x(x-1)\int_0^x dt = -x^2(x-1),
\end{aligned}$$

and, for $x \in (-\frac{4}{7}, 0)$ we have $G_0(x) = x - \sigma < 0$, $G_0'(x) = 1 - \sigma' > 0$. Hence, the hypothesis (H2) is satisfied. Next, we compute

$$\tau(x) = \frac{G_1(x)}{G_0(x)} = \frac{\sigma^2 + \sigma x + x^2}{3}, \qquad \tau'(x) = \frac{W_3(x,\sigma)}{50421\,(x-1)},$$

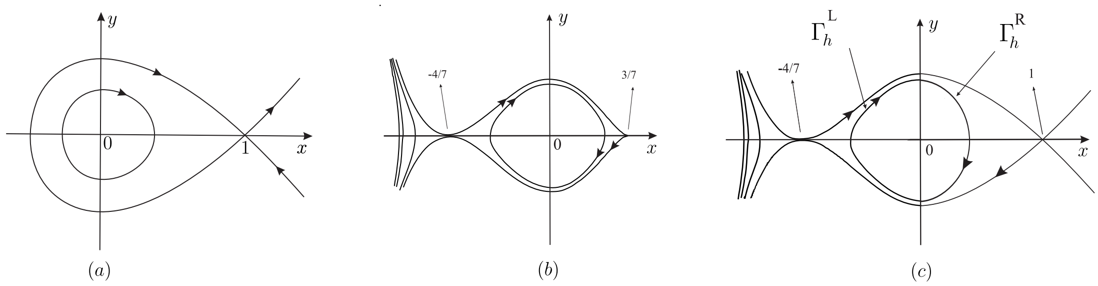

Figure 6: The level curves of Example 3.4.

where

$$\begin{aligned} W_3(x,\sigma) =& -16807\,x^6 + (-33614\,\sigma - 14406)\,x^5 + (-28812\,\sigma + 5145)\,x^4 + (10290\,\sigma + 5684)\,x^3 \\ & + (11368\,\sigma + 33278)\,x^2 + (16135\,\sigma - 34190)\,x - 17959\,\sigma. \end{aligned}$$

The resultant between $N_3(x,\sigma)$ in (3.10) and $W_3(x,\sigma)$ with respect to σ is given by

$$Res(W_3, N_3, \sigma) = \frac{-x^2}{100842} K_3(x),$$

where

$$\begin{aligned} K_3(x) =& -547146968897910864\,x^{20} - 1954096317492538800\,x^{19} - 1875932464792837248\,x^{18} \\ & + 1295286701880768576\,x^{17} + 4295090412676372442\,x^{16} + 3079524763670566722\,x^{15} \\ & - 1697976225600279534\,x^{14} - 4097075130928955910\,x^{13} - 2260505507341693962\,x^{12} \\ & - 442488715458751269\,x^{11} + 2209988706934004913\,x^{10} + 3888994893764985732\,x^9 \\ & + 2447153169061768944\,x^8 - 1724047389021142927\,x^7 - 6013298897467269219\,x^6 \\ & - 26296274718665532\,x^5 + 2902087535161414334\,x^4 + 3176139737322402900\,x^3 \\ & - 2036094102632248218\,x^2 - 1645007996775259508\,x + 1048493313487181388. \end{aligned}$$

Sturm's Theorem implies that the polynomial $K_3(x)$ has no zero in $(-\frac{4}{7}, 0)$. So $W_3(x,\sigma)$ and $N_3(x,\sigma)$ have no common zeros in $(-\frac{4}{7}, 0)$. Applying Theorem 2.2, we conclude that $S(h)$ is monotone in $h \in (0, \frac{1}{6})$.

Example 3.5. Consider

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{y^2}{2} - \frac{x^2}{20}(4x^3 - 15x^2 + 20x - 10), & x > 0, \\ H^-(x,y) = \frac{y^2}{2} + \frac{x^2}{45}(9x^3 - 15x^2 - 5x + 15), & x \leqslant 0. \end{cases} \tag{3.11}$$

We show that the function

$$S(h) = \frac{J_1(h)}{J_0(h)} = \frac{\int_{\Gamma_h^L} \frac{g_1^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_1^+(x)}{y}dx}{\int_{\Gamma_h^L} \frac{g_0^-(x)}{y}dx + \int_{\Gamma_h^R} \frac{g_0^+(x)}{y}dx},$$

is monotone for $h \in (0, \frac{4}{45})$. The semi-orbits Γ_h^L and Γ_h^R in this example are defined for $h \in (0, \frac{4}{45})$; see Fig. 7. We have

$$\Psi_2 = \Psi_1 \circ \sigma \Longleftrightarrow N_4(x,\sigma) = \frac{1}{5}x^5 - \frac{1}{3}x^4 \frac{1}{9}x^3 + \frac{1}{3x^2} + \frac{1}{5}\sigma^5 - \frac{3}{4}\sigma^4 + \sigma^3 - \frac{1}{2}\sigma^2 = 0, \tag{3.12}$$

where $-\frac{2}{3} < x < 0 < \sigma(x) < 1$. Also,

$$\begin{aligned} x\Psi_1'(x) &= -x^2(x-1)^3 > 0, && \forall x \in (0,1), \\ x\Psi_2'(x) &= x^2(x-1)^2(x+\frac{2}{3}) > 0, && \forall x \in (-\frac{2}{3}, 0). \end{aligned}$$

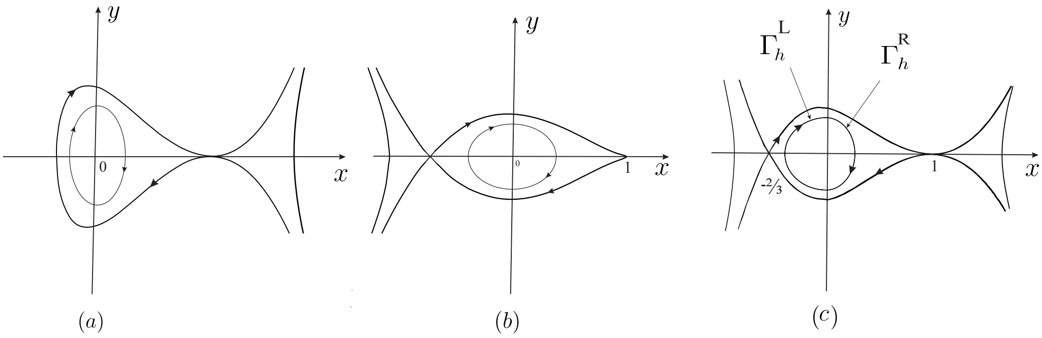

Figure 7: The level curves of Example 3.5.

Thus, the hypothesis (H1) holds. Let $f_0(x)=1$ and $f_1(x)=x^2$. Then

$$\begin{aligned}
g_1^-(x) &= \psi_2'(x)\int_0^x f_1(t)dt = x(x-1)^2(x+\frac{2}{3})\int_0^x t^2dt = \frac{1}{3}x^4(x-1)^2(x+\frac{2}{3}),\\
g_1^+(x) &= \psi_1'(x)\int_0^x f_1(t)dt = -x(x-1)^3\int_0^x t^2dt = -\frac{1}{3}x^4(x-1)^3,\\
g_0^-(x) &= \psi_2'(x)\int_0^x f_0(t)dt = x(x-1)^2(x+\frac{2}{3})\int_0^x dt = x^2(x-1)^2(x+\frac{2}{3}),\\
g_0^+(x) &= \psi_1'(x)\int_0^x f_0(t)dt = -x(x-1)^3\int_0^x dt = -x^2(x-1)^2,
\end{aligned}$$

and for $x\in(-\frac{2}{3},0)$ we have $G_0(x)=x-\sigma<0$ and $G_0'(x)=1-\sigma'>0$. Hence, the hypothesis (H2) is satisfied. Next, we compute

$$\tau(x)=\frac{G_1(x)}{G_0(x)}=\frac{\sigma^2+\sigma x+x^2}{3},\qquad \tau'(x)=\frac{W_4(x,\sigma)}{9\sigma(\sigma-1)},$$

where

$$\begin{aligned}
W_4(x,\sigma)=&-3\,\sigma^5+(6\,x-9)\,\sigma^4+(-18\,x+9)\,\sigma^3+(18\,x-3)\,\sigma^2\\
&+\left(-6\,x^4+8\,x^3+2\,x^2-10\,x\right)\sigma-3\,x^2\,(x-1)^2\left(x+\frac{2}{3}\right).
\end{aligned}$$

The resultant between $N_4(x,\sigma)$ in (3.14) and $W_4(x,\sigma)$ with respect to σ is given by

$$Res(W_4,N_4,\sigma)=\frac{-x^4}{777600000}K_4(x),$$

where

$$\begin{aligned}
K_4(x)=&-1934917632\,x^{21}+7255941120\,x^{20}+83980800\,x^{19}-17731146240\,x^{18}-48065740200\,x^{17}+168002281845\,x^{16}\\
&+2931568200\,x^{15}-406114723620\,x^{14}+261742158360\,x^{13}+393831940830\,x^{12}-431084163956\,x^{11}\\
&-163954269780\,x^{10}+309289609260\,x^9+12766683285\,x^8-115053566580\,x^7+10804892940\,x^6\\
&+21267175860\,x^5-2705984280\,x^4-1342118160\,x^3+6342300\,x^2-57153600\,x+33242400.
\end{aligned}$$

Sturm's Theorem implies that the polynomial $K_4(x)$ has no zero in $(-\frac{2}{3},0)$. So $W_4(x,\sigma)$ and $N_4(x,\sigma)$ have no common zeros in $(-\frac{2}{3},0)$. Applying Theorem 2.2, we conclude that $S(h)$ is monotone in $h\in(0,\frac{4}{45})$.

Example 3.6. Consider

$$H(x,y)=\begin{cases} H^+(x,y)=\frac{y^2}{2}+\frac{1}{42}x^2(7x^4+54x^3+159x^2+216x+120), & x>0,\\ H^-(x,y)=\frac{y^2}{2}+\frac{1}{2058}x^2(2401x^4-2058x^3-4263x^2+448x+1440), & x\leqslant 0.\end{cases}\tag{3.13}$$

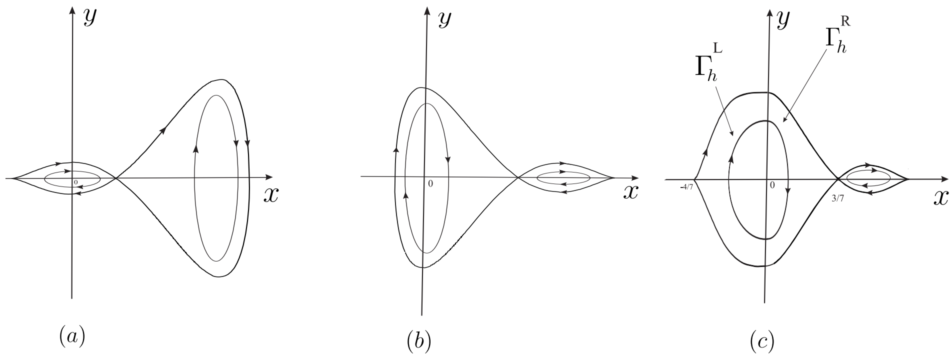

Figure 8: The level curves of Example 3.6.

We show that the function

$$S(h) = \frac{J_1(h)}{J_0(h)} = \frac{\int_{\Gamma_h^L} \frac{g_1^-(x)}{y} dx + \int_{\Gamma_h^R} \frac{g_1^+(x)}{y} dx}{\int_{\Gamma_h^L} \frac{g_0^-(x)}{y} dx + \int_{\Gamma_h^R} \frac{g_0^+(x)}{y} dx},$$

is monotone in $h \in (0, \frac{8}{21})$. The semi-orbits Γ_h^L and Γ_h^R in this example are defined for $h \in (0, \frac{8}{21})$; see Fig. 8. Note that

$$\Psi_2 = \Psi_1 \circ \sigma \Longleftrightarrow N_5(x, \sigma) = 0, \tag{3.14}$$

where

$$N_5(x, \sigma) = \frac{7}{6} x^6 - x^5 - \frac{29}{14} x^4 + \frac{32}{147} x^3 + \frac{240}{343} x^2 - \frac{1}{6} \sigma^6 + \frac{9}{7} \sigma^5 - \frac{53}{14} \sigma^4 + \frac{36}{7} \sigma^3 - \frac{20}{7} \sigma^2$$

and $-\frac{4}{7} < x < 0 < \sigma(x) < \frac{3}{7}$. We have

$$x\Psi_1'(x) = \frac{x^2}{7}(x+1)(7x+10)(x+2)^2 > 0, \quad \forall x \in (0, \frac{3}{7}),$$
$$x\Psi_2'(x) = 7x^2(x - \frac{10}{7})(x - \frac{3}{7})(x + \frac{4}{7})^2 > 0, \quad \forall x \in (-\frac{4}{7}, 0).$$

Thus, the hypothesis (H1) holds. Now, choose the functions $f_0(x) = 1$ and $f_1(x) = x$. Then

$$g_1^-(x) = \psi_2'(x) \int_0^x f_1(t) dt = \frac{x}{7}(1-x)(10-7x)(2-x)^2 \int_0^x t dt = \frac{x^3}{14}(1-x)(10-7x)(2-x)^2,$$
$$g_1^+(x) = \psi_1'(x) \int_0^x f_1(t) dt = 7x(x - \frac{10}{7})(x - \frac{3}{7})(x + \frac{4}{7})^2 \int_0^x t dt = \frac{7}{2} x^3 (x - \frac{10}{7})(x - \frac{3}{7})(x + \frac{4}{7})^2,$$
$$g_0^-(x) = \psi_2'(x) \int_0^x f_0(t) dt = \frac{x}{7}(1-x)(10-7x)(2-x)^2 \int_0^x dt = \frac{x^2}{7}(1-x)(10-7x)(2-x)^2,$$
$$g_0^+(x) = \psi_1'(x) \int_0^x f_0(t) dt = 7x(x - \frac{10}{7})(x - \frac{3}{7})(x + \frac{4}{7})^2 \int_0^x dt = 7x^2(x - \frac{10}{7})(x - \frac{3}{7})(x + \frac{4}{7})^2,$$

and

$$G_0(x) = x - \sigma < 0, \qquad G_0'(x) = 1 - \sigma' > 0,$$

for $x \in (-\frac{4}{7}, 0)$ and hence, the hypothesis (H2) is satisfied. Next, we compute

$$\tau(x) = \frac{G_1(x)}{G_0(x)} = \frac{1}{2}(x + \sigma), \qquad \tau'(x) = \frac{W_5(x, \sigma)}{2\sigma\,(7\sigma - 10)\,(7\sigma - 3)\,(7\sigma + 4)^2},$$

where

$$W_5(x, \sigma) = 2401\,\sigma^5 + 343\,x^5 - 1715\,\sigma^4 - 2205\,x^4 - 2842\,\sigma^3 + 5194\,x^3$$
$$+ 224\,\sigma^2 - 5292\,x^2 + 480\,\sigma + 1960\,x.$$

The resultant between $N_5(x,\sigma)$ in (3.14) and $W_5(x,\sigma)$ with respect to σ is given by

$$Res(W_5, N_5, \sigma) = \frac{-x^2}{1333584} K_5(x),$$

where

$$\begin{aligned}
K_5(x) = & 5522138368426873911111007597\, x^{28} - 18836448563787541900223799600\, x^{27} \\
& - 54718357563249790505749014 69\, x^{26} + 28445084880534378203062762476\, x^{25} \\
& - 56541343489315515176926113 45\, x^{24} - 126129956295650250232677730104\, x^{23} \\
& + 164336432417843098110307085945\, x^{22} + 187723126533533802031872319212\, x^{21} \\
& - 71698427035172897795002043158 8\, x^{20} + 1180124647197971742536923730104\, x^{19} \\
& - 1577235227802221733779196747780\, x^{18} + 5974407384295323006990448 68912\, x^{17} \\
& + 718308276961496212248382283088\, x^{16} + 2591686059465472315521576028320\, x^{15} \\
& - 4864012798198011583700951709648\, x^{14} - 3362595185680239138978351328640\, x^{13} \\
& + 8585844713431718735534889142464\, x^{12} + 6034415910236849464479747340 80\, x^{11} \\
& - 5683531198807713774265950432064\, x^{10} - 2460800574606330474249987526656\, x^{9} \\
& + 5947428254135543071519507863552\, x^{8} + 320189545983017810261364414464\, x^{7} \\
& - 3541138465670160577034486994432\, x^{6} + 948882618263998795632279650304\, x^{5} \\
& + 720462880579177519610025271296\, x^{4} - 123407800930406208541048012 80\, x^{3} \\
& - 519606440221753974198780518400\, x^{2} + 296641724987874470951190528000\, x \\
& - 5465969890193662832885760 0000.
\end{aligned}$$

Sturm's Theorem implies that the polynomial $Res(W_5, N_5, \sigma)$ has no zero in $(-\frac{4}{7}, 0)$. So $W_5(x,\sigma)$ and $N_5(x,\sigma)$ have no common zeros in $(-\frac{4}{7}, 0)$. Using Theorem 2.2, we conclude that $S(h)$ is monotone in $h \in (0, \frac{8}{21})$.

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